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TRANSITION TO HIGH-DIMENSIONAL CHAOS THROUGH QUASIPERIODIC MOTION

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In this contribution we report on a transition to high-dimensional chaos through three-frequency quasiperiodic behavior. The resulting chaotic attractor has a one positive and two null Lyapunov exponents. The transition occurs at the point at which two symmetry related three-dimensional tori merge in a crisis-like bifurcation. The route can be summarized as: 2D torus → 3D torus → high-dimensional chaotic attractor.

Nonlinear dynamical systems with chaotic behavior exhibit a number of well characterized routes to chaotic behavior [Ott, 1993; Bergé *et al.*, 1986]. These routes present universal properties, allowing to classify nonlinear systems in universality classes formed by systems with completely different microscopic interactions. A lot of effort has been devoted to study transitions to chaos through quasiperiodic motion, starting with the pioneering work of Ruelle and Takens [Ruelle & Takens, 1971; Newhouse *et al.*, 1978]. So far, many workers have studied the transition from two-frequency quasiperiodic behavior to low-dimensional ($d < 3$) chaotic behavior (where the dimension d is usually taken as *some* representative of the dimension spectrum D_q [Ott, 1993], typically the capacity dimension, D_0, the information dimension, D_1, or the correlation dimension, D_2, that for *typical* situations are found not to differ much among them; in this study we have chosen to work with D_1). This transition to chaotic behavior usually proceeds through the interaction of resonances (mode-locking), that lead to a wrinkling or corrugation of the torus, and ultimately to a strange attractor [Curry & Yorke, 1977]. Much light on the quantitative and universal features of this transitions has been gained through studies on the simplified circle map [Feigenbaum *et al.*, 1982]. However, it has been shown that 2D-tori can lead to a strange attractor without mode-locked states [Moon, 1997].

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The situation is not so clear regarding the transition to chaos mediated by three-dimensional tori (the very existence of these attractors appears unlikely in the light of the Newhouse–Ruelle–Takens (NRT) Theorem [Newhouse *et al.*, 1978]). However, both experiments [Gollub & Benson, 1980; Linsay & Cummings, 1989; Alaggio & Rega, 2000] and numerical studies [Grebogi *et al.*, 1983a, 1985; Battelino, 1988] give support to the existence of these attractors, although the apparent departure from the NRT Theorem is unclear at this level. Some further light has been shed on this topic after the work of Feudel *et al.* [Feudel *et al.*, 1993; Anishchenko *et al.*, 1994; Feudel *et al.*, 1996], who have presented convincing arguments on the stability of these 3D-tori on systems with certain types of symmetries. The rationale behind the stability of these attractors is the fact that the perturbations that affect these attractors are not generic due to the symmetry of the system (NRT Theorem says that three-dimensional tori are unstable when subjected to certain generic perturbations).

Not much is known regarding the possibility of direct transitions to chaotic behavior from a three-dimensional torus. One of the reported routes that involves a 3D-torus is: 2D-Torus → 3D-Torus → 2D-Torus → Chaos [Feudel *et al.*, 1993; Anishchenko *et al.*, 1994; Feudel *et al.*, 1996; Yang, 2000]. On the other hand, a recent study reports a gluing bifurcation of two 3D-tori to another (nonchaotic) 3D-torus [Lopez & Marques, 2000; Marques *et al.*, 2001].

