PARAMETRICALLY FORCED CGLE

- **General framework: spatio-temporal dynamics in systems with broken symmetry ⟶ nucleation of spatio-temporal patterns**

- **Resonant couplings between the forcing and oscillatory modes may lead to quasi-periodicity, frequency lockings, devil's staircases, chaos and intermittency.**

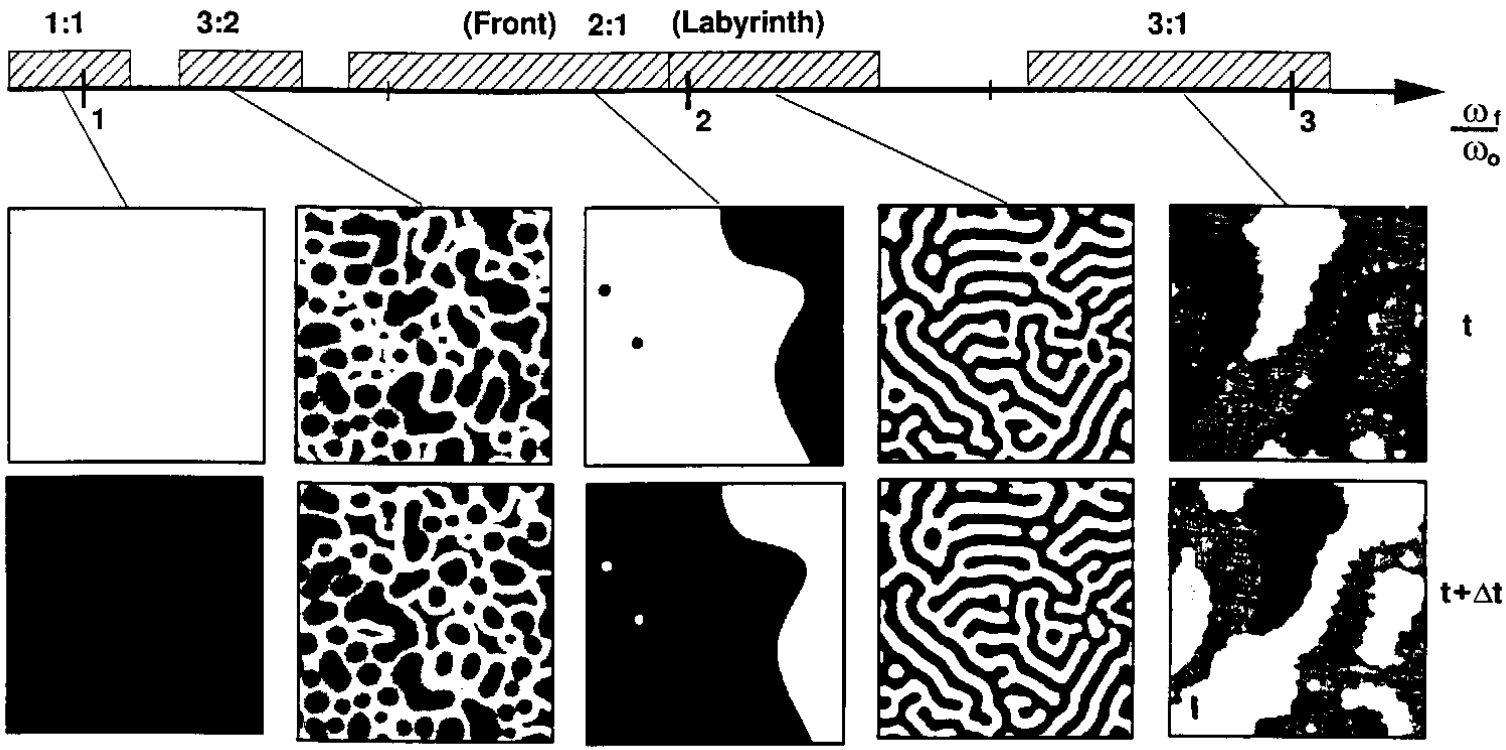


Fig.: Periodically perturbed ruthenium-catalized Belousov-Zhabotinsky reaction-diffusion system [A.L. Lin *et. al.*, Proceedings of IMA Workshop on Pattern Formation and Nonlocal Effects, U Minn (1998)]

- **Temporal modulation of Hopf bifurcation in extended systems $\Longrightarrow$ CGLE with broken phase symmetry**

$$\boxed{\partial_t A = \mu A + (1+i\alpha)\nabla^2 A - (1+i\beta)A|A|^2 + \gamma_n \bar{A}^{n-1}}$$

  - $\gamma_n = 0$: ***Phase spirals around zeroes of*** A
  - $\gamma_n > \gamma_c$: n ***equivalent states with different fixed phase*** ***(frequency locked solutions)***

