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The surface statistics of a granular aggregate

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The problem of the surface fluctuations in a settled granular material is posed. A simple model is given which describes the process by which a particle settles and comes to rest on the existing surface of the packing and from this a set of Langevin equations for the Fourier modes of the surface are derived. These equations imply that the Fourier amplitudes behave like the velocities of a set of independent Brownian particles. We show that this results in logarithmically divergent surface fluctuations if the flux of particles onto the surface is random, the divergence being removed by a more accurate description of the settling material, for example by having the granules fall through a sieve.

1. Introduction

Suppose that we take a bin and gently and uniformly pour in a granular material. As the material in the bin builds up we can identify a surface and ask the question. 'What is the magnitude of the fluctuation in the height of surface (measured from the base of the bin)?' Also of interest is the length scale of the surface fluctuations and how they behave dynamically as more material is added.

The statement that the material is added gently and uniformly is a statement about the flux of material into the bin. This statement will have to be mathematized eventually but we make the following assumptions which lead to the simplest problem, but which are physically realizable:

(i) The flux is sufficiently weak so that one can ignore any correlation between incoming particles.

(ii) The particles settle gently (as if sedimenting from a viscous liquid) in such a way that once each particle has settled under gravity it does not then move when other particles settle above it. Clearly a packing made according to this prescription will be more dilute than one which is shaken where a cooperative reorganization of the particles is allowed to take place.

These assumptions can be considered as part of the definition of the problem.

In §2 we go on to develop a model of the surface behaviour but first we must define the surface and the probability of finding it in some particular shape.

It is possible to devise an operational definition for a surface e.g. for each triplet of particles in the ensemble construct the plane joining the three centres of mass (the plane is not extended to infinity but is a finite triangle whose vertices are the centres

of mass). The surface is formed by those portions of each plane which do not lie vertically below part of another plane.

Having defined the surface we describe it using a single valued function $z(\boldsymbol{\rho}, N)$, where z is the height of the surface vertically above a point on the base of the bin, defined by the two dimensional vector $\boldsymbol{\rho} = x, y$ and N is the number of spheres in the bin at the time the surface is described.

After N spheres have been added to the bin there will be some probability of finding the surface described by a function $z(\boldsymbol{\rho})$. This probability is a function of the value of z at every point $\boldsymbol{\rho}$, i.e. it depends on the whole function $z(\boldsymbol{\rho})$. The probability is a functional of z and is denoted

$$p[z(\boldsymbol{\rho}), N].$$

In addition we can define a Green functional by asking for the probability that the surface is $z(\boldsymbol{\rho})$ after N spheres have been added given that after N' spheres it was $z'(\boldsymbol{\rho})$. This is denoted by

$$G[z(\boldsymbol{\rho}), N; z'(\boldsymbol{\rho}), N'].$$

In the usual way we can write down a functional integral equation which expresses the law of compound probabilities

$$p[z(\boldsymbol{\rho}), N] = \int \delta z'(\boldsymbol{\rho})\, G[z(\boldsymbol{\rho}), N; z'(\boldsymbol{\rho}), N']\, p[z'(\boldsymbol{\rho}), N'], \tag{1.1}$$

where the integral is over all possible functions $z'(\boldsymbol{\rho})$.

In §2 we construct a microscopic model of the Green functional for $N = N' + 1$. From this we derive a Langevin equation and hence derive the statistics of the surface following the theory of Brownian motion.

2. The blob model

When a particle lands and settles on the existing ensemble of particles, the surface changes. Consider the addition to the surface as a blob. Some of this blob will be made up of particle and some of the blob will consist of space. Ignore the distinction between particle and space and think only of blobs being added to the surface. The shape of blobs added to the surface will depend on the shape of the surface, just before the moment the blob lands, as well as the shape of the particle landing. Also the position of each blob added will depend on the shape of the surface locally because newly added particles will tend to seek favourable locations on the surface.

All the information relating the probable shape and position of a blob added to the surface to the shape of the surface just before the blob is added is contained in the Green functional

$$G[z(\boldsymbol{\rho}), N; z'(\boldsymbol{\rho}), N-1]$$

introduced in §1.

Now the Green functional cannot depend on N explicitly as the laws which determine the behaviour of the system do not change with N. (The Green functional

may depend on the number of particles in the bin but only in as much as $z'(\boldsymbol{\rho})$ is a function of that number.) We also define a variable $v(\boldsymbol{\rho})$ as $z(\boldsymbol{\rho}) - z'(\boldsymbol{\rho})$, which represents the shape of a blob. Thus the Green functional becomes

$$G[z(\boldsymbol{\rho}), N; z'(\boldsymbol{\rho}), N-1] \equiv W[v(\boldsymbol{\rho}); z'(\boldsymbol{\rho})],$$

where $W[v(\boldsymbol{\rho}); z'(\boldsymbol{\rho})]$ is the probability functional that a blob added to a surface described by $z'(\boldsymbol{\rho})$ will have shape $v(\boldsymbol{\rho})$. The functional equation (1.1) for ΔN equal to 1 becomes

$$p[z(\boldsymbol{\rho}), N+1] = \int \delta v(\boldsymbol{\rho})\, W[v(\boldsymbol{\rho}); z(\boldsymbol{\rho}) - v(\boldsymbol{\rho})]\, p[z(\boldsymbol{\rho}) - v(\boldsymbol{\rho}), N]. \tag{2.1}$$

At this point we must input the microscopic physics of the process of building up the surface one particle at a time and hence find a model for the functional W. The crudest possible form of W is obtained by completely ignoring the dependence of the function $v(\boldsymbol{\rho})$, which describes the blob, on the function $z(\boldsymbol{\rho})$ which describes the surface. In these circumstances $v(\boldsymbol{\rho})$ becomes a random variable. A suitable model for $W[v(\boldsymbol{\rho})]$ would then be

$$W[v(\boldsymbol{\rho})] = \frac{1}{L^2}\int \mathrm{d}^2\rho_0\, \delta(v(\boldsymbol{\rho}) - f(\boldsymbol{\rho} - \boldsymbol{\rho}_0)). \tag{2.2}$$

The interpretation of this equation is as follows.