The aim of the present paper is to describe the direct transition from two symmetry related 3D-tori to a high-dimensional chaotic attractor in a realistic system corresponding to three coupled Lorenz oscillators. One of the features of this system is that it is autonomous, i.e. all the frequencies that appear are created by the system dynamics. This attractor has one positive, and two null Lyapunov exponents, and, thus, its information dimension D_1 is larger than three, although the attractor is not hyperchaotic. The information dimension, D_1, has been estimated according to Kaplan–Yorke conjecture [Kaplan & Yorke, 1979; Ott, 1983] from the knowledge of the Lyapunov spectrum. In our case, in the chaotic region in Fig. 2 the sum of the four largest exponents is positive, while the sum of the largest five is negative, implying that $D_1 > 4$. This does not imply that the whole dimension spectrum obeys $D_q > 4$ for any q, but one can argue that $D_q > 3$ for any q, as there are three degrees of freedom with a non negative Lyapunov exponent, that need, at least, three dimensions to be spanned, while one of these degrees of freedom induces fractalization. As this type of attractor has not been so far reported in many studies we call it from now on high-dimensional chaotic attractor (following also [Yang, 2000]). In previous studies this attractor was reported in rings of unidirectionally coupled Chua's oscillators [Matías *et al.*, 1997; Sánchez *et al.*, 2000], and then also in rings of Lorenz oscillators [Sánchez & Matías, 1999], that is the setting studied in the present work. As shown in the previous studies, the extra null Lyapunov exponent is created in a symmetric Hopf bifurcation that takes place because of the invariance of the ring under the cyclic group Z_n [Collins & Stewart, 1994]. This rotational degree of freedom, involving a spatio-temporal symmetry in the array, corresponds to a simultaneous shift to a neighboring oscillator and advance in time by a period divided by N, is conserved in the region under study in this contribution, and this leads to the third frequency in the torus and to the high-dimensionality of the chaotic attractor. Thus, and following the previous discussion on the validity of the NRT Theorem in this case, this spatio-temporal symmetry allows to have a stable T^3 attractor over a finite parameter range.

Now we pass to discuss in more detail this transition. The 3N-dimensional system considered in this study can be written as,

$$\left.\begin{aligned} \dot{x_j} &= \sigma(y_j - x_j) \\ \dot{y_j} &= R\,\underline{x_j} - y_j - x_j\,z_j \\ \dot{z_j} &= x_j\,y_j - b\,z_j \end{aligned}\right\} \quad j = 1, \ldots, N\,, \qquad (1)$$

$\underline{x_j} = x_{j-1}$ for $j \neq 1$, introduces the coupling and the (periodic) boundary conditions entering through $\underline{x_1} = x_N$. The results presented correspond to the case $N = 3$ (the dimension of the dynamical system is, then, 9), and the parameters for the Lorenz system are the same as in [Sánchez & Matías, 1999], i.e. $\sigma = 20$, $b = 3$, while $R \in [35, 35.4]$. The system (1) has been chosen because a hardware implementation of three coupled Lorenz systems following this prescription is available (it is described in [Sánchez & Matías, 1998, 1999]).

In the previous (experimental) study [Sánchez & Matías, 1999], it was already shown that there is a route to chaos through quasiperiodic behavior, although the transition was not resolved with sufficient details to show the presence of three-frequency quasiperiodicity. This is due to the fact