- **AIMS (n=3): Study of**

  (1) Analogies with the dynamics of systems with competing fields

  (2) Transition between phase spirals and amplitude spirals

UNIFORM SOLUTIONS

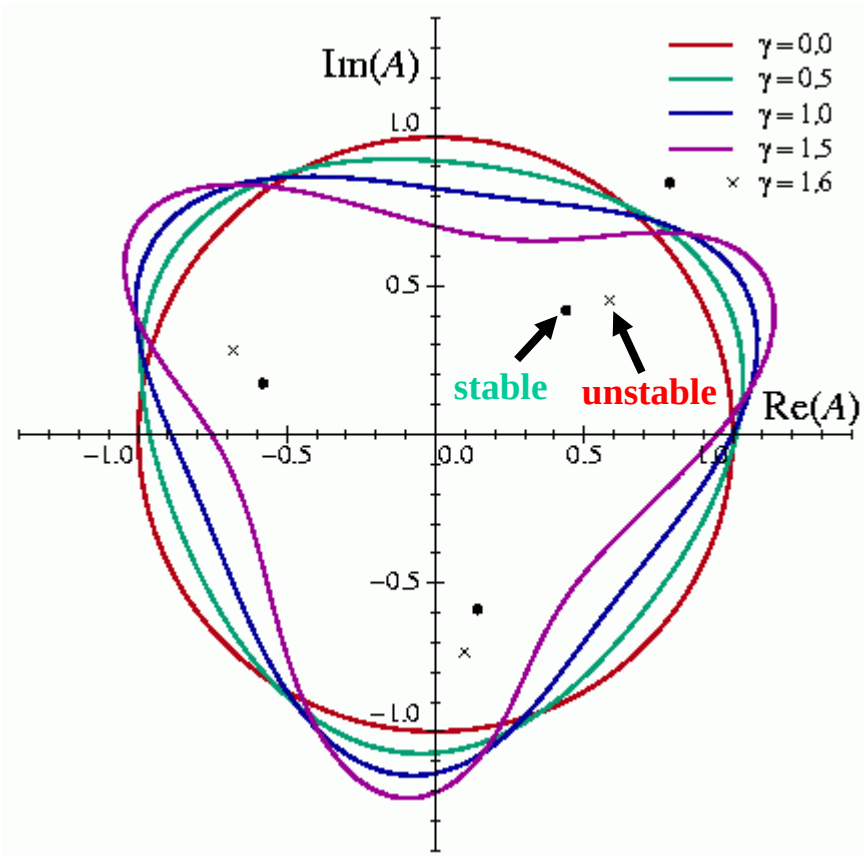
$$R_0\, e^{i\Phi_0} \rightarrow \begin{cases} \dot{R}_0 = \mu R_0 - R_0^3 + \gamma R_0^2 \cos 3\Phi_0 \\ \dot{\Phi}_0 = -\beta R_0^2 - \gamma R_0 \sin 3\Phi_0 \end{cases}$$

$$R_\pm^2 = \frac{1}{2(1+\beta^2)}\left[2\mu + \gamma^2 \pm \sqrt{\gamma^4 + 4\mu\gamma^2 - 4\mu^2\beta^2}\right]$$

$$\cos 3\Phi_0 = \frac{R_0^2 - \mu}{\gamma R_0}, \qquad \sin 3\Phi_0 = \frac{-\beta R_0}{\gamma}$$

Stationary solutions exist provided that $\gamma^2 > \gamma_c^2 \Rightarrow$ ***Critical value:*** $\gamma_c^2 \equiv 2\mu\left(\sqrt{1+\beta^2} - 1\right)$

- **OSCILLATORY REGIME ($\gamma < \gamma_c$): asymptotic solutions correspond to temporal oscillations of limit cycle type**



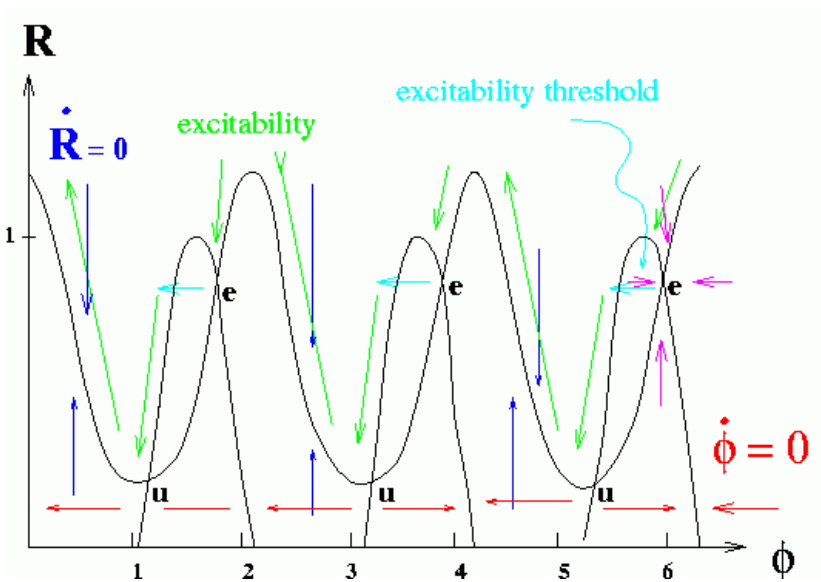
- **EXCITABLE REGIME ($\gamma > \gamma_c$): asymptotic solutions are fixed points**

