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Sticky grains do not change the universality class of isotropic sandpiles

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We revisit the sandpile model with "sticky" grains introduced by Mohanty and Dhar [Phys. Rev. Lett. **89**, 104303 (2002)] whose scaling properties were claimed to be generically in the universality class of directed percolation for both isotropic and directed models. While for directed models this conclusion is unquestionable, for isotropic models we present strong evidence that the asymptotic scaling in the self-organized regime (in which a stationary critical state exists in the limit of slow driving and vanishing dissipation rate) is, like other stochastic sandpiles, generically in the Manna universality class. This conclusion is drawn from extensive Monte Carlo simulations, and is strengthened by the analysis of the Langevin equations (proposed by the same authors to account for this problem), argued to converge upon coarse-graining to the well-established set of Langevin equations for the Manna class.

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Toy models of sandpiles are the archetypical examples of self-organized criticality [1–3]. Sandpiles come in many different flavors (deterministic or stochastic rules [4], discrete or continuous variables [5], with or without height restrictions [6], etc.), but they usually consist in adding grains one by one until a local threshold (typically a condition on some slope or height) is reached, triggering a series of redistribution events, i.e., "avalanches," which may lead to dissipation of sandgrains at the open boundaries. Their numerical study is notoriously difficult and first led to a largely unsatisfactory situation in which "microscopic details" were believed to influence scaling properties, in contradiction with universality principles [7,8]. Major progress in favor of universality came when sandpiles were put into the broader context of standard nonequilibrium absorbing-state phase transitions [9–15]. Indeed, switching off both dissipation (open boundaries) and driving (slow addition of grains) the total amount of sand or "energy" is conserved and becomes a control parameter for these "fixed energy sandpiles." For large amounts of sand the system is in an active phase with never-ending relaxation events, while for small energies it gets trapped with certainty into some absorbing state (all sites below threshold). Separating these two regimes there is a critical energy which was shown [10–12] to coincide with the stationary energy-density in the corresponding original sandpile (which corresponds to the limit of driving and dissipation rates going to zero, with the ratio of these two rates going also to zero [16,17]). In this way, the exponents characterizing sandpiles can be related to standard critical exponents in an absorbing-state phase transition [18]. An alternative route, not dicussed here, is to map sandpiles into standard pinning-depinning interfacial phase transitions [13].

Using this approach, it was determined that stochastic sandpiles [20] do *not* belong generically to the very robust directed percolation (DP) class, prominent among absorbing phase transitions, but to the so-called "conserved-DP" or Manna class (hereafter C-DP/Manna) characterized by the coupling of activity to a static conserved field directly representing the local conservation of sandgrains [10–12,14,21]. The field theory or mesoscopic Langevin equations describing this class reads

$$\partial_t \rho = a\rho - b\rho^2 + \omega\rho E + D\nabla^2\rho + \sigma\sqrt{\rho}\,\eta,$$

$$\partial_t E = D_E \nabla^2 \rho, \qquad (1)$$

where $\rho(\mathbf{x},t)$ is the activity field (characterizing the density of grains above threshold), $E(\mathbf{x},t)$ is the locally conserved energy field, a, b, ω, D, σ, and D_E are parameters, and $\eta(\mathbf{x},t)$ is a Gaussian white noise. Equation (1) represents a robust and well-established universality class including not only stochastic sandpiles, but also some reaction-diffusion systems [21,14].

Recently, Mohanty and Dhar (MD) [22] have introduced a new type of sandpiles where, with some probability, $1-p$, grains may remain stable even if the local threshold is passed. Owing to this, for small values of p the average energy grows unbounded and no stationary state is reached, while for p larger than a critical value, p_c^*, a *self-organized critical state* with a finite average energy and critical avalanches is reached. MD claimed that these sandpiles should be generically in the DP class. They presented clean analytical and numerical evidence that indeed this is the case for a *directed* two-dimensional system, which happens to be mappable into an effective one-dimensional directed site-percolation dynamics. Also, for isotropic (*undirected*) models with stickiness it was shown that DP scaling holds right at p_c^*.