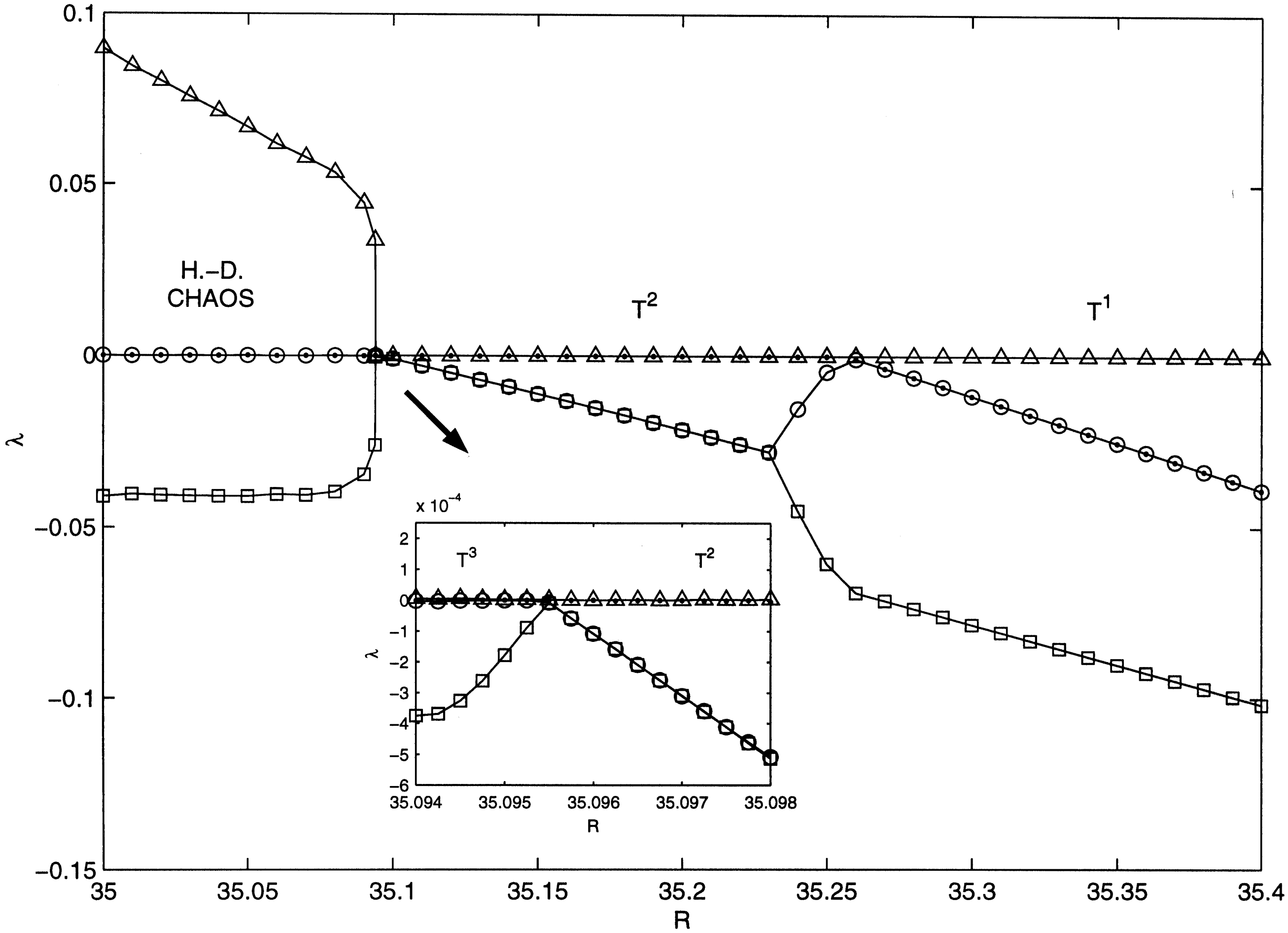
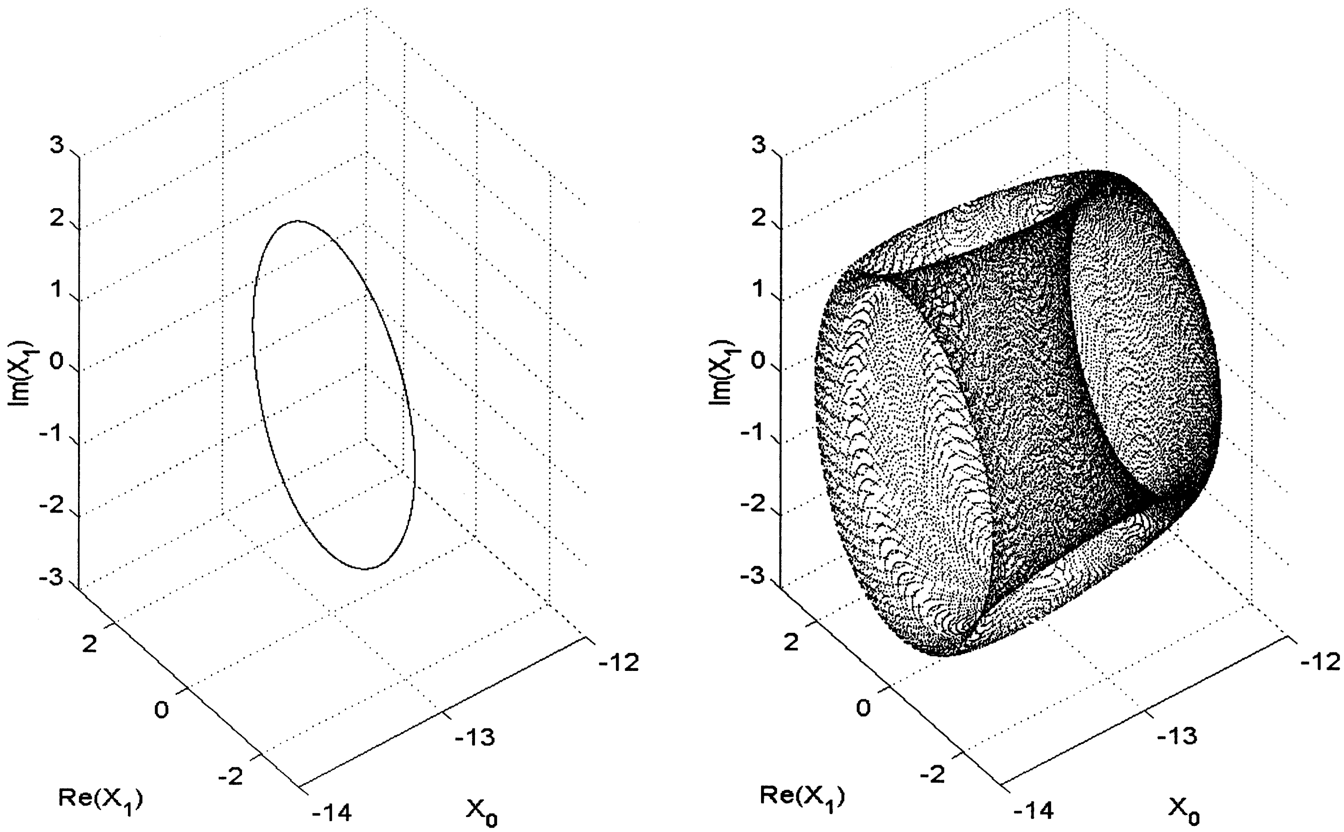