$$\Phi^u = R_e^{i\Phi_-^{(k)}}, \; k = 1,2,3$$ **always linearly unstable**

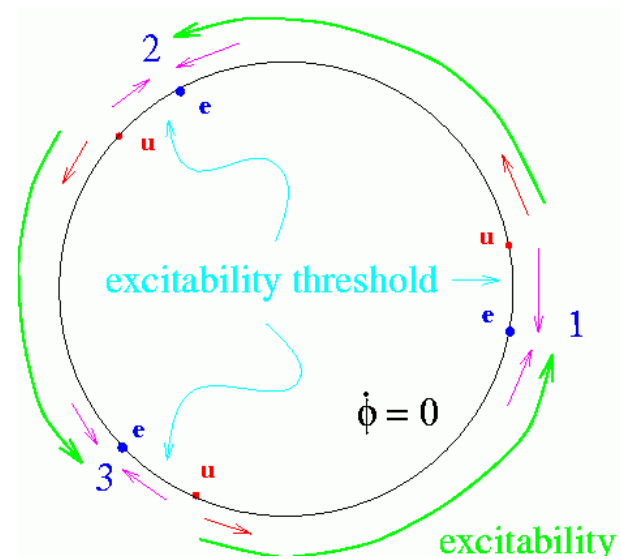
$$\Phi^e = R_+ e^{i\Phi_+^{(k)}}, \; k = 1,2,3$$

$$\begin{cases} |\beta| < \sqrt{3}, & \text{stable} \\ |\beta| > \sqrt{3}, & \text{oscillatory unstable } (k = 0,\ \omega \neq 0) \end{cases}$$

if $\gamma_c \lesssim \gamma_1 < \gamma < \gamma_2$. **Stable for** $\gamma > \gamma_2$



$\gamma == \beta = 4\mu == 1$ [rect. coord.]



$\gamma == \beta = 0.01$, $\mu == 0.25$ [polar coord.]

$$\beta, \gamma << \mu \Rightarrow \Delta\Phi = \frac{\pi}{3} - \frac{2}{3}\frac{\gamma_c}{\gamma}$$ (excitability threshold)

PHASE APPROXIMATION

- **OSCILLATORY REGIME** $(\gamma < \gamma_c)$

$$\left.\begin{aligned} R &= R_0(t) + \rho(\vec{r},t) \\ \Phi &= \Phi_0\big(t + \phi(\vec{r},t)\big) \end{aligned}\right\} \xrightarrow[\text{of } \rho(\vec{r},t)]{\text{Adiabatic elimination}}$$

Phase dynamics

$$\partial_t \phi = (1+\alpha\bar{\beta})\nabla^2\phi + \kappa(\vec{\nabla}\phi)^2 + \Lambda$$

$$\kappa = \frac{\int_0^T dt\left(\frac{2\beta R_0 + \gamma\sin 3\Phi_0}{2R_0 - \gamma\cos 3\Phi_0} - \alpha\right)\dot{\Phi}_0^3}{\int_0^T dt\,\dot{\Phi}_0^2}, \quad \bar{\beta} = \frac{\int_0^T dt\,\frac{2\beta R_0 + \gamma\sin 3\Phi_0}{2R_0 - \gamma\cos 3\Phi_0}\dot{\Phi}_0^2}{\int_0^T dt\,\dot{\Phi}_0^2}$$

- $\gamma \longrightarrow 0 \Longrightarrow$ **Burgers equation**: $\partial_t\bar{\phi} = (1+\alpha\beta)\nabla^2\bar{\phi} + (\alpha-\beta)(\vec{\nabla}\bar{\phi})^2 + \Lambda\ ,\ \bar{\phi} = \beta\mu\phi$

- $1+\alpha\bar{\beta} > 0$: **stable (phase) spirals waves**, $q \propto \kappa$

  $1+\alpha\bar{\beta} < 0$: **turbulence regimes may be expected**

- **System presents the same complexity and the same spatio-temporal behavior than self-oscillating systems**

EXCITABLE REGIME ($\gamma > \gamma_c$)

If $\gamma, \beta \ll \mu,\ \beta \ll 1 \Longrightarrow R^2 \cong \mu,\ \ \gamma_c \cong |\beta|\sqrt{\mu}$:

$$\partial_t \Phi = -\sqrt{\mu}(\gamma_c + \gamma \sin 3\Phi) + (1+\alpha\beta)\nabla^2\Phi - (\alpha-\beta)(\vec{\nabla}\Phi)^2 + \frac{\alpha^2(1+\beta^2)}{2\mu}\nabla^4\Phi$$

- $\alpha = \beta = 0$: **relaxational dynamics. Isolated 1D front at rest**

  $\alpha = \beta \neq 0$: **relaxational dynamics but 1D front moves**

  $\alpha \neq \beta$: **nonpotential front motion**

- **Φ small $\Longrightarrow$ damped Kuramoto-Sivashinsky phase equation. Frequency locked solutions stable regardless of the sign of $1+\alpha\beta$ $\Longrightarrow$ no pattern forming instabilities.**

