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Efficiency optimization in forced ratchets due to thermal fluctuations

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Abstract

The average current and efficiency of an overdamped Brownian particle, moving in an asymmetric potential and subject to an external driving force is studied by Langevin equation simulations. We found that there is a regime where the efficiency can be optimized at finite temperatures, contradicting the earlier findings (H. Kamegawa et al. Phys. Rev. Lett. 80 (1998) 5251). This, in fact, proves that thermal fluctuations contribute to the efficiency of the forced thermal ratchets. The conditions for achieving maximum flux and efficiency are different as claimed in previous investigations. We also discuss the influence of these quantities on the period of the external driving force. We argue that the theoretical results are valid only in the limiting case where the period of the external driving force is infinity. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Much of the recent interest in non-equilibrium induced transport processes is concentrated on stochastically driven ratchets. For reviews see Refs. [1,2]. The development of this subject has been motivated by the challenge to explain unidirectional transport in biological systems [3–5], as well as by their potential novel technological applications ranging from classical non-equilibrium models [6,7] to quantum systems [8–11].

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The energetics of such ratchet systems, which rectify the zero-mean fluctuations are under thorough investigation in the recent years [12,13]. To define optimal models for such ratchet systems, the maximization of the efficiency of energy transformation is inevitable [12,14].

A large amount of interest was initiated by the elegant piece of work done by Magnasco [15] which showed that a Brownian particle, subject to external fluctuations, can undergo a non-zero drift while moving under the influence of an asymmetric potential. The temperature dependence of the current has been studied and it has been shown that the current can be maximized at finite temperatures. He claimed that there is a region of operating regime where the efficiency can be maximized at finite temperatures and the existence of thermal fluctuations facilitate the efficiency of energy transformation.

This work was followed by Kamegawa et al. [13] (to be referred as KHT) who did an energetic analysis of the same model used by Magnasco for a driving force having a square waveform with amplitude A. The energetic analysis was done with arguments developed by Sekimoto [12]. The important conclusion made by KHT is that the efficiency of energy transformation cannot be optimized at finite temperatures and that the thermal fluctuations does not facilitate it. A recent investigation by Dan and co-workers [16] showed that the efficiency can be optimized at finite temperatures in inhomogeneous systems with spatially varying friction coefficient in an adiabatically rocked ratchet. The important question of whether the thermal fluctuations actually facilitate the energy transformation in forced, homogeneous ratchet systems is still unknown and is the subject of the current investigation.

By integrating the Langevin equation, we show that it is possible to optimize the efficiency of the energy transformation of a forced thermal ratchet at finite temperatures, and that it is facilitated by thermal fluctuations.

The rest of the article is organized as follows. In Section 2 we describe the ratchet model. In Section 3 a summary of the results is presented. We measure the current and the efficiency for a ratchet system subject to an external driving force of square waveform operating at different frequencies. We also analyze the case when the driving force is of the form $F(t)=A\sin(\omega t)$. We also consider the behavior of the system with the amplitude as a variable, and conclude with the study of a system subject to a driving force of non-zero mean. The conclusions are presented in Section 4.

2. The model

The Brownian dynamics of the overdamped particle, moving under the influence of an asymmetric potential $V_0(x)$ and subject to an external force field $F(t)$ at temperature T, is described by the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial}{\partial x}[V_0(x) + V_L(x)] + F(t) + \sqrt{2k_BT}\,\xi(t)\,, \tag{1}$$

where $\xi(t)$ is a randomly fluctuating Gaussian white noise with zero mean and with autocorrelation function $\langle\xi(t)\xi(s)\rangle = \delta(t-s)$. $V_L(x)$ is a potential against which the work is done and $\partial/\partial x V_L = \lambda$ is the load force. The damping coefficient β is absorbed in the time scale and temporal symmetry is applied.