Each blob is a shape $f(\boldsymbol{\rho})$ centred at a point $\boldsymbol{\rho}_0$, where $\boldsymbol{\rho}_0$ is a random point in the bin, whose area is L^2. The function $f(\boldsymbol{\rho} - \boldsymbol{\rho}_0)$ might be some given function or it may itself be distributed with some probability functional. The only important feature of $f(\boldsymbol{\rho} - \boldsymbol{\rho}_0)$ is that it must have a sharp cut off at $|\rho - \rho_0| = a$, where a is the width of the blob. For example

$$f(\boldsymbol{\rho} - \boldsymbol{\rho}_0) = \frac{h}{2\pi a^2} \exp\left(-\frac{(\boldsymbol{\rho} - \boldsymbol{\rho}_0)^2}{2a^2}\right), \tag{2.3}$$

where h is the total volume of each blob, would be a suitable model for the blob shape.

In § 3, we shall show that the crude model of W given by (2.2) gives rise to an indefinite increase in the magnitude of the surface fluctuations as the packing builds up. The root mean square fluctuations increase like the square root of the depth of the packing, in fact. This behaviour stems purely from the fact that we have put $v(\boldsymbol{\rho})$ equal to a random variable, uncorrelated with $z(\boldsymbol{\rho})$.

The crucial difference between the real system and the one represented by (2.2) is the fact that particles will tend to settle in local minima of the existing surface. If a particle descends centred at $\boldsymbol{\rho}_0$ its final resting place on the surface will be near $\boldsymbol{\rho}_0$ but will be in a local minimum of $z(\boldsymbol{\rho})$. We might, at first sight, expect that particles will tend to find resting positions of low potential energy in a global sense, i.e. will tend to sit in regions of the surface where $z(\boldsymbol{\rho})$ is smaller than average. This, however, is not the case. The phenomenon of settling is entirely local as we have assumed that the particles settle without kinetic energy and they move, on the

surface, only so far as to find a resting position in which they are stable under gravity; i.e. a local minimum of the surface. Thus $v(\boldsymbol{\rho})$ which describes the blob added to the surface is no longer a random variable but depends on $z(\boldsymbol{\rho})$.

A very simple model for W which describes this situation can be obtained by dividing the bin into a square lattice. Thus $z(\boldsymbol{\rho})$ becomes a discrete set of values z_a and similarly $v(\boldsymbol{\rho})$. In this representation a model for W in which $v(\boldsymbol{\rho})$ is uncorrelated with $z(\boldsymbol{\rho})$ would then be

$$W(\{v_a\}) = \frac{1}{M^2}\sum_a \prod_{\beta\neq\alpha} \delta(v_\beta)\,\delta(v_\alpha - l), \tag{2.4}$$

where M^2 is the number of lattice squares in the bin. For the function (2.4), the value of z in the lattice square labelled by $\boldsymbol{\alpha}$ increases by l and all the other values of z are unchanged. The selected lattice square $\boldsymbol{\alpha}$ is chosen randomly.

Suppose we allow a change in the values of z_{a-1x}, z_{a+1x}, z_{a-1y} and z_{a+1y} (the four nearest neighbours of the lattice square $\boldsymbol{\alpha}$), as well as z_a, then we effectively allow some local movement of the particle. If in addition we increase say z_{a-1x} if z_{a+1x} is less than z_a then this represents a net movement of the blob towards a local minimum. For instance consider

$$\begin{aligned} W(\{v_\alpha\};\{z_\alpha\}) = \frac{1}{M^2}\sum_a \prod_{\beta\sim a} \delta(v_\beta)\,\delta(v_\alpha - l + \tfrac{1}{2}(4z_a - z_{a-1x} - z_{a+1x} - z_{a-1y} - z_{a+1y})) \\ \times \delta(v_{a-1x} - \tfrac{1}{2}(z_a - z_{a-1x}))\,\delta(v_{a+1x} - \tfrac{1}{2}(z_a - z_{a+1x})) \\ \times \delta(v_{a-1y} - \tfrac{1}{2}(z_a - z_{a-1y}))\,\delta(v_{a+1y} - \tfrac{1}{2}(z_a - z_{a+1y})). \end{aligned} \tag{2.5}$$

Notice that for (2.5) the total volume increase of the system is l multiplied by the lattice spacing squared. Thus the form of W represented by (2.5) allows the movement and spreading out of a fixed volume blob. Suppose that z_a were large in comparison with its four neighbours, then (2.5) allows the blob to spread out equally over the four neighbouring lattice squares, whereas a more physical model would allow the blob to move at random to one of the four neighbours. The model is therefore only crude but improving the details will not change the results derived below except perhaps for numerical constants. Given the model function (2.5) we need to relate the imaginary lattice spacing to a physical length. The lattice spacing is the typical distance moved by the particle between landing on the surface and finding a suitable resting place. For a packing of identical hard spheres this will be approximately equal to the sphere radius.

At this point we could solve (2.1) with W given by (2.5), however it is more useful to extract a Langevin equation from the model and then evoke the central limit theorem to calculate the probability functionals which describe the surface. In this way it is clear that the exact specifications of the model functional $W[v(\boldsymbol{\rho});z(\boldsymbol{\rho})]$ are unimportant.

We proceed by considering the finite set of complex fourier coefficients

$$z_k = a^2 \sum_a \mathrm{e}^{+\mathrm{i}k\cdot a a} z_a, \tag{2.6}$$

where a is the lattice spacing. Hence the number of lattice squares is given by $M^2 = L^2/a^2$. The inverse of (2.6) is

$$z_a = \frac{1}{L^2} \sum_k z_k \, e^{-ik \cdot aa}. \tag{2.7}$$

(There is never any ambiguity in using the same symbol z for these two functions.)