FRONTS AND SPIRALS

1D

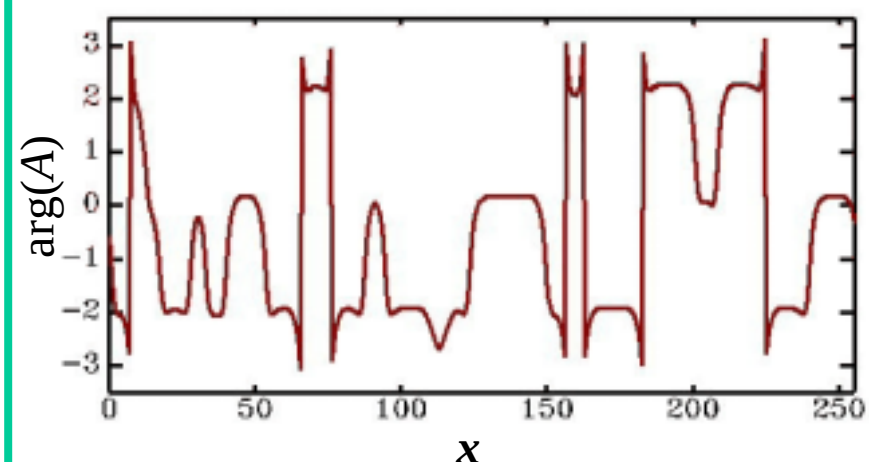
- **Isolated kink moves at a constant velocity due to non-potential dynamics**

$$v = \frac{2\pi\gamma_c\sqrt{\mu} + 3(\alpha-\beta)\int_{-\infty}^{+\infty}(\partial_x \Phi_{ij}^0)^3}{3\int_{-\infty}^{+\infty}(\partial_x \Phi_{ij}^0)^2}$$

- $\alpha > \beta, \quad \Phi_1^e < \Phi_2^e < \Phi_3^e$

$$\Phi_{12}, \Phi_{23}, \Phi_{31} \longrightarrow v$$

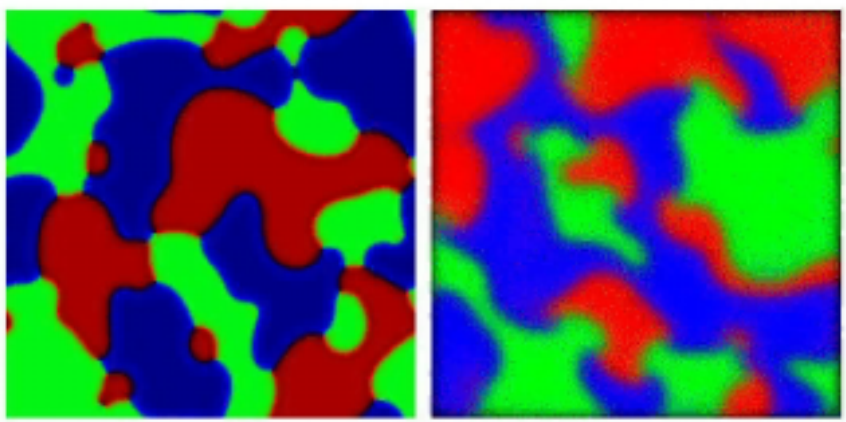
$$\Phi_{21}, \Phi_{13}, \Phi_{32} \longleftarrow v$$



2D

- **Formation of vertex points**
- **Rotation of interfaces around vertices $\Longrightarrow$ no coarsening**
- **Analogies with systems with competing fields**

