PHYSICAL REVIEW E **69**, 045105(R) (2004)

Numerical study of the Langevin theory for fixed-energy sandpiles

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(Received 16 July 2003; published 30 April 2004)

The recently proposed Langevin equation, aimed to capture the relevant critical features of stochastic sandpiles and other self-organizing systems, is studied numerically. The equation is similar to the Reggeon field theory, describing generic systems with absorbing states, but it is coupled linearly to a second conserved and static (nondiffusive) field. It has been claimed to represent a different universality class, including different discrete models: the Manna as well as other sandpiles, reaction-diffusion systems, etc. In order to integrate the equation, and surpass the difficulties associated with its singular noise, we follow a numerical technique introduced by Dickman. Our results coincide remarkably well with those of discrete models claimed to belong to this universality class, in one, two, and three dimensions. This provides a strong backing for the Langevin theory of stochastic sandpiles, and to the very existence of this meagerly understood universality class.

DOI: 10.1103/PhysRevE.69.045105 PACS number(s): 02.50.−r, 05.50.+q, 64.60.Ht, 05.70.Ln

Aimed at shedding some light at the origin of *order* in Nature, some different routes to organization have been proposed in the past 15 years or so. In particular, the concept of *self-organization*, as exemplified by *sandpiles* [1–3] (for reviews see Refs. [4–6]), one of the canonical instances of self-organizing systems, has generated a rather remarkable outburst of interest. In order to rationalize sandpiles in particular, and *self-organized criticality* (SOC) in general, and to understand their critical properties, it has been recently proposed to look at them as systems with many absorbing states [6–10]. The underlying idea is that in the absence of external driving sandpile models get eventually trapped into stable configurations from which they cannot escape, i.e., absorbing states (AS) [11,12]. In order to make this connection more explicit the notion of *fixed-energy sandpiles* was introduced. These modified sandpiles share the microscopic rules with their standard (slowly driven and dissipative) counterparts, but with neither driving (no addition of sand grains) nor dissipation; i.e., the total amount of sand (energy) becomes a conserved quantity acting as a control parameter. In this way, if a standard sandpile in its stationary critical state has an average density of grains (or energy) ζ_c, it can be shown that its fixed-energy counterpart exhibits a transition from an active to an absorbing phase at precisely ζ_c, while it is in an absorbing (active) state below (above) this value. *Slow driving and dissipation define a mechanism which is able to pin the system to its critical point* [6–9].

Using this analogy to systems with AS [13], a field theoretical description of *stochastic sandpiles* has been proposed [6,7,9], which includes the two more relevant features of stochastic sandpiles: (i) *the presence of infinitely many AS* and (ii) *the global conservation of the total energy*. The phenomenological field theory (Langevin equation) aimed at capturing the relevant critical features of this type of systems is similar to the well-known Reggeon field theory (RFT) [12,14] (describing generic systems with AS) but it is coupled linearly to a conserved nondiffusive energy field, namely [7,9],

$$\partial_t \rho = D_a \nabla^2 \rho - \mu\rho - \lambda\rho^2 + \omega\rho\phi + \sigma\sqrt{\rho}\,\eta(x,t),$$

$$\partial_t \phi = D_c \nabla^2 \rho, \tag{1}$$

where D_a, D_c, μ, λ and ω are constants, $\rho(x,t)$ and $\phi(x,t)$ are the activity and the energy field respectively, η is a zero-mean Gaussian white noise.

Soon after the introduction of the previous Langevin equation its range of applicability was extended, as it was conjectured to describe all systems with many AS and an auxiliary conserved and nondiffusive (or static) field [15]. In particular, for a reaction-diffusion model in this family an equation similar to Eq. (1) was derived rigorously by using standard Fock-space formalism techniques [15–17]. To be more precise, we should mention that the derived set of equations includes some higher order (irrelevant) terms such as noise crossed-correlations, whose role in the asymptotic properties is not clear.