We consider a piecewise linear but asymmetric ratchet potential $V_0(x)$ of periodicity $L = 1$,

$$V_0(x) = \begin{cases} s_1 x, & x \leqslant a, \\ s_2(1 - x), & a < x \leqslant L \end{cases} \tag{2}$$

with slopes $s_1 = Q/a$ and $s_2 = Q/(1 - a)$. Throughout the work, we kept fixed $a = 0.8$ and $Q = 1$. The periodic external driving force $F(t)$ has a square waveform,

$$F(t) = \begin{cases} A, & n\tau \leqslant t < n\tau + \tau_1, \\ -A, & n\tau + \tau_1 \leqslant t < (n + 1)\tau \end{cases} \tag{3}$$

with a period τ larger than the time scale of the Brownian particles in the bath environment, but smaller than the diffusion time of the particle over the potential barriers.

A quantity of central interest is the time-averaged current J, or the velocity of the particles in the stationary state given by the relation $\langle \dot{x}(t) \rangle_{st} = LJ$. The latter can be evaluated from Eq. (1) and, in general, can be written as:

$$\langle \dot{x}(t) \rangle_{st} = -\lambda + \frac{1}{\tau} \lim_{t \to \infty} \int_t^{t+\tau} \langle f(x(s)) + F(s) \rangle \, \mathrm{d}s \,, \tag{4}$$

where

$$f(x) = -\frac{\partial}{\partial x} V(x) \tag{5}$$

and $V(x) = V_0(x) + V_L(x)$.

We have also examined the efficiency of the energy transformation following the method of stochastic energetics [12,13,16]. As usual, the efficiency η is defined as the ratio of the useful work W accomplished by the system in pumping particles against the load force λ, to the input of energy E_i from the external fluctuation. Thus,

$$\eta = W/E_i$$

with

$$W = \frac{1}{\tau} \int_{x(n\tau)}^{x((n+1)\tau)} \mathrm{d}V[x(t)] = \lambda J \tag{6}$$

and

$$E_i = \frac{1}{\tau} \int_{x(n\tau)}^{x((n+1)\tau)} F(t) \, \mathrm{d}x(t) \,. \tag{7}$$

We have integrated Eq. (1) by using a stochastic integral algorithm [17], with a time step of $\Delta t = 1.e - 4$. The data has been averaged over 10^4 different trajectories, each trajectory evolving over 50 periods (50τ).

3. Results and discussion

We have calculated the average current and the efficiency as a function of the ratio k_BT/Q for the case where a zero-mean external driving force is applied ($\tau_1 = \tau/2$). The time period is fixed at $\tau = 6$ and several values of the amplitude A have been considered. The load term has been fixed to $\lambda = 0.01$. The results are plotted in Figs. 1 and 2, respectively.

From Fig. 1, we observe three different regimes, in agreement with earlier observations [13,15]: (1) $A < Q/a + \lambda$; (2) $Q/a + \lambda < A < Q/(1-a) - \lambda$; (3) $A > Q/(1-a) - \lambda$. The first two regimes correspond to low and moderate forcing. In these regimes, an increase of the temperature first results in a rise and then a fall in the average current, thus, it is possible to find a region where the current is optimized at finite temperatures. In the third regime, for high amplitudes of the forcing, the current decreases monotonically as a function of the temperature, as the ratchet effect becomes unimportant in this regime.

The corresponding behavior of the efficiency, as it is shown in Fig. 2, is much more interesting. For a period $\tau = 6$ we can observe that for low amplitudes of the forcing, the efficiency attains a maximum at a finite temperature. As the amplitude increases this effect starts to disappear. At very high amplitudes of the forcing, the efficiency is too small and an increase in temperature reduces it to almost zero. In contradiction with the results of KHT we found a region where the efficiency can be optimized with respect to the temperature. This indicates that thermal fluctuations may facilitate the