3ω Forced CGLE **Busse-Heikes Model**



NUMERICAL SIMULATIONS

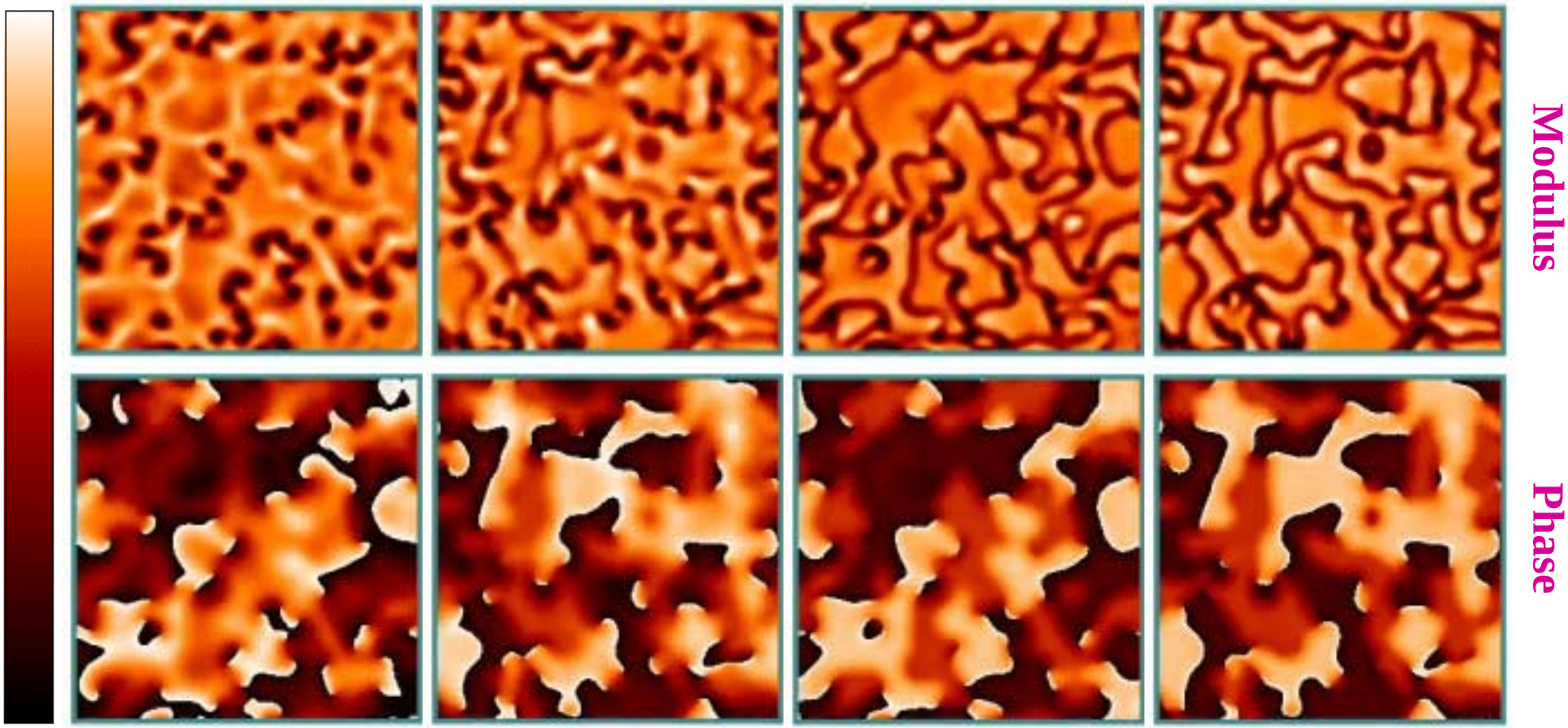
- ***PHASE APPROXIMATION VALID***

μ	α	β	γ_c	$1+\alpha\beta$	REGIME ($\gamma = 0$)
1	2	–0.2	0.199	>0	***Frozen States I***
1	5.5	–0.2	0.199	<0	***Frozen States II***

- ***PHASE APPROXIMATION INVALID***

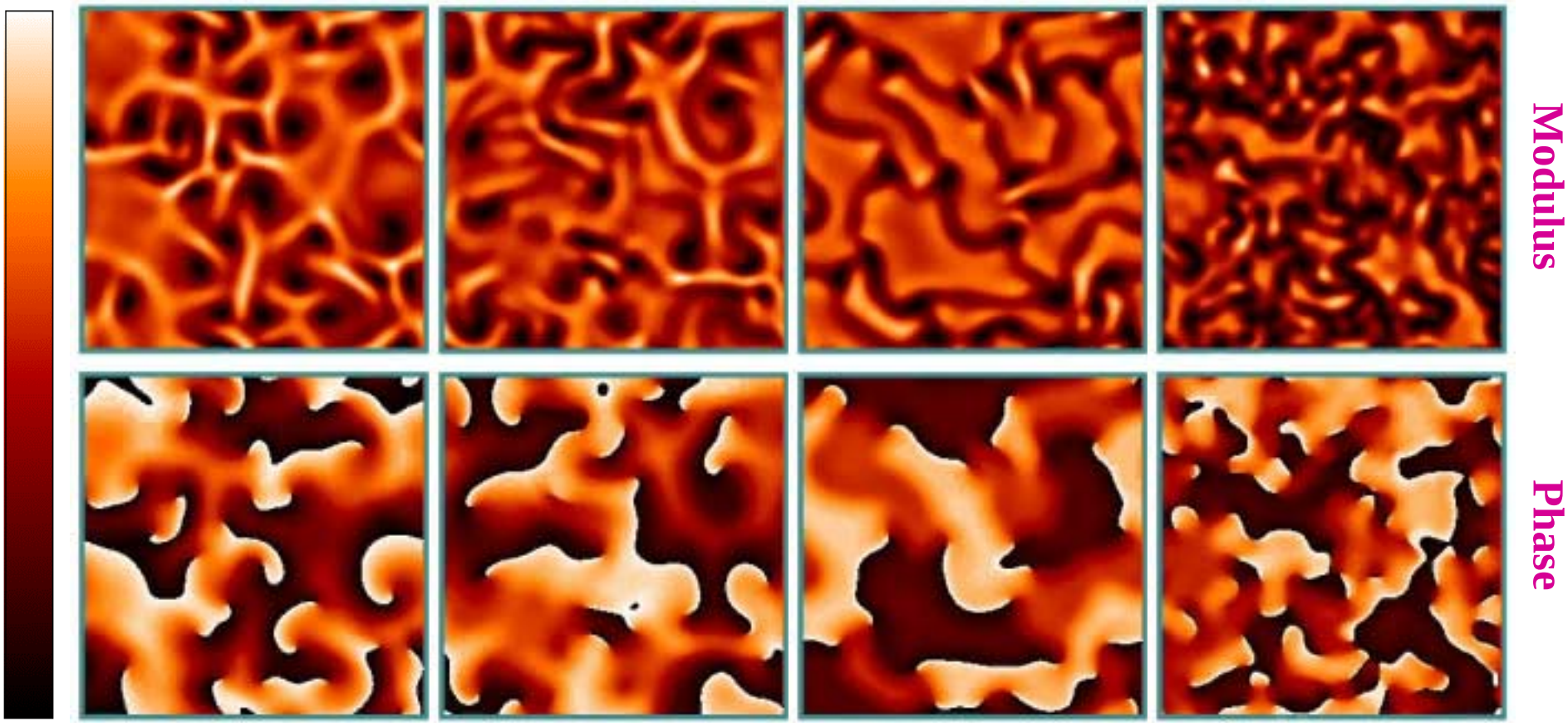
μ	α	β	γ_c	$1+\alpha\beta$	REGIME ($\gamma = 0$)
1	2.0	–1.00	0.910	<0	***Defect Turbulence I***
1	0.0	–1.80	1.450	>0	***Defect Turbulence II***
1	2.0	–0.76	0.720	<0	***Phase Turbulence***