Fig. 1. Representation of the four largest Lyapunov exponents (λ) of the system (1) with $\sigma = 20$ and $b = 3$ as R is varied in the range shown in abscissae. The inset shows a blowup in which the three-frequency torus can be clearly seen. Symbols correspond to Lyapunov exponents as follows: λ_1, $\triangle$; λ_2, $\cdot$; λ_3, $\bigcirc$; λ_4, $\square$; with $\lambda_j \geq \lambda_{j+1}$.

that the third frequency has a much slower time scale than the others, which is a problem with the number of points that the oscilloscope uses. In addition, this region is relatively difficult to locate in parameter space. Figure 1 contains a representation of the four largest Lyapunov exponents for the nine-dimensional dynamical system (1). In particular, the inset illustrates the region with three-frequency quasiperiodic behavior. The quantitative characterization of these exponents is made quite difficult due to the fact that there are very long chaotic transients. Thus, for a time $t = 10^6$ time units the dynamical system is left to evolve, and then the Lyapunov exponents are calculated during a time of the order of $t = 10^6$ t.u. This allows to resolve unambiguously the exponents, showing that there are three null Lyapunov exponents, the fourth being slightly negative (for shorter calculation times the fourth exponent is of the order of the other three). The exponents have been calculated by extending the procedure of [Wolf *et al.*, 1985].