What is not so clear is what is the generic behavior of these, isotropic, sandpiles in the *self-organized regime*. For this case, no rigorous mapping exists, and MD presented some Monte Carlo simulation evidence that the model could still be in the DP class. Additionally, they justified this claim by proposing that an adequate set of Langevin equations for sandpiles with stickiness should include a coupling of the form $\omega\rho\Theta(E-\rho-E_c)$, where Θ is the Heaviside step function and E_c is the instability threshold, substituting the bilinear coupling $\omega\rho E$ in Eq. (1). The logic behind such a term is,

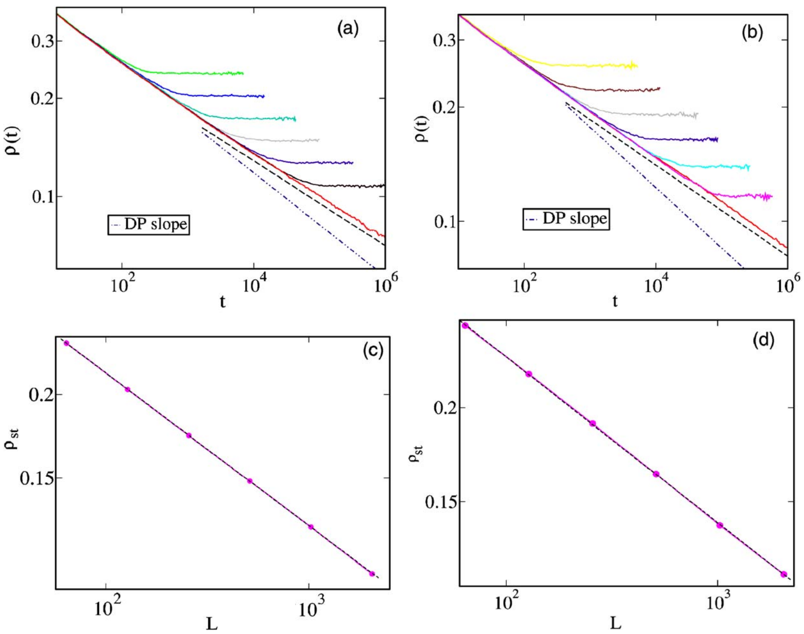

FIG. 1. (Color online) (a) and (b) Log-log plot of the time-decay of the order parameter (activity density) for different system sizes (from top to bottom: $L=64$, 128, 256, 512, 1024, 2048, and $L=2^{18}$) and for the (a) stochastic rule and (b) the deterministic one, at their corresponding critical points $p_c=0.849\,37(2)$ and $p_c=0.767\,50(3)$. From the slopes, we determine $\theta=0.120(8)$ and $\theta=0.115(8)$, respectively (DP slopes are plotted for comparison). In (c) and (d) we plot the saturation values at the previously determined critical points for the stochastic (c) and deterministic (d) rules, respectively. From the scaling at the critical point we determine $\beta/\nu_\perp=0.22(1)$ for both of them.

in principle, reasonable [19] and it is argued in [22] that considering this coupling, one could leave the C-DP/Manna universality class and return to DP. In such a case, the conservation law should be irrelevant in the presence of "stickiness" (see the schematic diagram in Fig. 4 of [22]).

Here we present strong evidence that, even in the presence of stickiness, the generic universality class of isotropic self-organized sandpiles remains the C-DP/Manna one. To justify this claim we first report on extensive simulations of the model studied in [22] in the fixed-energy ensemble, from which we conclude that isotropic sticky sandpiles are in the C-DP/Manna class. Afterward, we integrate the set of Langevin equations with a Θ function coupling, which we find to be also in the C-DP/Manna class, and we perform a numerical renormalization treatment and illustrate how, upon coarse-graining, these Langevin equations evolve to Eq. (1).