Frozen States I: $\mu = 1$, $\alpha = 2$, $\beta = -0.2$, $\gamma_c = 0.199$



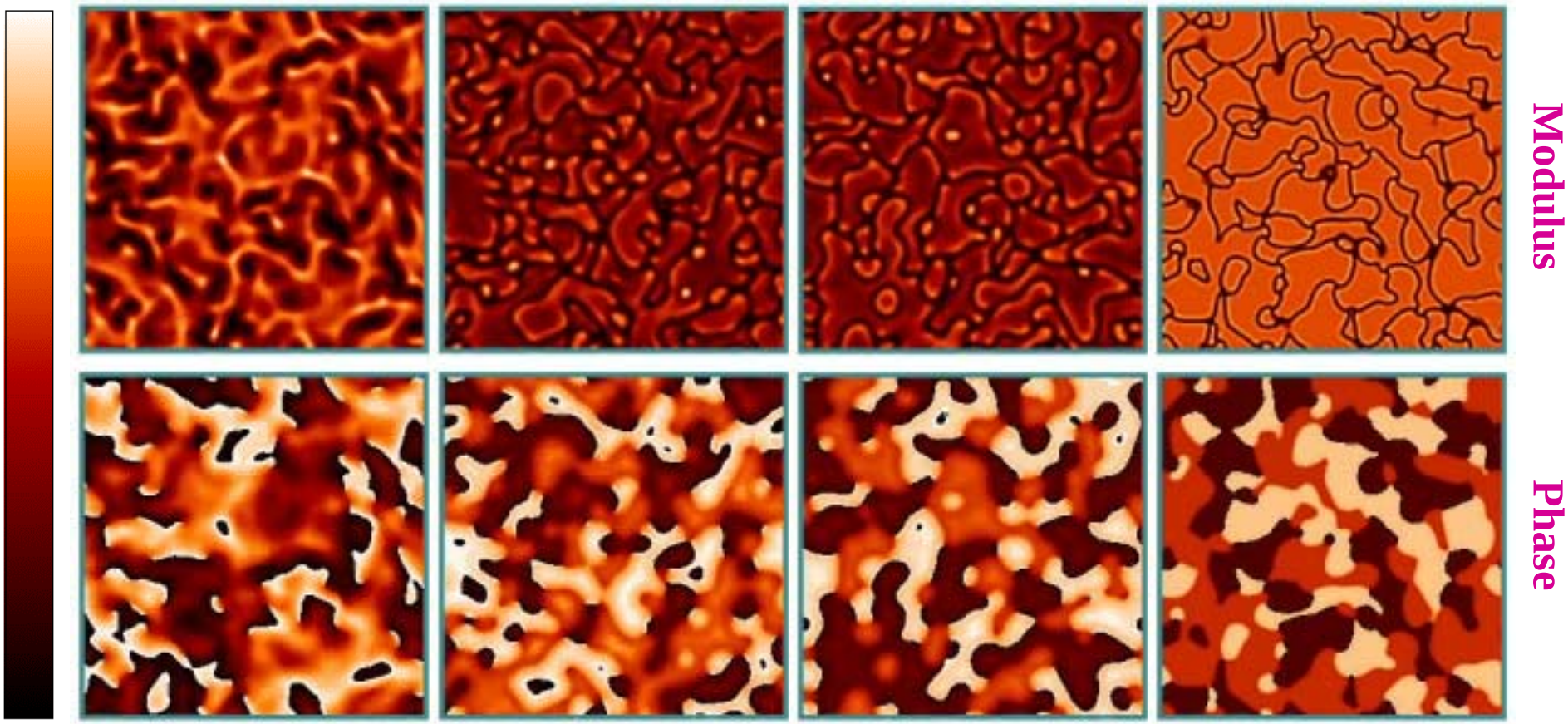
$\gamma = 0$ $\gamma < \gamma_c$ $\gamma \cong \gamma_c$ $\gamma > \gamma_c$