We now calculate $W[\{v_k\}; \{z_k\}]$, the probability function for the set of v_k given the set of z_k. This is given by

$$W[\{v_k\}; \{z_k\}] = \int \prod_a dv_a \, W[\{v_a\}; \{z_\alpha\}] \prod_k \delta(v_k - a^2 \sum_a e^{+ik \cdot aa} v_a),$$

where clearly the set of z_k is given if the set of z_a is.

The average of v_{k_1}, for some particular fourier mode $\boldsymbol{k}_1$, is given by

$$\langle v_{k_1} \rangle = \int \prod_k dv_k \, W[\{v_k\}; \{z_k\}] \, v_{k_1}.$$

For the function $W[\{v_a\}; \{z_a\}]$ given by (2.5) this average is

$$\langle v_{k_1} \rangle = \frac{1}{M^2} \sum_a a^2 \, e^{ia a \cdot k_1} \left(l + \frac{1}{L^2} \sum_{k'} x_{k'} \, e^{-ia \cdot k' a} \tfrac{1}{2}[2 \cos k'_x a + 2 \cos k'_y a - 4] \right)$$
$$+ \frac{1}{M^2} \sum_a a^2 \, e^{ia a \cdot k_1} \, e^{iak_{1x}} \frac{1}{L^2} \sum_{k'} e^{-ia \cdot k' a} \tfrac{1}{2}[1 - e^{-iak'_x}]$$

plus three similar terms representing contributions from v_{a-1x}, v_{a-1y} and v_{a+1y}.

Performing the sum over $\boldsymbol{\alpha}$ gives

$$\langle v_{k_1} \rangle = la^2 \, \delta_{k_1} + a^2 z_{k_1} / 2L^2 \, [4(\cos k_{1x} a + \cos k_{1y} a) - 8].$$

Now for $|\boldsymbol{k}|$ much smaller than $1/a$ we can expand the cosines to second order in k to obtain

$$\langle v_{k_1} \rangle = h \delta_{k_1} - a^4 k_1^2 z_{k_1} / L^2, \tag{2.8}$$

where h is the total volume increase of the system (i.e. $h = la^2$).

For values of $|\boldsymbol{k}_1|$ close to $1/a$ the lattice model breaks down as it is not a precise description, of the process of settling, over very short length scales. In the lattice model wavelengths less than the lattice spacing are not included in the sum over $\boldsymbol{k}$ in (2.7). In a blob model without a lattice the shortest wavelengths allowed correspond to the width of the blob. Thus we conclude that (2.8) is a good approximation for most of the $\boldsymbol{k}$ modes of interest.

The first term on the right hand side of (2.8) represents the volume increase of the system per particle added. The second term shows that for any z_k which becomes large there is a tendency for the arrival of additional spheres to reduce this value of z_k. This effect is most pronounced for large values of $\boldsymbol{k}$ (corresponding to short wavelength fluctuations of the surface).

We now rewrite (2.8) in real space. Multiplying by $e^{-i\boldsymbol{k}_1\cdot\boldsymbol{\rho}}/L^2$ and summing over $\boldsymbol{k}_1$ gives

$$\frac{1}{L^2}\sum_{k_1}\langle v_{k_1}\rangle e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}} = \frac{h}{L^2}\sum_{k_1}\delta_{k_1}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{\rho}} - \frac{a^4}{L^4}\sum_{k_1} z_{k_1}k_1^2 e^{-i\boldsymbol{k}_1\cdot\boldsymbol{\rho}},$$

or

$$\langle v(\boldsymbol{\rho})\rangle = \frac{h}{L^2} + \frac{a^4}{L^2}\nabla_\rho^2 z(\boldsymbol{\rho}). \tag{2.9}$$

At this point we introduce a time variable, t, and an average rate of landings per unit time per unit area, r. Because of the assumptions about the flux of particles being weak and particles not moving once settled we do not expect r to appear in the final formula for the fluctuations.

The expected change in $z(\boldsymbol{\rho})$ per unit time is thus

$$\langle \partial z(\boldsymbol{\rho})/\partial t\rangle = rL^2\langle v(\boldsymbol{\rho})\rangle = rh + ra^4\nabla_\rho^2 z(\boldsymbol{\rho}). \tag{2.10}$$

This equation states that two effects contribute to the expected rate of change of $z(\boldsymbol{\rho})$. First there is the average steady increase in $z(\boldsymbol{\rho})$ equal to the volume of blobs added to the system per unit area per unit time (i.e. rh). Secondly for regions of $\boldsymbol{\rho}$ where $\nabla^2 z$ is large we expect $z(\boldsymbol{\rho})$ to increase more than average. This is because where $\nabla^2 z$ is large there is a minimum in the function $z(\boldsymbol{\rho})$ and particles are expected to settle in minima and hence $z(\boldsymbol{\rho})$ is expected to increase. In regions where $\nabla^2 z$ is large and negative, $z(\boldsymbol{\rho})$ has a maximum and is expected to increase more slowly than average.

We could have written down (2.10), for the rate of change of $z(\boldsymbol{\rho})$, from the original discussion of what was required of the model functional $W[v(\boldsymbol{\rho})]$. The particular model (2.5) merely shows that the coefficient of $ra^4\nabla^2 z$ is one. A slightly different model would give a different coefficient of order one.

Returning to $W[\{v_k\};\{z_k\}]$, we can calculate the second moment given by

$$\langle v_{k_1} v_{k_2}\rangle = \frac{1}{M^2}\sum_{\alpha} l^2 a^4 e^{+i\boldsymbol{\alpha}\cdot\boldsymbol{k}_1 a} e^{+i\boldsymbol{\alpha}\cdot\boldsymbol{k}_2 a} + \text{terms much smaller than this for } L/a \text{ large.} \tag{2.11}$$

Summing over $\boldsymbol{\alpha}$ gives $\langle v_{k_1} v_{k_2}\rangle = h^2\delta_{k_1+k_2}$.