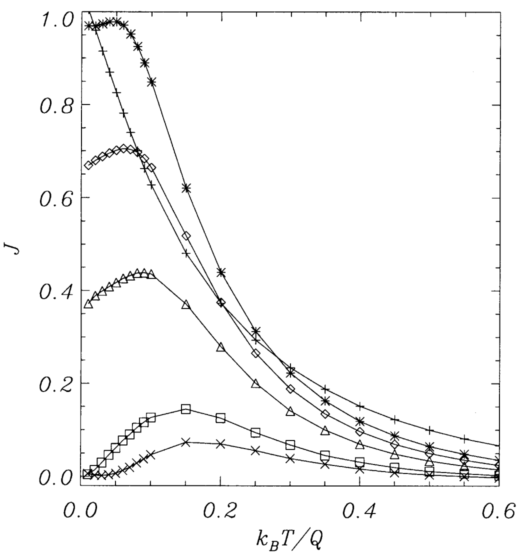

Fig. 1. Current as a function of k_BT/Q, for different values of the amplitude of the driving force A: $A = 1.0$ ($\times$); $A = 1.3$ ($\square$); $A = 2.0$ ($\triangle$); $A = 2.5$ ($\diamond$); $A = 3.0$ ($*$); $A = 6.0$ ($+$).

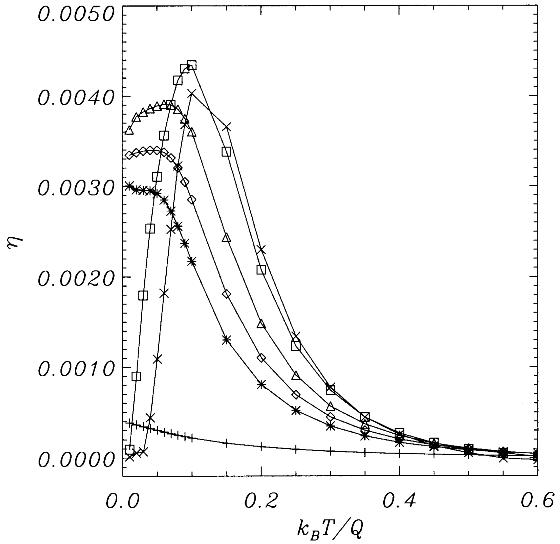

Fig. 2. Plot of efficiency as a function of k_BT/Q. The conditions are the same as in Fig. 1.

energy conversion by ratchet systems for low amplitude values of the driving force, proving the claim by Magnasco to be correct.

In order to understand this discrepancy we briefly review the arguments followed by KHT. For the equation of motion for the Brownian ratchet (1), we can write the associated Fokker–Planck equation in the form of a conservation law for the probability density $P(x,t)$, $\partial_t P(x,t) + \partial_x j(x,t) = 0$, and a probability current $j(x,t)$ obeying:

$$j(x,t) = \left(f(x) + F(t) - k_B T \frac{\partial}{\partial x} \right) P(x,t) \,. \tag{8}$$

Assuming the solution for the probability density to be periodic in time and space, $P(x,t) = P(x+L,t) = P(x,t+\tau)$, it is possible to solve analytically $j(x,t)$ for a *constant* driving force [15]. The average current, under the assumption that $F(t)$ changes slowly enough compared to any other frequency in the problem, and for an applied external field of a square waveform of amplitude A, reads [12] as

$$J = \frac{1}{2}[J(A) + J(-A)]$$

and the efficiency

$$\eta = \frac{\lambda[J(A) + J(-A)]}{A[J(A) - J(-A)]}$$

which in turn, being $J(-A) < 0$, can be written as

$$\eta = \frac{\lambda}{A} \left(\frac{1 - |J(-A)/J(A)|}{1 + |J(-A)/J(A)|} \right) \,. \tag{9}$$

The current depends on the sum of $J(A)$ and $J(-A)$. For small and moderate amplitudes of the external fluctuation, $|J(-A)|$ increases slower than $|J(A)|$ as k_BT

increases, thus, a maximum in J is expected at finite temperatures. Eq. (9) shows that the efficiency η depends on the ratio $|J(-A)/J(A)|$. If this function is monotonically increasing, η should be a monotonically decreasing function of the temperature. KHT found the factor $|J(-A)/J(A)|$ to be always a monotonically increasing function of the temperature, causing the efficiency to be always a monotonically decreasing function of temperature. However, our results show a much richer behavior. For small amplitudes we clearly see, from Fig. 2, that the efficiency is not a monotonically decreasing function of temperatures.