Frozen States II: $\mu = 1$, $\alpha = 5.5$, $\beta = -0.2$, $\gamma_c = 0.199$



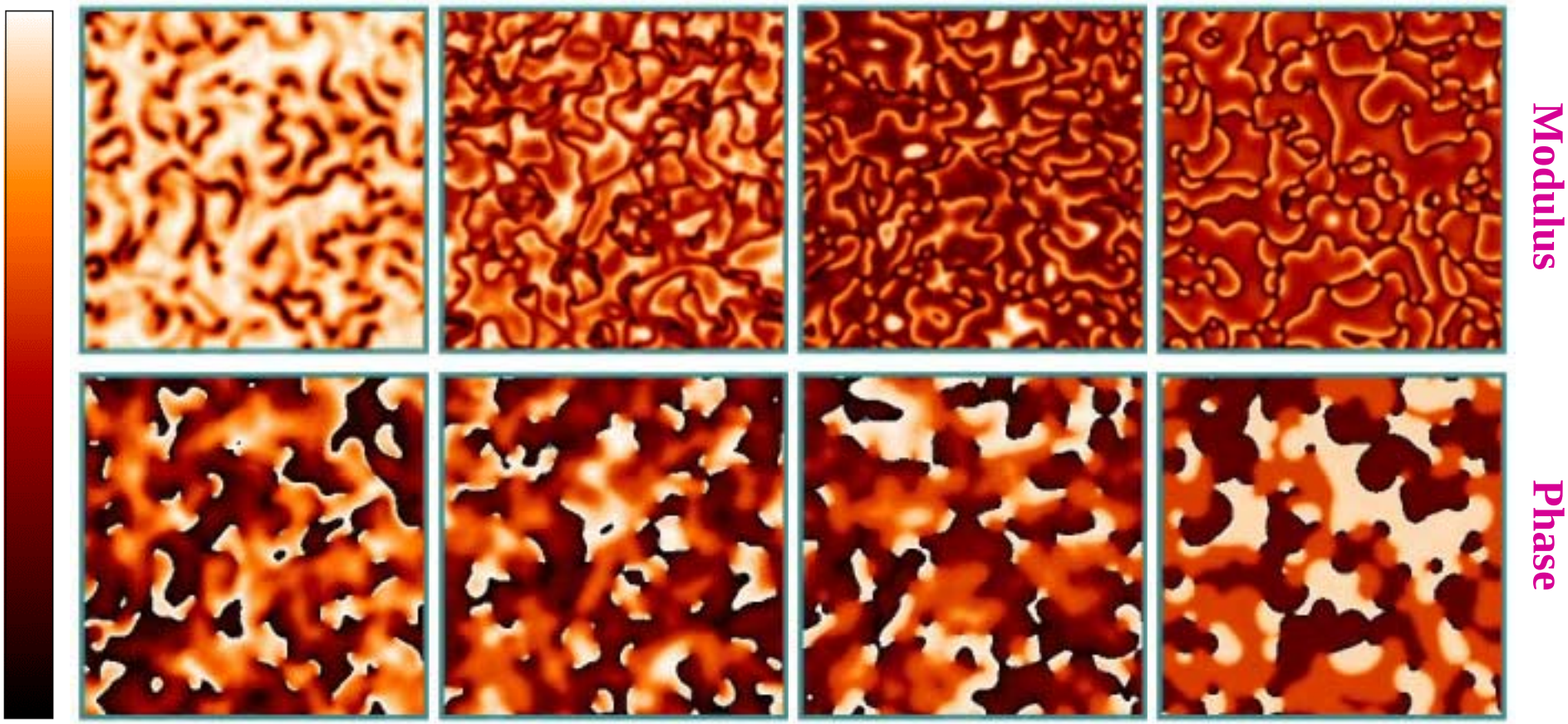
$\gamma = 0$ $\gamma < \gamma_c$ $\gamma \cong \gamma_c$ $\gamma > \gamma_c$

Defect Turbulence I: $\mu = 1$, $\alpha = 2$, $\beta = -1.0$, γ_c=0.910



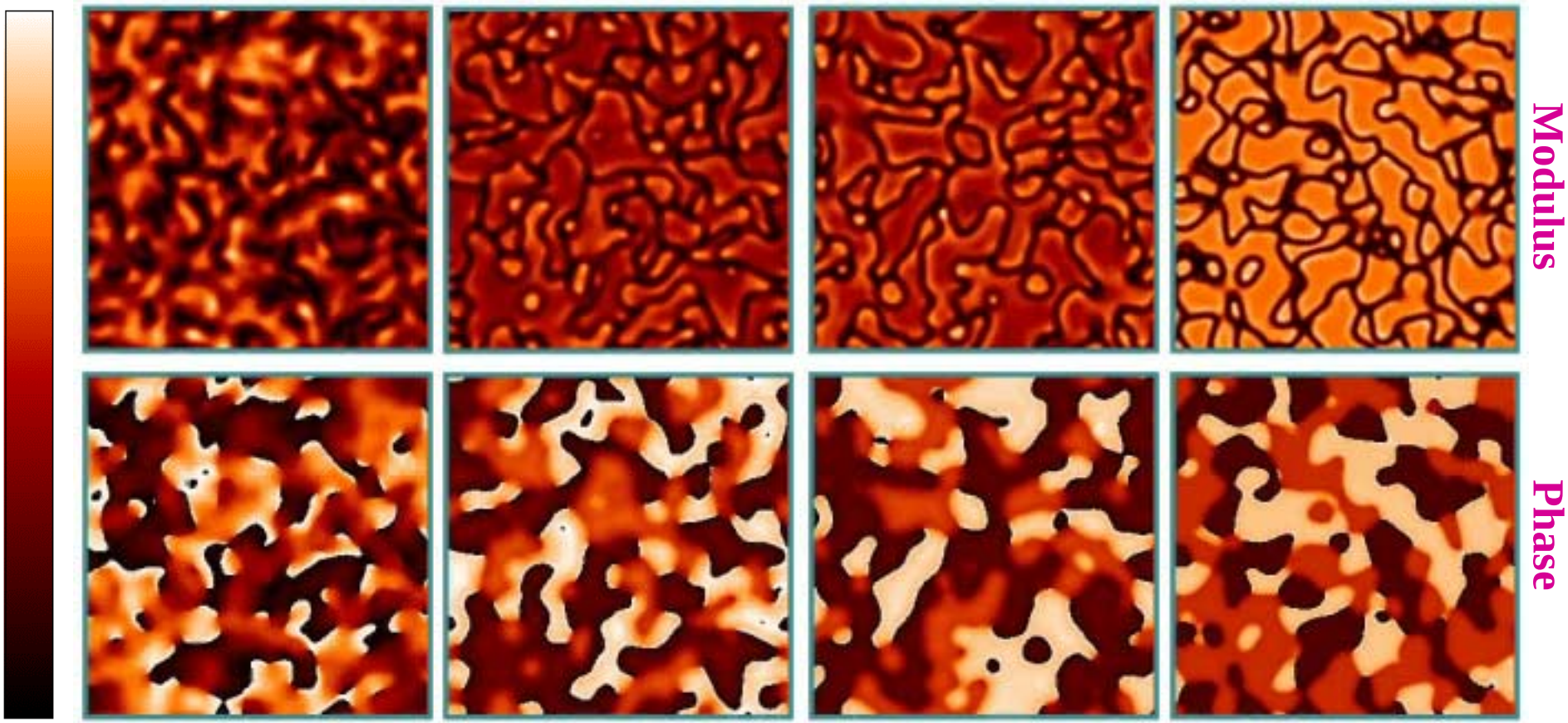
$\gamma = 0$ $\gamma < \gamma_c$ $\gamma \cong \gamma_c$ $\gamma > \gamma_c$