The model proposed in [22] is a variation of the Manna model: a discrete sandpile, defined on a one-dimensional lattice, with a height threshold h_c, slow sand addition and small dissipation, but including a (sticking) probability $1-p$ for grains to remain stable even if they are above threshold. This model self organizes to a critical state in the double limit of vanishing driving and dissipation, with the ratio of both going also to 0. Here we consider only the limit which possesses a critical point, i.e., the bulk-dissipation rate (δ in the notation of [22]) is set to zero. Active sites (at which $h>h_c$) are updated in parallel with the toppling occurring with probability p. Following the strategy in [10–12], we analyze the "fixed energy" version of the model: we suppress grain addition and dissipation, fix the total energy E, and use p as a control parameter. Note that, owing to the existence of a nonvanishing sticking probability, arbitrarily large heights are allowed; upon approaching p_c^* the average height diverges and the fixed energy ensemble cannot even be defined for $p\leqslant p_c^*$.

We have implemented two different versions in which each toppling event redistributes two grains to the two nearest-neighbor sites, either randomly (stochastic rule) or regularly, with one grain onto each neighbor (deterministic rule). The methodology followed is standard for absorbing phase transitions. First, we fixed a given energy, $E=2$, and locate the critical point by studying the decay of activity from some initial condition varying p in a large system: for large p values, activity saturates (active phase), while for small p activity vanishes (absorbing phase). At the critical point, separating these two phases, $p_c(E)$, activity decays asymptotically as a power-law with an exponent $\theta=\beta/\nu_\parallel$. For the stochastic and the deterministic rules, we find, respectively, $p_c=0.849\,37(2)$ with $\theta=0.120(8)$ and $p_c=0.767\,50(3)$ with $\theta=0.115(8)$ [Figs. 1(a) and 1(b)]. These estimates of θ are in good agreement with the best evaluations for the C-DP/Manna class in one dimension, i.e., $\theta=0.125(2)$ [23], and clearly incompatible with the DP value $\theta\approx0.159$.

Next, using the critical value determined above, the variation of the stationary saturation value of activity at the critical point for smaller system sizes is recorded. From the expected scaling law $\rho_{st}(p=p_c)\sim L^{-\beta/\nu_\perp}$, we determine $\beta/\nu_\perp=0.22(1)$ for both the stochastic and the deterministic rule as shown in Figs. 1(c) and 1(d). Again, this value is in good agreement with available estimates for the C-DP/Manna class $\beta/\nu_\perp=0.215(5)$ [15,24], and incompatible with the DP value $\beta/\nu_\perp\approx0.252$.

Also, spreading experiments (not shown) fully confirm this result. In addition, we verified that taking the previous critical points for both the stochastic and the deterministic rule, and performing standard simulations without fixing the energy (i.e., with open boundaries and slow addition of grains) the system self-organizes to an average energy $E=2$ with C-DP/Manna exponents.

Finally, we have considered also larger values of E as $E=3$, and, even if with longer transient effects, simulation results exclude DP behavior and support again C-DP/Manna scaling. For all the studied values in the self-organized phase we obtain this same conclusion.