In Fig. 3, we present the fluxes obtained for the case under study of a field $F(t)$ of a square waveform with amplitude $A=1.3$. Fig. 3(a) corresponds to a periodicity of the driving force of $\tau=6$. It is clearly seen that the ratio $|J(-A)/J(A)|$ displays a clear minimum at the same value of the temperature which corresponds to the maximum of η in Fig. 2.

In order to explain the difference with the analytical result, we have analyzed the effect on the efficiency of the period of the driving force τ. We have then considered a period of $\tau=50$. The results for the efficiency are represented in Fig. 4. The curves corresponding to $A=2$ and 2.5, which were giving clear optimum values in Fig. 2 are now monotonically decreasing with the temperature as found by KHT. We also note that as τ increases, the efficiency approaches the theoretical limiting value of λ/A. For smaller A values the peaks in the efficiency curve still exist and we expect it will follow the theoretical result only for much longer periods. The current in each half period has been analyzed and the result is shown in Fig. 3(b). We can still observe a shallow minimum for $|J(-A)/J(A)|$ at finite temperatures. Finally, for a constant force (Fig. 3(c)) we were able to reproduce exactly the analytical results in agreement with that of KHT.

For increasing amplitude of the driving force A, the results obtained by KHT are qualitatively reproduced at finite τ values, and the ratio $|J(-A)/J(A)|$ is monotonically increasing. As an example, the results for $A=3$ and $\tau=6$ are plotted in Fig. 5.

Our results conclude that the presence of thermal fluctuations are important to improve the efficiency of energy transformation at low amplitudes of the forcing and that the analytical result is valid only in the limiting case of $\tau \to \infty$.

We have also performed numerical computations for a sinusoidal driving force $F(t)=A\sin(\omega t)$. As in the square wave case, the efficiency can be optimized at a finite temperature. The results for current and efficiency show qualitatively the same kind of behavior and are shown in Figs. 6 and 7 for different amplitudes.

For the square wave forcing as well as for the sinusoidal one, observe that the temperature associated with the maximum current J does not correspond with the one at which η is maximum. From this we can conclude that the conditions for obtaining optimum current and efficiency are different as previously noted in other ratchet systems [16].

We have also studied the behavior of the current and efficiency as a function of the amplitude A of a external driving force of a square waveform with periodicity $\tau=6$. The results are shown in Figs. 8 and 9, respectively. It is clear that both the current