A priori, it is not straightforward to decide from a field theoretical point of view whether the extra conservation law induces a critical behavior different from that of RFT or if, on the contrary, it is an irrelevant perturbation at the RFT renormalization group fixed point [14]. From the theory side, it has been recently argued by van Wijland that the Langevin equation is renormalizable in $d_c=6$ [21], while other authors have previously claimed $d_c=4$ [6,7,9]. Some mean field results and simulations in high dimensions of discrete models [19] and also a new method recently proposed by Lübeck and Heger to determine the upper critical dimension of systems with AS [20] lead rather convincingly to $d_c=4$, but we are still far from a full clarification of these issues at a theoretical level. In any case, it is accepted, from numerical evidence, that this constitutes a different universality class, usually called Manna class, or C-DP (in the spirit of Hohenberg and Halperin [22]) [6,9,15–20].

In order to shed some light on these questions, it is our purpose here to integrate numerically Eq. (1) in one, two, and three dimensions. In this way we will verify whether this set of Langevin equations describes correctly the critical properties of the discrete models reported to belong to this class. We employ an integration scheme introduced by Dickman some years back [23] which, to the best of our knowledge, is the only working method for Langevin equations including a RFT-like type of noise. We will verify that indeed the Langevin equation as it is reproduces remarkably well the known exponents (as measured in discrete models in this class), thus providing us with a sound base for further theoretical analyses of this universality class and of the role of conservation in self-organizing systems [5].

THE MODEL

We integrate numerically Eq. (1) [6,7,9]. A technical problem appears when a standard (Euler [24]) discretization scheme is used: due the symmetry of η around zero and the fact that the noise term dominates the evolution whenever the density field is sufficiently small, negative (unphysical) local values of the density field can be generated. In order to overcome this difficulty a different, nontrivial integration scheme was proposed by Dickman for the RFT [23]. It consists in discretizing the density field ρ as well as time and space. The *quanta* of density of activity can be taken proportional to the discrete time step, $\Delta\rho=\Delta t$, in such a way that the continuous model is recovered in the limit $(\Delta t,\Delta x \to 0)$ [23]. The activity density at a given site i and time t is then given by $\rho(i,t)=m(i,t)\Delta\rho$, where $m(i,t)$ takes integer values. Note that $m(i,t)$ diverges as the continuous limit $(\Delta\rho\to 0)$ is approached (which makes a strong diffence with respect to intrinsically discrete, particle models). Further details of the scheme, which has been successfully applied to both the RFT and to systems with many AS [25], leading to good estimations of phase diagrams and critical properties, can be found in Ref. [23]. In order to extend the algorithm to our problem, the second equation of Eq. (1) is integrated using an usual Euler scheme with a continuously varying field $\phi(i,t)$, while for the equation of ρ, we follow Dickman's ideas. First, we calculate

$$\hat{f}(i,t+\Delta t)-f(i,t)=\Delta t[D_a\nabla_d^2 m(i,t)-\mu m(i,t)-\lambda\Delta\rho m^2(i,t)+\omega m(i,t)\phi(i,t)]+\sigma m^{1/2}(i,t)\eta'(i,t), \tag{2}$$

where $\Delta t=\Delta\rho$, η' is a zero-mean Gaussian white noise, ∇_d^2 is the discrete Laplacian operator, $f(i,t)$ is an auxiliary continuous field, and $\hat{f}(i,t)$ is an intermediate stage of $f(i,t)$ (just before the new quanta of activity have been substracted). Then, after each integration step, the number of quanta of ρ, $m(i,t)$, is updated according to

$$m(i,t+\Delta t)=m(i,t)+\int[\hat{f}(i,t+\Delta t)],$$

$$f(i,t+\Delta t)=\hat{f}(i,t+\Delta t)-\int[\hat{f}(i,t+\Delta t)]. \tag{3}$$

Initial conditions are taken as follows: (i) $\phi(x,t=0)=\phi_0[1+a\nabla^2\varepsilon(x)]$, where ε is a normalized Gaussian noise with zero average and a is a constant establishing the range of relative variation allowed to ϕ with respect to its mean value ϕ_0. ϕ_0 is the control parameter, and except for transient effects results should not depend on a. (ii) The initial condition for ρ is chosen by randomly distributing active-field quanta, in such a way that $\rho(x,t=0)\leqslant\phi(x,t=0)$ everywhere.