As above, values of $\boldsymbol{k}_1$ greater than $1/a$ are not allowed so we write

$$\langle v_k v_{-k}\rangle = h^2 e^{-k^2a^2}.$$

The cut off is exact for the functional W given in (2.2) by using (2.3) for the blob shape $f(\boldsymbol{\rho}-\boldsymbol{\rho}_0)$.

We are now in a position to write down a set of Langevin equations for the fourier modes of the surface;

$$\partial z_k/\partial t = hrL^2\delta_k - ra^4k^2 z_k + \xi_k, \tag{2.12}$$

where ξ_k is a random variable, uncorrelated with z_k and with zero mean. The variable ξ_k for $\boldsymbol{k}=\boldsymbol{0}$ is zero for identical blobs. (It will be non-zero for blobs with a

volume distribution. The contribution to the surface fluctuations from $\xi_{k=0}$ is discussed in § 3, vi.)

Thus
$$\langle \xi_k \xi_{-k} \rangle = h^2 \exp(-k^2 a^2), \quad \boldsymbol{k} \neq \boldsymbol{0},$$
$$= 0, \quad \boldsymbol{k} = \boldsymbol{0}.$$

The equivalent real space Langevin equation is

$$\partial z(\boldsymbol{\rho}, t)/\partial t = rh + ra^4 \nabla^2 z(\boldsymbol{\rho}, t) + \xi(\boldsymbol{\rho}, t). \tag{2.13}$$

The terms $rh + \xi(\boldsymbol{\rho}, t)$ represent the flux of particles onto the surface, where rh is the steady flux and ξ is the zero mean, random fluctuation in the flux. The term in $\nabla^2 z(\boldsymbol{\rho}, t)$ represents the fact that the change in the surface depends on the existing surface as well as the flux, but note that this term is proportional to r and thus is connected with the behaviour of incoming particles.

The discussion above gives some justification for what is essentially a phenomenological equation which describes the behaviour of the surface. There is some further discussion in § 5 but we note here an important difference between the blob model and the physical system of a granular material. The blob model allows local movement of the blob on settling and allows a change of shape in the blob to accommodate the existing surface. The volume of each blob is however independent of its surface landing position whereas the increase in volume of the real system will depend on where that particle settles.

3. The calculation of the surface fluctuations from the Langevin equation

The Langevin equations having been constructed for each of the fourier modes of the surface the problem becomes mathematically equivalent to a set of independent Brownian particles in which the fourier amplitude z_k behaves like the velocity of a Brownian particle with a viscous damping proportional to k^2. Consequently many results can be written down at once from the theory of Brownian motion (see for instance Chandrasekhar (1943), Reif (1965) and Resibois & De Leener (1978)). For example the steady state probability distribution for the set of fourier amplitudes, $\{z_k\}$, is given by

$$p(\{z_k\}) \propto \exp\left(-\sum_k \frac{2a^4 k^2}{L^2} \frac{z_k z_{-k}}{\langle f_k f_{-k} \rangle}\right),$$

and hence changing variables (see for instance Edwards (1973))

$$p[z(\boldsymbol{\rho})] \propto \exp\left(-a^4 \int \mathrm{d}^2\rho \int \mathrm{d}^2\rho' \, \nabla z(\boldsymbol{\rho}) \cdot \nabla z(\boldsymbol{\rho}') \, g^{-1}(\boldsymbol{\rho} - \boldsymbol{\rho}')\right),$$

where
$$g(\boldsymbol{\rho} - \boldsymbol{\rho}') = \frac{1}{4\pi^2} \int \mathrm{d}^2 k \, \mathrm{e}^{+\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} \langle f_k f_{-k} \rangle.$$

Here we have ignored the distribution for the total volume of the system ($z_{k=0}$).

Where this problem differs from that of Brownian motion is in the physical interpretation of the moments of the probability distributions. We therefore derive these now directly from the Langevin equation

$$\partial z(\boldsymbol{\rho},t)/\partial t = rh + ra^4\nabla^2 z + \xi(\boldsymbol{\rho},t). \tag{2.13}$$

This can be simplified by redefining z so that it is measured relative to its average value rather than the base of the bin. Thus the transformation

$$z + rht \to z$$

reduces the equation to

$$\partial z/\partial t = ra^4\nabla^2 z + \xi(\boldsymbol{\rho},t). \tag{3.1}$$

If we define the complex fourier coefficients $z_{k,\omega}$ by

$$z_{k,\omega} = \int_0^T \mathrm{d}t \int_0^L \int_0^L \mathrm{d}^2\rho\, z(\boldsymbol{\rho},t) \exp \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\rho} + \omega t),$$

where T is some very long time, then by the Wiener–Khintchine theorem (see for instance Reif (1965)), in the limit of large L and T, the correlation function $\langle z(\boldsymbol{\rho},t)\, z(\boldsymbol{\rho}+\boldsymbol{\rho}', t+t')\rangle$ is given by

$$\langle z(\boldsymbol{\rho},t)\, z(\boldsymbol{\rho}+\boldsymbol{\rho}', t+t')\rangle = \frac{1}{8\pi^3}\int \mathrm{d}^2k \int \mathrm{d}\omega\, \frac{\langle z_{k,\omega}\, z_{-k',-\omega'}\rangle}{L^2T} \exp \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\rho}' + \omega t'). \tag{3.2}$$

Fourier transforming the Langevin equation (3.1) gives

$$\mathrm{i}\omega\, z_{k,\omega} + ra^4k^2 z_{k,\omega} = \xi_{k,\omega},$$

and hence the ensemble average required for (3.2) is given by

$$\langle z_{k,\omega}\, z_{-k',\,-\omega'}\rangle = \frac{\langle \xi_{k,\omega}\, \xi_{-k',-\omega'}\rangle}{\omega^2 + r^2a^8k^4}. \tag{3.3}$$

Thus to calculate the correlation function $\langle z(\boldsymbol{\rho},t)\, z(\boldsymbol{\rho}+\boldsymbol{\rho}', t+t')\rangle$ we require the ensemble average $\langle \xi_{k,\omega}\, \xi_{-k',\omega'}\rangle$.