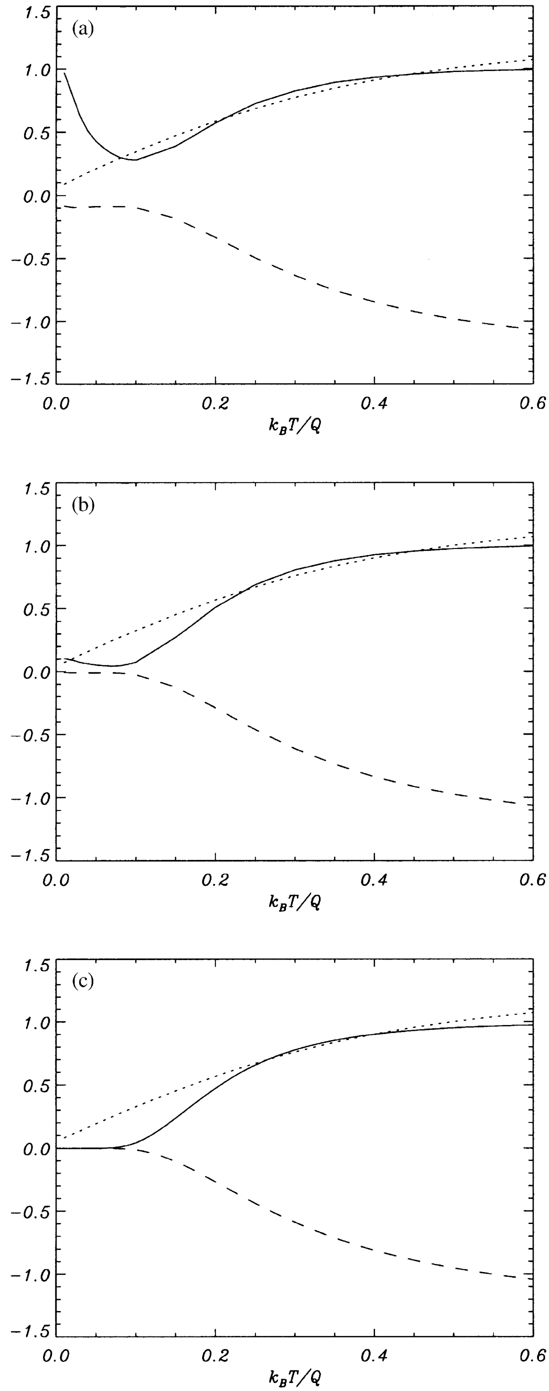

Fig. 3. Plot of the currents for a field $F(t)$ of a square waveform with amplitude $A = 1.3$ at different periods τ: $J(A)$ (dotted line); $J(-A)$ (dashed line); $|J(-A)/J(A)|$ (solid line). From top to bottom: (a) $\tau = 6$; (b) $\tau = 50$; (c) $\tau = \infty$.

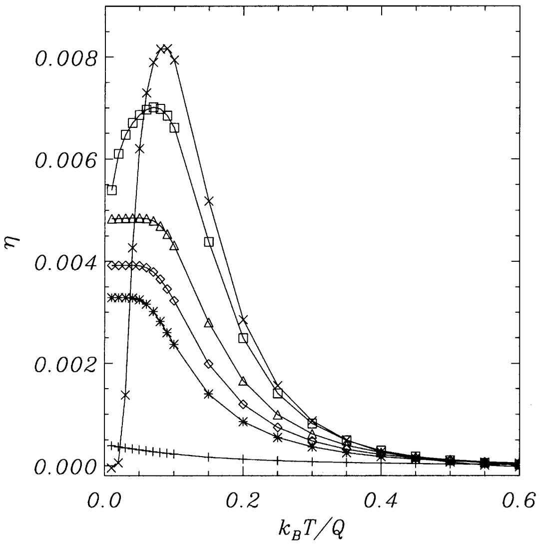

Fig. 4. Same as Fig. 2 for a forcing period $\tau = 50$.

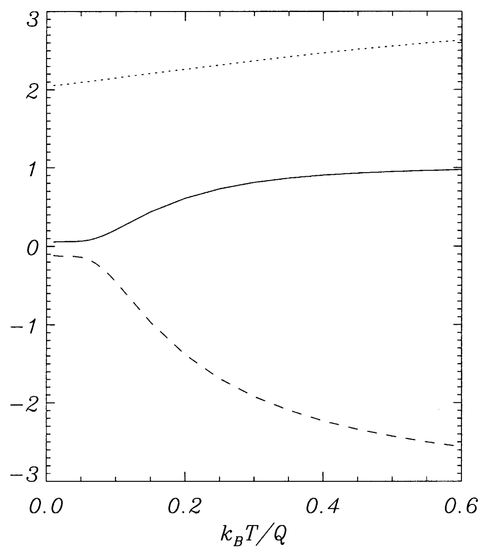

Fig. 5. Same as Fig. 3 for $A = 3$ and $\tau = 6$.

and efficiency can be maximized with respect to the amplitude A. Smaller amplitudes are always preferred for having better efficiency and, as the amplitude goes to higher values, the ratchet becomes almost inefficient.