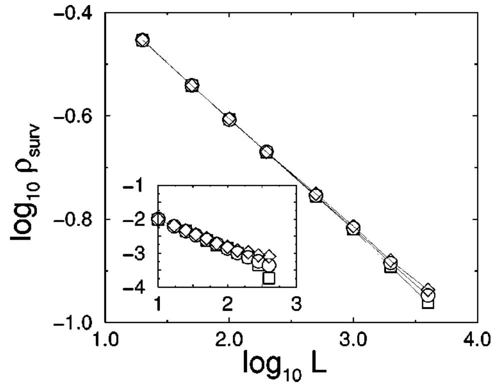

FIG. 1. Stationary value of ρ_{surv} for different system sizes in 1D. Squares correspond to $\phi_0=1.6369$, circles to $\phi_0=1.6371$, and diamonds to $\phi_0=1.6373$. In the inset, the same graph is displayed but for 2D data; squares are for $\phi_0=0.631$, circles for $\phi_0=0.6325$, and diamonds for $\phi_0=0.635$.

We have carried out extensive simulations of the coupled equations (1) in one-, two-, and three-dimensional lattices. In all the cases, the time mesh has been fixed to $\Delta t=0.01$, and $\Delta x=1$ (we have verified that our estimations of critical exponents are not significantly affected upon further decreasing these constants). This choice implies $\Delta\rho=0.01$. As initial conditions, we usually start [in one dimension (1D)] with 100 quanta per site; the evolution of the system drives this quantity to much lower values at the critical point. We also fix $D_a=D_c=5$ and $\mu=\lambda=\omega=a=1$. The noise amplitude σ is taken different for the various dimensions in order to fix the transition in a reasonable (but arbitrary) value of ϕ_0: $\sigma=1$ in 1D, $\sigma=0.5$ in 2D, and $\sigma=0.35$ in 3D. We have verified that the total energy is conserved within the considered precision, in all cases. The number of runs goes from 10^2 up to 10^5 depending on system size.

RESULTS

As we vary ϕ_0, a continuous transition separating the absorbing (small ϕ_0) from active phase (large ϕ_0) is observed at a critical threshold ϕ_c. The usual scaling laws $\rho\sim(\phi_0-\phi_c)^{\beta}$, $\xi\sim(\phi_0-\phi_c)^{-\nu_\perp}$, and $\tau\sim(\phi_0-\phi_c)^{-\nu_\parallel}$, where ξ (τ) is the correlation length (time), are expected to hold [11,12]. This leads to the definition of the dynamic exponent as $\tau\sim\xi^z$, with $z=\nu_\parallel/\nu_\perp$. It is also expected that at the critical point the density of activity presents a power law decay with time, $\rho\sim t^{-\theta}$. However, in some models, an anomalous critical time behavior of ρ has been reported [9,10,15–17]. We shall later return to this issue. As usual, the finite size of simulated systems induces the possibility of falling into the AS even for $\phi_0>\phi_c$. This fact has two consequences. The density of activity ρ does not reach a stationary state close to the critical point. Hence, we are forced to consider the den-