We now turn to a study of the coupled Langevin equations

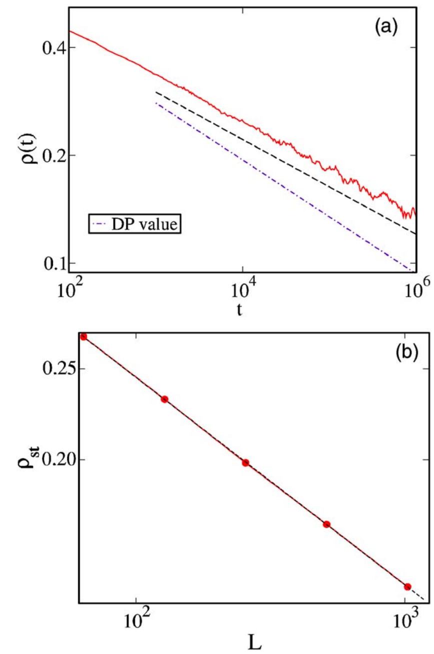

FIG. 2. (Color online) Direct numerical integration of the Langevin equations proposed in [22]. Black dashed lines correspond to C-DP/Manna scaling. (a) Decay experiments at the critical point $a=0.723\,08$ [parameters: system size $L=2^{20}$, $\langle E(x,t=0)\rangle=0.5$, $b=1$, $h_c=0.5$, $D=D_E=0.25$, $\omega^2=\sigma^2=2$, integrated with time-step $dt=0.25$]. (b) Finite size scaling at criticality (more details in the text).

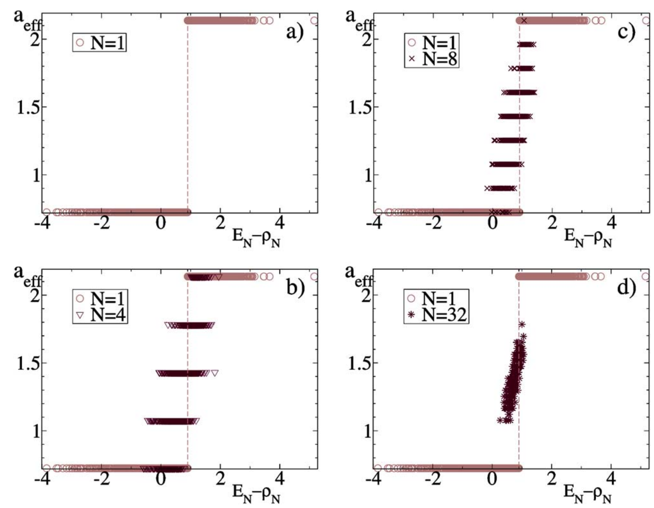

FIG. 3. (Color online) Effective mass as defined by Eq. (2) as a function of the field difference averaged in Kadanoff blocks of size N, for $a=0.723\,13$ in the active phase and $\omega=\sqrt{2}$. The vertical line corresponds to the threshold value $h_c=0.9$. Observe that the larger the block size, the smoother the effective-mass dependence on the coarse-grained field difference.

proposed by MD to describe their coarse-grained dynamics. The stochastic equations proposed in [22] is Eq. (1), i.e., those of the C-DP/Manna class, except for the coupling term $\omega\rho E$ which is replaced by $\omega\rho\Theta(E-\rho-E_c)$ where E_c is the (microscopic) toppling threshold, and $E-\rho$ the density of nonactive grains. The presence of this microscopic feature and of the step function Θ is surprising in so far as Langevin equations are usually understood as resulting from some coarse-graining of microscopic dynamics. In particular, the step function is unlikely to be a robust mesoscopic description, as it will be modified (probably transformed into a smoother function) upon coarse-graining.

Discontinuous functions are notoriously difficult to handle in the framework of renormalization group analysis. Moreover, even the "simple" equation (1) resists standard perturbative renormalization attempts [25]. The only available strategy then is direct numerical integration. The presence of the (square-root) multiplicative noise term makes this *a priori* difficult [26], but this technical difficulty was recently circumvented by the fast sampling method introduced in [23].

We have used this numerical scheme to integrate the equations proposed by MD. The simulations yielded two sets of results: following the protocol recalled above, we studied the absorbing phase transition observed when varying the linear coefficient and analyze critical properties. Also, we introduced a local effective "mass" coefficient measuring whether there is fostering of activity creation (site above threshold) or not at each site,

$$a_{\text{eff}}(\mathbf{x},t) = a + \omega\Theta(E(\mathbf{x},t) - \rho(\mathbf{x},t) - E_c), \qquad (2)$$

and studied the behavior of its average upon coarse-graining in numerical simulations. With this, we illustrate how the Θ-function evolves upon coarse-graining, and clarify the connection between the two sets of Langevin equations.