Consider a function $\xi(\boldsymbol{\rho},t)$ given by

$$\xi(\boldsymbol{\rho},t) = \sum_{\text{events } i} \delta(t-t_i) f_i(\boldsymbol{\rho}-\boldsymbol{\rho}_i), \tag{3.4}$$

which represents a series of incoming blobs, of shape f_i, arriving centred at position $\boldsymbol{\rho}_i$ and at time t_i. Initially we assume that each event (particle landing) can occur at a time t_i randomly distributed in $0 \leqslant t_i \leqslant T$. Similarly we assume that $\boldsymbol{\rho}_i = x_i, y_i$ is found at random in $0 \leqslant x_i \leqslant L$ and $0 \leqslant y_i \leqslant L$.

Then the average value of $\xi_{k,\omega}$ is given by

$$\langle \xi_{k,\omega}\rangle = \int_0^L \prod_i \frac{\mathrm{d}^2\rho_i}{L^2} \int_0^T \prod_i \frac{\mathrm{d}t_i}{T} \Big[\sum_i \exp \mathrm{i}(\omega t_i + \boldsymbol{k}\cdot\boldsymbol{\rho}_i)\Big] \langle f_k\rangle,$$

where $\langle f_k \rangle$ is the $\boldsymbol{k}$ fourier component of the shape of a blob, averaged over all possible blob shapes. To remove the average flux we put $\langle f_{k=0} \rangle$ equal to zero and hence

$$\langle \xi_{k,\omega} \rangle = 0 \quad \text{for all } \boldsymbol{k} \text{ and } \omega.$$

The average of $\xi_{k,\omega} \xi_{-k',-\omega'}$ is

$$\langle \xi_{k,\omega} \xi_{-k',-\omega'} \rangle = \sum_i \sum_j \int \prod_i \frac{\mathrm{d}^2 \rho_i}{L^2} \int \prod_i \frac{\mathrm{d}t_i}{T} \exp \mathrm{i}(\omega(t_i - t_j) + \boldsymbol{k}(\boldsymbol{\rho}_i - \boldsymbol{\rho}_j)) \langle f_k f_{-k'} \rangle \delta(\omega - \omega').$$

The integrals are only non zero in the case where $i = j$ and hence

$$\langle \xi_{k,\omega} \xi_{-k',-\omega'} \rangle = \sum_i \sum_j \delta_{ij} \langle f_k f_{-k'} \rangle \delta(\omega - \omega') = rL^2 T \langle f_k f_{-k'} \rangle \delta(\omega - \omega'),$$

where rL^2T is the total number of events under consideration.

The function $\langle f_k f_{-k'} \rangle$ is equal to the average square volume of the blobs multiplied by a sharp cut off at $|\boldsymbol{k}|$ equal to the reciprocal of a typical blob width. For identical circularly symmetric Gaussian blobs for which $f(\boldsymbol{\rho} - \boldsymbol{\rho}_0)$ is given by

$$f(\boldsymbol{\rho} - \boldsymbol{\rho}_0) = \frac{h}{2\pi a^2} \exp\left(- \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}_0)^2}{2a^2} \right),$$

and $\langle f_k f_{-k'} \rangle$ is then given by

$$\langle f_k f_{-k'} \rangle = h^2 \exp(-k^2 a^2) \delta(\boldsymbol{k} - \boldsymbol{k}').$$

Thus the function $\langle \xi_{k,\omega} \xi_{-k',-\omega'} \rangle$ is

$$\langle \xi_{k,\omega} \xi_{-k',-\omega'} \rangle = rL^2 T h^2 \exp(-k^2 a^2) \delta(\boldsymbol{k} - \boldsymbol{k}') \delta(\omega - \omega'). \tag{3.5}$$

Combining (3.2), (3.3) and (3.5) yields

$$\langle z(\boldsymbol{\rho}, t) z(\boldsymbol{\rho} + \boldsymbol{\rho}', t + t') \rangle = \frac{rh^2}{8\pi^3} \int \mathrm{d}^2 k \int \frac{\mathrm{d}\omega \exp \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{\rho}' + \omega t') \exp(-k^2 a^2)}{(\omega^2 + r^2 a^8 k^4)}.$$

Performing the ω integral gives

$$\langle z(\boldsymbol{\rho}, t) z(\boldsymbol{\rho} + \boldsymbol{\rho}', t + t') \rangle = \frac{h^2}{8\pi^2 a^4} \int \frac{\mathrm{d}^2 k}{k^2} \exp \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{\rho}') \exp(-k^2 a^2 (1 + rt' a^2)). \tag{3.6}$$

The integral as it stands is divergent.

We must remember that $\langle \xi_k \xi_{-k} \rangle$ is zero for $k = 0$ reflecting the fact that the average of the flux is treated separately (i.e. it is not included in ξ). The sum over $\boldsymbol{k}$ is from $|\boldsymbol{k}| = \pi/L$ only. (The factor is π not 2π because we impose the boundary conditions that ∇z is zero at the edge of the box. Thus the first allowed fourier components are $\cos(\pi x/L)$ and $\cos(\pi y/L)$. However this does not mean that we have included only one quarter of the fourier components in the sum over $\boldsymbol{k}$ as we have allowed negative $\boldsymbol{k}$ as well as positive. The only effect is to change the bottom limit of the integral. Note also that one of the components of $\boldsymbol{k}$ is allowed to be zero provided the other is not.) Thus we replace, what is correctly, the sum over $\boldsymbol{k}$ by an integral over $\boldsymbol{k}$ with $|\boldsymbol{k}| > \pi/L$.