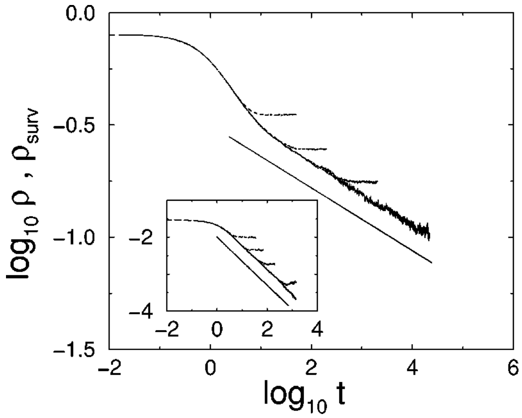

FIG. 2. Evolution of ρ (continuous line) and ρ_{surv} (dashed lines) for several system sizes in 1D. The curve of ρ is for $L=4000$ and those of ρ_{surv} for (from top to bottom) $L=20,100$, and 500. The slope of the straight line is $\theta=0.14$. In the inset, the same data in 2D are represented; ρ_{surv} curves correspond from top to bottom to $L=10,25,70$ and $L=280$, and ρ to $L=280$. The slope of the line is $\theta=-0.65$ [27].

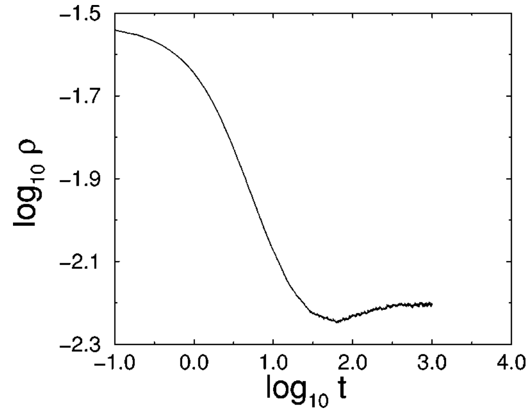

FIG. 3. Anomalous time decay of the activity density in 2D, for $L=280$ and $\phi_0=0.711$.

sity of surviving trials, ρ_{surv}, in order to realize the finite size analysis of ρ. On the other hand, this provides us with a method to measure the dynamic exponent z, by determining a characteristic decaying time as a function of system size. We have studied systems of linear size up to $L=4000$ in 1D, $L=400$ in 2D, and $L=80$ in 3D. The dependence of the stationary activity density on system size for several values of ϕ_0 in a one-dimensional system is shown in Fig. 1. From this picture, we deduce the critical point location in 1D, $\phi_c(1\mathrm{D})=1.6371(2)$ (numbers in parentheses correspond to the statistic uncertainty in the last digit). From the slope of the log-log plot, we obtain $\beta/\nu_\perp(1\mathrm{D})=0.214(8)$. The exponent β may be estimated in an independent way from the scaling of the stationary value of ρ_{surv} for large system sizes, as a function of $(\phi_0-\phi_c)$ above the critical point. This gives $\beta(1\mathrm{D})=0.28(2)$. By studying the time evolution of the characteristic time of the surviving probability $P(t)$ at criticality, we obtain $z(1D)=1.47(4)$. Finally, the exponent $\theta(1\mathrm{D})=0.14(1)$ may be measured from the critical power law decay of ρ in time, as may be seen in Fig. 2. Errors in these exponents mostly come from the uncertainty in the determination of the critical point. Repeating this process in higher dimensions, we find $\phi_c(2\mathrm{D})=0.6325(5)$ and $\phi_c(3\mathrm{D})=0.456(1)$, together with the critical exponents listed in Table I. In the table, we have also included the critical exponents of discrete models claimed to belong to the same universality, and also (for comparison) those of the directed percolation (DP) class. Observe the rather remarkable agreement (within error bars) between all the measured exponents and their counterparts in discrete models. Let us remark that, for those exponents for which the differences with DP values are larger, our values also deviate from DP. As we have already mentioned, some models in the Manna universality class may present an anomalous behavior in the time decay of the activity density at the critical point [9,10,15–17] $\rho(t,\phi_c)$. This anomaly implies that, apparently, the scaling relation $\beta=\theta\nu_\parallel$ fails [9,10,15–17], and that $\rho(t)$ may decay in a nonmonotonous way at criticality. In our case, there is no anomalous decay in $d=1$ (Fig. 2). However, the anomaly is present both in $d=2$ and in $d=3$. As can be seen in Fig. 3, the