The best way of displaying the three-frequency quasiperiodic attractor is by reducing its dimensionality with the use of the Poincaré section technique [Hénon, 1982]. The Poincaré sections of the two- and three-frequency quasiperiodic attractors of the system are presented in Fig. 2. As expected, the dimensionality of these attractors is reduced and one sees a limit cycle and a two-frequency quasiperiodic attractor, respectively. Due to the symmetry of the problem, a representation in terms of the spatial (discrete) Fourier modes [Matías *et al.*, 1997] has been found more adequate. The definition of these modes is the following,

$$\mathbf{X}_k = \frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j \exp\left[\frac{2\pi i(j-1)k}{N}\right] \qquad (2)$$



(a) (b)

Fig. 2. Plot of a Poincaré cross-section for the system (1) with $\sigma = 20$ and $b = 3$: (a) $R = 35.1$, for which the system exhibits two-frequency quasiperiodic behavior (one-frequency periodic in the Poincaré section); (b) $R = 35.095$, for which the system exhibits three-frequency quasiperiodic behavior (two-frequency quasiperiodic in the Poincaré section). The representation has been carried out by using the (complex) mode representation of the system (2), and the three axes correspond to the x coordinate and are the uniform ($k = 0$) mode, and the real and imaginary parts of the $k = 1$ mode. The Poincaré cross-section has been defined by the condition: $Z_0 = 34 \approx R - 1$ with $\dot{Z}_0 > 0$.

with $N = 3$, where i is the imaginary unit, and where $\mathbf{X}$ and $\mathbf{x}$ represent the (x, y, z) coordinates as a vector for the modes and the original coordinate representations, respectively.

A feature of the reported transition from two symmetry-related three-frequency quasiperiodic attractors to a high-dimensional chaotic attractor is the presence, at the chaotic and quasiperiodic sides of the transition, of laminar bursts and chaotic transients, respectively, that indicates that the transition has intermittent features. This can be seen for the chaotic side of the transition from Fig. 3, in which a time series is reported for $R = 34.093 < R_c \sim 34.093(8)$. The laminar bursts correspond to a long residence time on each side of the attractor: chaos is introduced by the jump to the other side. Although it appears that this type of attractor-merging transition has not been reported so far, it has some attributes of a crisis [Grebogi *et al.*, 1982, 1983b], although in this case at one of the sides of the transition the attractors are periodic. On the other hand, Lopez and Marques [2000] have recently reported the merging of two symmetry related three-frequency quasiperiodic attractors, but this yields a nonchaotic (quasiperiodic) attractor. Up to this point it is not clear the mechanism that creates the high-dimensional chaotic attractor. The idea of a crisis-like mechanism is supported by the fact that the 3D-tori enlarge their size as R is decreased. One can speculate that this mechanism implies the simultaneous collision of both (symmetry related) 3D-tori with the (unstable) chaotic invariant manifold corresponding to the uniform mode or a homoclinic (or heteroclinic)