Phase transition. Starting from a homogeneous, active, initial condition, we studied the time decay of spatially averaged activity varying the control parameter a. As expected, algebraic decay is found at the critical value separating exponential decay (absorbing phase) from saturation (active phase). The estimated decay exponent $\theta=0.130(5)$ is in perfect agreement with the C-DP/Manna class value [Fig. 2(a)]. At the critical point, $a=0.723\,08(5)$, the scaling of the stationary activity for finite size systems yields the estimate $\beta/\nu_\perp=0.22(1)$ [Fig. 2(b)] again in agreement with the C-DP/Manna value. These estimates are thus incompatible with the DP class values. We have also performed spreading experiments [9] by following the standard procedure: we perturb a natural absorbing state (one generated by the system dynamics) to generate a small amount of localized activity and analyze how it spreads out at the previously determined critical point. We measured $\eta=0.39(3)$, $\delta=0.167(5)$, and $z=1.39(3)$ exponents for the number of active sites, surviving probability, and average square-radius critical, respectively [9]. These values are in good agreement with the best estimations for the C-DP/Manna class [14] and differ from their corresponding DP values ($\eta\approx0.313$, $\delta\approx0.159$, and $z\approx1.258$ [18]).

Numerical coarse-graining. The above results are easily understood when observing the behavior of the MD Langevin equations coarse-grained numerically. To do this, we build scatter plots of $\langle a_{\text{eff}}\rangle_N$ vs the field difference $\langle E-\rho\rangle_N$, where the averages are taken on Kadanoff-blocks of length N. For $N=1$ [Fig. 3(a)], we obviously observe the Θ-function form. For $N=4$ [Fig. 3(b)], the effective coupling can take intermediate discrete values between a and $a+\omega$, depending on the number of above-threshold microscopic sites in the block. When the block size is larger and larger [Figs. 3(c) and 3(d)], a smooth function gradually appears. By retaining just the two leading terms in a Taylor expansion of such an analytical function around the origin, we recover, at a coarse-grained level, the Langevin equation for the C-DP/Manna class Eq. (1), i.e., a linear coupling term and a correction to the linear term, a in Eq. (1). Higher order terms in the Taylor expansion can be argued to be irrelevant from standard naive power counting arguments. Therefore, it is not surprising that the set of Langevin equations including a Heaviside Θ function should exhibit the same asymptotic behavior as the original C-DP/Manna class Langevin equations, Eq. (1).

In summary, introducing "stickiness" in isotropic sandpile models does not change their universality class, which remains generically that of the Manna model for $p>p_c^*$. This conclusion is supported by extensive numerical simulations of microscopic models and Langevin equations proposed in [22] to describe these sandpiles. We showed in addition that these Langevin equations "flow" toward those of the C-DP/Manna class under some numerical coarse-graining procedure. Fully understanding the origin of the discrepancies between our results and the claim in MD that the self-organized regime is DP-like (which might be either due to the presence of long crossovers or related to the way the driving and dissipation rates tend to zero [17]) remains an open task.

One can also wonder what is the reason why this conclusion does not hold for the *directed* sandpiles studied also by Mohanty and Dhar in [22] (see also [27]), which they proved to be in the DP class. These directed models, defined on a two-dimensional lattice include an isotropic direction and a fully anisotropic one, in the sense that sand goes "downward" in that direction but not "upward." This makes it possible to map the problem on DP in $(1+1)$ dimensions, i.e., the anisotropic dimension can be taken as "time." The local conservation of energy is present also in these models, but "local" here means in "space-time" neighborhoods, while *energy is not conserved* in the isotropic spatial direction. Hence, C-DP/Manna behavior does not show up while DP scaling, the usual one in the absence of spatial energy conservation, emerges, as indeed was proved in [22].

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