TABLE I. Critical exponents for steady state experiments in $d=1,2$, and 3. Figures in parentheses indicate the statistical uncertainty in the last digit. C-DP exponents are from Refs. [16,18] and DP exponents from Refs. [26]. In 1D, $\beta/\nu_\perp$ for C-DP has been calculated using the scaling relation $\beta/\nu_\perp=z\theta$; and $\nu_\parallel$ is derived for both Eq. (1) and C-DP from $z=\nu_\parallel/\nu_\perp$.

D	Model	β	$\beta/\nu_\perp$	z	$\nu_\parallel$	θ
	Eq. (1)	0.28(2)	0.214(8)	1.47(4)	1.95(15)	0.14(1)
1	C-DP	0.29(2)	0.217(9)	1.55(3)	2.07(10)	0.140(5)
	DP	0.276...	0.252...	1.580...	1.733...	0.159...
	Eq. (1)	0.66(2)	0.85(8)	1.51(3)	1.27(7)	0.50(5)
2	C-DP	0.64(2)	0.78(2)	1.55(3)	1.29(8)	0.51(1)
	DP	0.583(4)	0.795(6)	1.766(2)	1.295(6)	0.450(2)
	Eq. (1)	0.84(5)	1.44(5)	1.69(4)	1.07(8)	0.93(3)
3	C-DP	0.88(2)	1.39(4)	1.73(5)	1.12(8)	0.88(2)
	DP	0.80(2)	1.39(1)	1.901(5)	1.105(5)	0.730(4)

activity density decays initially faster than a power law, showing afterwards a nonmonotonous behavior. The fast decay explains the anomalous values of θ, not satisfying scaling relations, usually reported in the literature [9]. Later on, ρ increases before reaching the steady state value, $\rho_{stat}(L)$ after a certain time $t_{\times}(L)$. The criterion to fix the latter is arbitrary; we have chosen it as the time when $\rho_{surv}(t)$ reaches a value that does not differ more than 5% of the final stationary estimation. If the points $(t_{\times}(L), \rho_{stat}(L))$ are represented in a log-log plot at the critical point, an alternative value for the exponent θ is found, which is related to the saturation time scale and satisfies the scaling laws. This value in two dimensions is $\theta \approx 0.50$ [27] (with a large statistic uncertainty), which is much closer to the more accurate measurements reported in the literature for models in this class ($\theta = 0.51$ for C-DP [18]). The common presence of anomalous behavior in discrete systems [9,10,15–17] and in the continuous theory reinforces the claim that both belong to the same universality class: *they share not only the critical behavior but also the dynamical anomalies*. A deeper study of the physical origin of this anomaly is still missing but, essentially, it is related to the existence of different relaxation time scales. On the other hand, this anomaly is absent for flat initial conditions [28].

CONCLUSIONS

This is the first time, to our knowledge, that the phenomenological Langevin equation, proposed some time ago to capture the criticality of this universality class [Eq. (1)], has been numerically integrated. Our results in one-, two-, and three-dimensional systems support the claim that it constitutes a sound minimal continuous representation of this class, sharing all the critical exponents as well as the dynamical anomalies with the discrete models. Therefore, no other higher order terms nor other noise correlations are needed to describe properly this class. Now that the situation has been clarified from the numerical side, further theoretical analyses are highly desirable in order to put this puzzling universality class under more firm bases.

We thank M. A. Santos, H. Chaté, R. Pastor-Satorras, and R. Dickman for useful comments, as well as P. Hurtado for his helpful participation in the early stages of this work. Support from the Spanish MCyT (FEDER) under Project No. BFM2001-2841, from the postdoctoral program of the "Centro de Física do Porto," and from the Portuguese Research Council under Grant No. SFRH/BPD/5557/2001 are acknowledged.

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