We proceed to evaluate the integral (3.6) for certain special cases:

(i) *The mean square value of $z(\boldsymbol{\rho})$ measured relative to its average value*

Putting $\boldsymbol{\rho}' = 0$ and $\tau' = 0$ in (3.6) gives

$$\langle z^2 \rangle = \frac{h^2}{8\pi a^4} \int_{\pi/L}^{\infty} \frac{2k\,\mathrm{d}k}{k^2} \exp(-k^2a^2),$$

which evaluated gives an exponential integral. For a/L small we can replace the Gaussian cut off by a top-hat cut-off to obtain

$$\langle z^2 \rangle = \frac{h^2}{4\pi a^4} \ln\left(\frac{L}{\pi a}\right). \tag{3.7}$$

The term $h^2/4\pi a^4$ is a quantity proportional to the square of the height of each blob, as we might expect. The term $\ln(L/\pi a)$ is at first sight somewhat surprising as the bin size appears in the formula for the surface fluctuations. The result is due to the assumption that each particle in the flux lands at random anywhere in the bin. Suppose that the particles were dropping from a sieve above or sedimenting after precipitation in a chemical reactor. In the sieve case the flux is uniform to the extent that over a long period of time the flux of particles into a region of the bin (small compared to the total area of the bin) is constrained to be equal to its expectation value. It is possible to define a length b such that modes of wavelength greater than b are constrained is such a way that the quantity $\langle \int_0^t \mathrm{d}t' \xi_k(t') \int_0^t \mathrm{d}t'' \xi_{-k}(t'') \rangle$ is not allowed to increase indefinitely as the time, t, increases. Consequently the quantity $\langle \xi_{k,\omega} \xi_{-k',-\omega'} \rangle$ is zero for $\omega = 0$ for these constrained $\boldsymbol{k}$ modes, and the divergence of the Wiener–Khintchine, integral over ω and $\boldsymbol{k}$, at $\omega = 0$, $\boldsymbol{k} = 0$ is removed. The effect is to replace the lower limit of the integral (3.6) by $2\pi/b$ instead of π/L. Thus the quantity $2L$ (representing the longest allowed wavelength present in the flux) in (3.7) should be replaced by the length b.

Returning to (3.7), it is interesting to insert some typical values to see just how weak the logarithmic divergence is.

Consider particles of size 1 mm in a bin of dimension 3 m. Then the logarithmic term in (3.7) gives rise to a factor of (2.6) in the root mean square fluctuation of the surface. Now suppose we increase the bin size to 3 km, then the factor increases to (3.7). Thus for a 1 million fold increase in the surface area of the bin the magnitude of the fluctuations increases by only 50 %.

Notice that the result (3.7) does not imply a surface 'roughness' dependent on the bin size. The result comes from very long wavelength fluctuations, in the surface, which have a small slope but a large amplitude.

(ii) *The static behaviour of the fluctuations*

Consider the function

$$\langle (z(\boldsymbol{\rho}, t) - z(\mathbf{0}, t))^2 \rangle.$$

This is the expectation value of the square of the height change observed when moving on the surface, in a straight line, distance $|\boldsymbol{\rho}|$. The origin, $\boldsymbol{\rho} = \mathbf{0}$, is chosen arbitrarily. From (3.6) the function is given by

$$\langle (z(\boldsymbol{\rho}, t) - z(\mathbf{0}, t))^2 \rangle = \frac{2h^2}{8\pi^2 a^4} \int \frac{\mathrm{d}^2 k}{k^2} [1 - \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho})] \exp(-k^2 a^2). \tag{3.8}$$

In this expression the divergence at the origin of the integral is removed because $(1 - J_0(\boldsymbol{k}\cdot\boldsymbol{\rho}))$ is of order k^2 for small k. Hence the bottom limit of the integral can again be replaced by zero provided $|\boldsymbol{\rho}| \ll L$.

The integral is given approximately by

$$\langle (z(\boldsymbol{\rho}, t) - z(\mathbf{0}, t))^2 \rangle = \frac{h^2}{2\pi a^4} \ln \left[\frac{(\rho^2 + a^2)^{\frac{1}{2}}}{a} \right]. \tag{3.9}$$

This is also a strange result. The value of $\langle (z(\boldsymbol{\rho}, t) - z(\mathbf{0}, t))^2 \rangle$ increases indefinitely as $\boldsymbol{\rho}$ increases. In practice the function will reach a steady value when $|\boldsymbol{\rho}|$ is a length of order L. Thus the fluctuations have no microscopic length scale.

Some physical justification for this result can be found as follows:

Consider a one dimensional surface in the lattice model. If the value of z_α at each lattice point were constrained to be $z_{\alpha-1} \pm l$, then the function $z(x)$ would make a random walk and for large x we obtain

$$\langle (z(x) - z(0))^2 \rangle = Xl^2/a,$$

which is Einstein's famous result.

In two dimensions the equivalent constraint becomes that for each lattice square the value of z differs by l from the value of z in each of the four nearest neighbour lattice squares. The lattice problem as posed is not easily soluble because the lattice square at the origin effects the lattice square at $\boldsymbol{\rho}$ by every possible path of nearest neighbours joining $\mathbf{0}$ to $\boldsymbol{\rho}$. The effect of this is that the fluctuations are strongly constrained as a result of the 'frustration' caused by all the paths. Thus the function x/a in one dimension is replaced by the function $\ln(|\boldsymbol{\rho}|/a)$ in two dimensions. Note that the equivalent problem in three dimensions which may represent temperature fluctuations in a block of metal, for instance, shows none of the divergent behaviour.