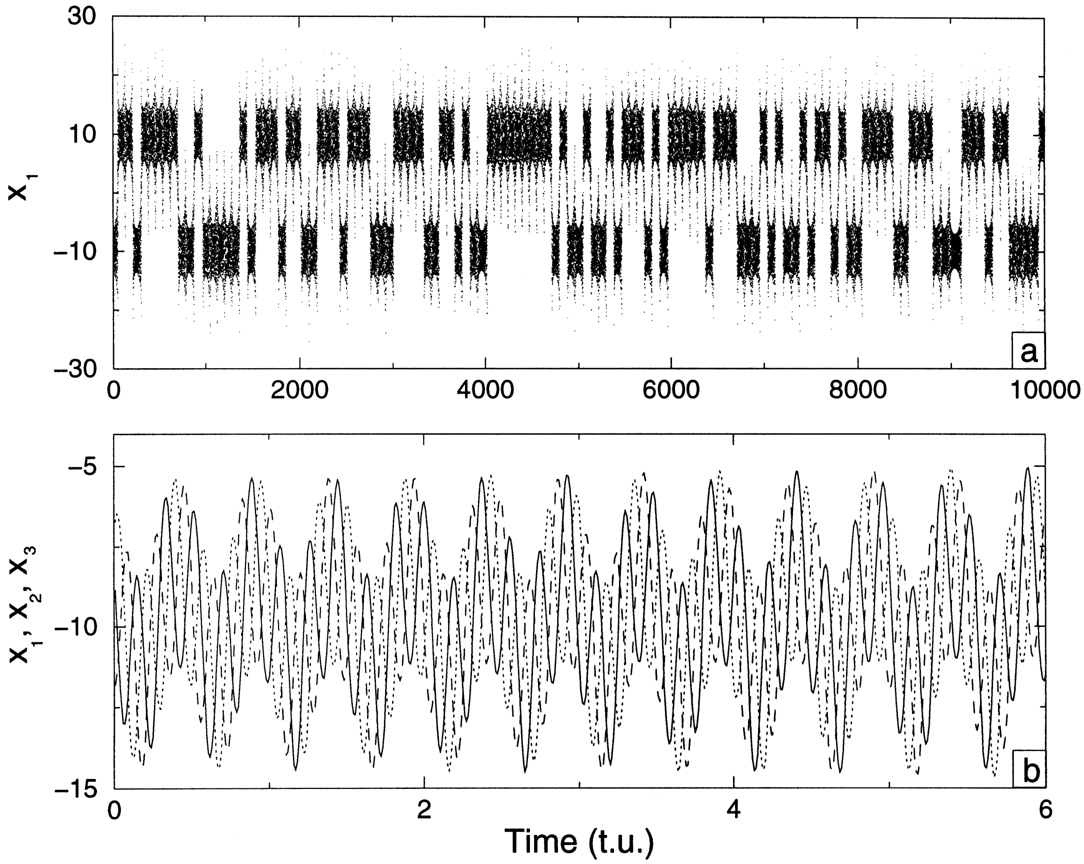


Fig. 3. (a) Time series, variable x in one of the oscillators versus time for $\sigma = 20$, $b = 3$ and $R = 35.093$. The system is already in the chaotic region, but exhibits long bursts of *laminar* behavior, corresponding to the three-frequency quasiperiodic behavior. (b) Time series, variable x in the three oscillators, i.e. x_1, x_2, and x_3 (solid, dotted and dashed lines respectively) versus time (for a short time interval) for the same parameters, showing the *laminar* character of one of the bursts.

connection. The detailed mechanism is still under investigation.

In conclusion, in this work we have reported a direct transition from three-frequency quasiperiodic behavior to high-dimensional chaotic behavior ($d > 3$) with a single positive and two null Lyapunov exponents. The system studied is autonomous, which implies that all the frequencies are generated by physical mechanisms, and there exists an experimental counterpart [Sánchez & Matías, 1998, 1999], and an experimental study of the phenomena studied here is underway [Sánchez *et al.*, 2001]. The transition does not involve any kind of frequency locking and interaction of resonances. Instead, it involves the (crisis-like) merging of two symmetry-related three-frequency quasiperiodic attractors with intermittent features and long transients at both sides of R_c.

The present work illustrates a novel route to (high-dimensional) chaos through the route: 2D-torus $\rightarrow$ 3D-torus $\rightarrow$ high-dim. chaos. Other routes to chaos recently reported in the literature, like those in [Moon, 1997; Yang, 2000] involve a two-frequency quasiperiodic attractor, while [Lopez & Marques, 2000] deals with the transition from three-frequency quasiperiodic behavior to a non-chaotic attractor. Probably, the fact that this type of unusual transition occurs in this system is due to both the dimensionality of the system, nine, and its degree of symmetry (rotations that permutate the oscillators).

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