Defect Turbulence II: $\mu = 1$, $\alpha = 0$, $\beta = -1.8$, $\gamma_c = 1.447$



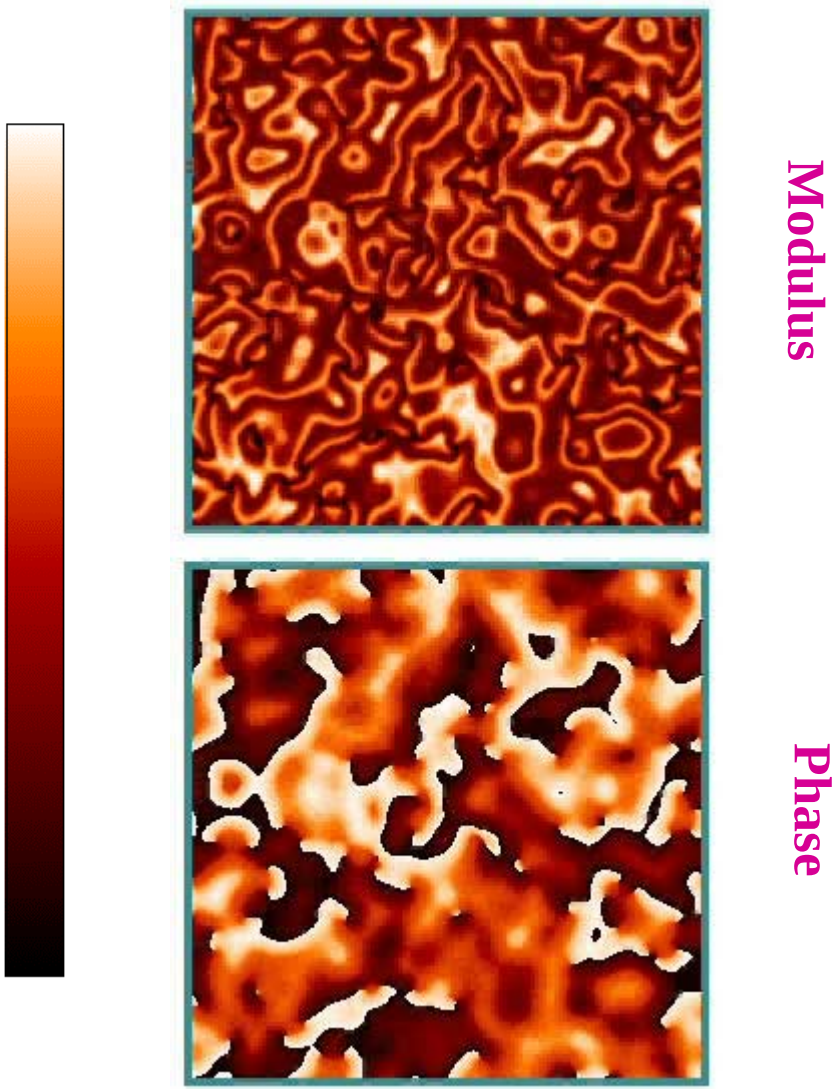
$\gamma = 0$ $\gamma < \gamma_c$ $\gamma \cong \gamma_c$ $\gamma > \gamma_c$

Phase Turbulence : $\mu = 1$, $\alpha = 2$, $\beta = -0.76$, $\gamma_c = 0.72$



$\gamma = 0$ $\gamma < \gamma_c$ $\gamma \cong \gamma_c$ $\gamma > \gamma_c$

Oscillatory instability at k = 0: $\mu = 1$, $\alpha = 0$, $\beta = -3.0$, $\gamma_c = 2.080$



$\gamma = 2.150$

MAIN NUMERICAL RESULTS

- $\gamma \simeq \gamma_c$:
  - ♦ **Transition observed when phase approximation holds**
  - ♦ **Other kind of structures may appear (targets, ...)**
- $\gamma < \gamma_c$: **Amplitude spirals may form. Complex patterns may appear**
- $\gamma \gg \gamma_c$: **Excitable regime is attained regardless of the phase approximation**