The results (3.7) and (3.9) are directly related to the divergence at $\boldsymbol{k}$ equals $\mathbf{0}$ of the integral (3.6). The long wavelength modes are not strongly damped because the slope of the surface associated with such a mode is very small for a given amplitude. The Langevin equation (4.1) implies that the system has a resistance to high slopes, i.e. high values of $|\nabla z|$. A Langevin equation of the form

$$\frac{\partial z}{\partial t}(\boldsymbol{\rho}, t) = -r\gamma z(\boldsymbol{\rho}, t) + ra^4 \nabla^2 z(\boldsymbol{\rho}, t) + \xi(\boldsymbol{\rho}, t), \tag{3.10}$$

would correspond to a resistance of the system to high values of z^2. (Remember that z here is defined relative to its average value.) The divergence of the integral (3.6) would be removed and a new scale of length, given by $a^2/\gamma^{\frac{1}{2}}$, would characterize the fluctuations.

The equation (3.10) would correspond to a tendency for incoming particles not to settle in areas where $z(\boldsymbol{\rho})$ is higher than average. Although this sounds reasonable it is in fact unphysical because a particle settling can only tell whether it is on a steep slope or at a minimum, etc. it has no 'knowledge' of its value of $z(\boldsymbol{\rho})$ in a global sense. The corresponding statement about the steady state probability functional $p[z(\boldsymbol{\rho})]$, is that it does not depend on $z(\boldsymbol{\rho})$ explicitly, only on the derivatives of $z(\boldsymbol{\rho})$.

The resistance of the system to high slopes is clearly related to the angle of repose for a granular material. The tangent of the angle of repose is the steepest slope the material can sustain under gravity.

(iii) *The dynamical behaviour of the fluctuations*

Consider the function $\langle (z(\boldsymbol{\rho}, t+t') - z(\boldsymbol{\rho}, t))^2 \rangle$,

where z is defined relative to its average value at time t. From (3.6) the function is given by

$$\langle (z(t+t') - z(t))^2 \rangle = \frac{2h^2}{8\pi a^4} \int \frac{\mathrm{d}^2 k}{k^2} [\exp(-k^2 a^2) - \exp(-k^2 a^2 (1 + ra^2 t'))].$$

The lower limit of the integral can be taken as zero provided $ra^2 t' \ll L^2/a^2$. The integral can be evaluated to give

$$\langle (z(t+t') - z(t))^2 \rangle = \frac{h^2}{4\pi a^4} \ln(1 + rt' a^2). \tag{3.11}$$

For t' much less than the time to add just enough particles to cover one layer on the packing ($t' \ll 1/ra^2$) (3.11) reduces to

$$\langle (z(t+t') - z(t))^2 \rangle = (h^2/4\pi a^2)\, rt'.$$

In other words for short times the function $z(\boldsymbol{\rho}, t)$ describes a random walk of the usual kind, with the fluctuation increasing like $t^{\frac{1}{2}}$. For longer times this walk is constrained as the value of z at some point $\boldsymbol{\rho}$ 'realizes' that it is effected by neighbouring values of z. The fluctuations then increase very slowly with time until $ra^2 t'$ is of order L^2/a^2. (That is the number of monolayers added to the packing is of order the number of particles in a monolayer.) After this very long time the fluctuation reaches its steady value given by (3.7).

(iv) *The fluctuations in the gradient of the surface*

By analogy with the derivation of (3.2) the correlation function in the slope or gradient of the surface is given by

$$\langle \nabla z(\boldsymbol{\rho}, t)\, \nabla z(\boldsymbol{\rho} + \boldsymbol{\rho}', t+t') \rangle = \frac{1}{8\pi^3} \int \boldsymbol{k}\boldsymbol{k}\, \mathrm{d}^2 k\, \mathrm{d}\omega \frac{\langle z_{k,\omega} z_{-k',-\omega'} \rangle}{L^2 T} \exp \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{\rho}' + \omega t')\, \mathrm{d}^2 k'\, \mathrm{d}\omega',$$

combining this equation with (3.3) and (3.5) and integrating over ω and $\boldsymbol{k}$ gives

$$\langle \nabla z(\boldsymbol{\rho}, t)\, \nabla z(\boldsymbol{\rho} + \boldsymbol{\rho}', t+t') \rangle = \frac{h^2}{8\pi a^6} \frac{I}{(1 + ra^2 t')} \exp\left(- \frac{\rho^2}{4a^2 (1 + ra^2 t')} \right), \tag{3.12}$$

where I is the unit tensor in two dimensions.

Thus the fluctuations in ∇z show none of the bad behaviour exhibited by the fluctuations in z. The fluctuations decay with a length scale and a timescale which are well defined in terms of microscopic properties.

(v) *The solution to the problem in which the blobs settle completely at random*

The process by which a surface builds up by blobs landing at random and simply resting where they land, is also of some interest. For instance if lumps of a substance, such as clay, are deposited at random and just stick where they land, remaining rigid, then the problem will be described by a blob model in which the settling position of the blobs is uncorrelated with the existing surface.

In this case we can derive a Langevin equation without a drag term. The fourier modes z_k behave like the positions of Brownian particles and consequently the surface fluctuations increase with time and are given by

$$\langle (z(\boldsymbol{\rho}) - \langle z \rangle)^2 \rangle = h \langle z \rangle / 4\pi a^2, \tag{3.13}$$

where $\langle z \rangle = rht$.

Other problems of interest are those where the blobs really do exist, for example blobs of a very viscous liquid. When these land their shape immediately changes to accommodate the existing surface but then the surface slowly relaxes. In addition a granular material may be added to a shaken container. In these problems a Langevin equation of the form (2.13) may still be valid except that the coefficient of $\nabla^2 z$ will no longer be proportional to the rate of addition of particles, r. Consequently as soon as the flux of particles onto the surface is turned off the steady state fluctuations will die away exponentially leaving a flat surface.

(vi) *The contribution of the mode $k = 0$ to the surface fluctuations*

Within the blob model the volume of each blob added is a random variable uncorrelated with the existing volume of the system. Consequently if the blobs have a volume distribution we should expect the fluctuation in the fourier amplitude $z_{k=0}$ to grow indefinitely with time and to be given by

$$\langle (z_0 - \langle z_0 \rangle)^2 \rangle = rL^2 t (\langle f_0^2 \rangle - \langle f_0 \rangle^2).$$

The $k = 0$ term should then be included in the Wiener–Khintchine sum and consequently contributes a term

$$(rt/L^2)(\langle f_0^2 \rangle - \langle f_0 \rangle^2), \tag{3.14}$$

to the mean square surface fluctuation.