Since the previous calculations were made with $\tau_1 = \tau/2$, it is interesting to see how the current and the efficiency varies with time periods τ_1 and $\tau_2 \equiv \tau - \tau_1$ over which

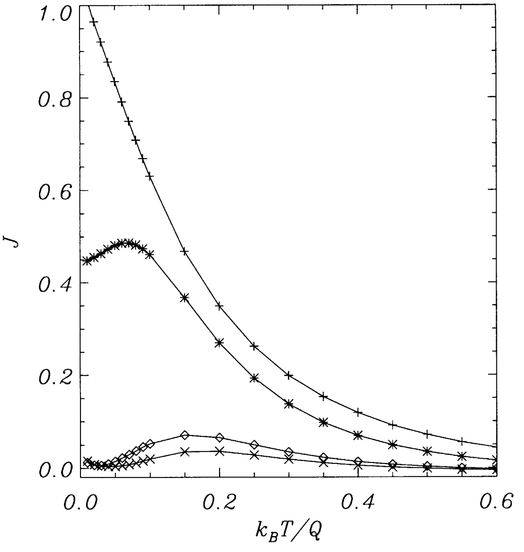

Fig. 6. Current as a function of k_BT/Q for $F(t) = A\sin(\omega t)$. We have chosen $\omega = \pi/3$ that corresponds to $\tau = 6$. $A = 1.0$ ($\times$); $A = 1.3$ ($\square$); $A = 3.0$ ($*$); $A = 6.0$ ($+$).

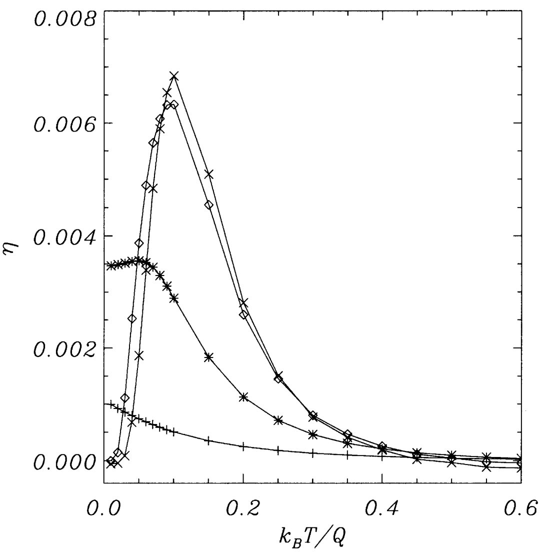

Fig. 7. Same as Fig. 6 for the efficiency.

the system experiences a positive and negative driving force. In Figs. 10 and 11, we give respectively, the current and the efficiency as a function of τ_1/τ_2 are plotted. The efficiency increases with the ratio in the same way the current does, and then attains a limiting value. The efficiency cannot be optimized with this ratio unlike in the case of thermal ratchet pumps [18] where it is maximized for ratios of τ_1/τ_2 less than unity.

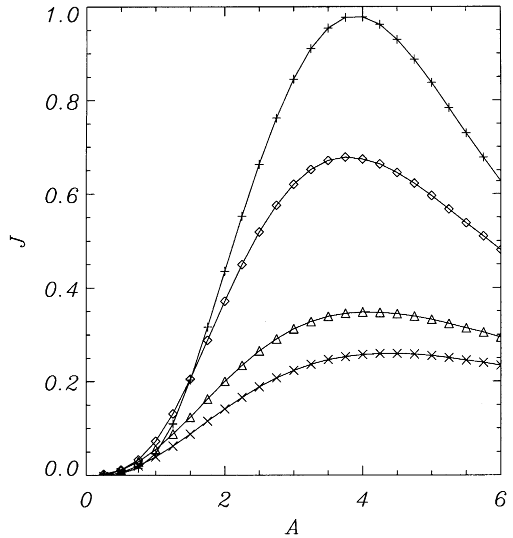

Fig. 8. Current as a function of A, for different k_BT values and $\tau = 6$. $k_BT = 0.10$ (+); $k_BT = 0.15$ ($\Diamond$); $k_BT = 0.25$ ($\triangle$); $k_BT = 0.30$ ($\times$).