However it is interesting to note the following points:

(*a*) For a packing whose average height is equal to its width (i.e. $rt\langle f_0 \rangle = L$) this term is of order $\langle f_0 \rangle / L$ and is consequently negligible in comparison with the expression given in (3.7).

(*b*) If we are studying a single system at a single time rather than an ensemble of systems we can measure the fluctuations in the surface relative to the average height of the surface for this particular system (i.e. Z_0/L^2). Fluctuations measured thus do not include a contribution of the form (3.14).

(c) For a real system we might expect there to be a local correlation in 'blob' volumes. Thus particles will settle in such a way that the increase in volume of the system is large or small depending on whether the recent volume increases occurring in the same neighbourhood were small or large respectively. Hence for a real system we might expect the volume fluctuations to be smaller than predicted by the blob theory.

4. Conclusions

The problem posed and the assumptions made in § 1 define a precisely specified physical problem. We have solved this by deriving a Langevin equation in § 2. This equation is of sufficiently simple structure to be soluble. Any criticism of the theory will therefore be made in §§ 1 or 2. We should ask whether the precisely defined physical problem, formulated in § 1, is the problem which is really of interest in practical applications. The following assumptions have been made in deriving the Langevin equation:

(i) That the local movement of particles on the surface is a sufficiently accurate description of the settling process. So long as this process is truly local the Langevin equations for the fourier amplitudes z_k will remain an uncoupled set. We might imagine the situation arising where a particle rolls a long way down a steep slope to find a minimum in which to rest. If one argues that this behaviour is not well described by a local movement of the particle, then the theory can be defended by pointing out that the blob model predicts that the probability of finding a long steep slope on the surface is very small. This is because the correlation function in the gradient of $z(\boldsymbol{\rho})$ is short ranged. Hence the blob theory is in this respect self consistent.

(ii) That terms linear in $z(\boldsymbol{\rho}, t)$, and hence $z_k(t)$, are a sufficiently accurate description of the settling process. For instance one might argue that a term such as $-\nabla z(\boldsymbol{\rho})\cdot\nabla z(\boldsymbol{\rho})$ should appear in the real space Langevin equation, in addition to $\nabla^2 z(\boldsymbol{\rho})$. Such terms make the problem considerably more difficult because fourier transformation will not diagonalize the problem.

(iii) That the term $-k^2 z_k$ or equivalently $\nabla^2 z(\boldsymbol{\rho})$ is a reasonable description of the settling process. In fact k^2 is only the leading term in an even power series in $\boldsymbol{k}$. The higher terms make no significant difference to the predictions of the blob theory because the divergence of the integral over $\boldsymbol{k}$ is unaffected. A term independent of $\boldsymbol{k}$ (i.e. $-z_k$) is ruled out for reasons explained in the text.

Of the various criticisms of the theory it is the question of linearity which is most difficult to defend on physical grounds.

If we accept the Langevin equation we can rigorously show that, in the limit of large bin area, the fourier modes behave as a set of independent Brownian quasi-particles, where the mode amplitude, z_k, corresponds to the velocity of the quasi-particle. We can also show that the surface behaviour is dominated by the long wavelength modes of the system. In a system in which the flux of particles has fluctuation modes with wavelengths equal to the bin dimension, the effect is to give surface fluctuations whose steady state, mean square value is proportional to the

logarithm of the bin dimension. Any particular fourier mode, however, has a well defined contribution to the fluctuations, which is independent of the bin dimensions. It is only when the sum over the fourier modes is performed that the bin dimension appears. The theory has a sort of self consistency in that it is most strictly valid for the long wavelength modes.

If the flux of particles is uniform to the extent that long wavelength fluctuations are not allowed to increase with time, then the bin size dependence of the surface fluctuations is removed. However, the longarithmic behaviour, which is typical of two dimensional systems, remains but only over a short range. The relevance of this point to such systems as a solid precipitating from a chemical reaction and settling is not clear. For practical purposes it makes very little difference whether the logarithmic dependence on bin size is present or not because of the incredibly slow increase in the root mean square surface fluctuation with bin area, which the square root of the logarithm of the square root of the area represents.

The steady state probability functional which describes the surface is predicted to be not explicitly dependent on the function $z(\boldsymbol{\rho})$ but to depend on the slope of the surface $\nabla z(\boldsymbol{\rho})$. (With the proviso that the total volume of the system is given.) This reflects the fact that a particle landing at a random position in the bin, ρ, is 'unaware' of the value of $z(\boldsymbol{\rho})$. Thus there is no tendency for the surface to increase more than expected in a region where $z(\boldsymbol{\rho})$ is low, i.e. in a global minimum of the surface. It is only the value of $\nabla^2 z(\boldsymbol{\rho})$, or the curvature of the surface, which is important.

The Langevin equation of § 2 is a precise description of the behaviour of the surface of a certain class of randomly deposited granular material and it is our contention that it incorporates most of the essential physics of the problem and, therefore, that its predictions are, at least qualitatively, correct for the more difficult case where nonlinear terms matter. Various approximate methods are available for handling this problem, but it is our belief that the precise definition of improvements in the description of the physics of the problem is more difficult than the resulting mathematical problem.

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References

Chandrasekhar, S. 1943 *Rev. mod. Phys.* **15**, 1.

Edwards, S. F. 1976 *Molecular fluids* (eds. R. Balian and G. Weill). New York: Gordon and Breach.

Reif, F. 1965 *Statistical and thermal physics*, ch. 15. New York: McGraw-Hill.

Resibois, P. & De Leener, M. 1978 *Classical kinetic theory of fluids*. Wiley.