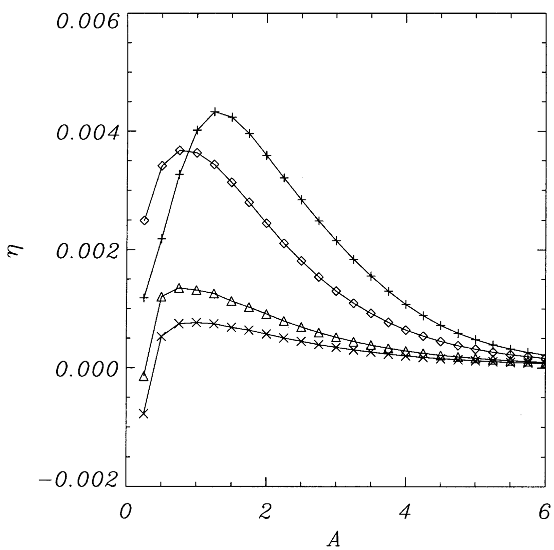

Fig. 9. Same as Fig. 8 for the efficiency.

4. Conclusion

Langevin equation simulations have been done to investigate the energetics of an overdamped forced ratchet. In contradiction with the previous findings, our results for a driving force of a square wave type and finite period τ, show that the efficiency of

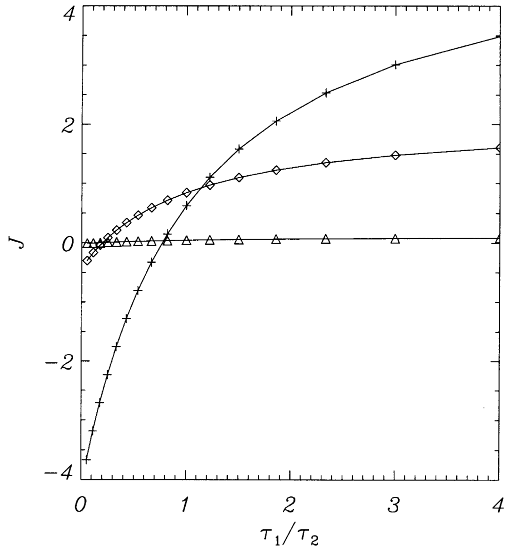

Fig. 10. Current as a function of τ_1/τ_2, for $k_BT = 0.1$ and $\tau = 6$ for different A values: $A = 1.0$ ($\triangle$); $A = 3.0$ ($\diamond$); $A = 6.0$ (+).

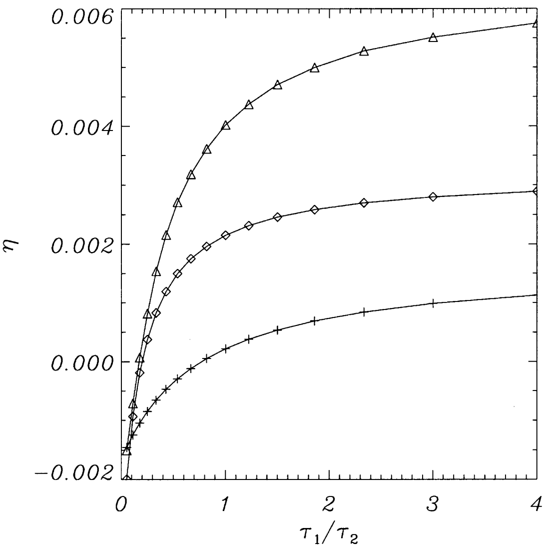

Fig. 11. Same as Fig 10 for the efficiency.

energy transformation can be optimized at finite temperatures in a spatially homogeneous system. This proves the claim made by Magnasco that there is a region of the operating regime where the efficiency can be optimized at finite temperatures. Similar observation has been done in the case of an inhomogeneous rocked system with spatially varying friction coefficient. We have also studied the case of a sinusoidal driving force and the special case of a non-zero mean force. We conclude that the thermal fluctuations facilitate the energy conversion of homogeneous forced ratchet systems, and that the period of the external driving force has enormous influence in deciding the contribution of the thermal fluctuations.

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