The Edwards-Wilkinson equation with drift term

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Overview

One appealing aspect of the Edwards-Wilkinson (EW) equation is that almost everything can be done exactly. In the following, results are presented for the Edwards-Wilkinson equation in one dimension with an extra drift term, for periodic and fixed boundary conditions. The key points are:

- **Contrary to suggestions in the literature, there is *no smoothing*.**
- **Although the problem is linear, with fixed boundary conditions, the extra drift term produces *anomalous dimensions*.**
- **Dimensional analysis, in conjunction with coarse graining arguments, fails.**

Introduction

We start out with an **Edwards-Wilkinson equation with drift term** living on the interval $[0, L]$:

$$\partial_t \phi(x,t) = D\partial_x^2 \phi(x,t) + v\partial_x \phi(x,t) + \eta(x,t) \tag{1}$$

where v denotes the drift velocity (pointing in the "wrong" direction). The noise, η, is chosen to be delta-correlated as usual:

$$\langle \eta(x,t)\eta(x',t')\rangle = \Gamma^2 \delta(x-x')\delta(t-t') \tag{2}$$

The whole problem is much more conveniently expressed in terms of **dimensionless variables**, $\tau = t/(L^2/D)$, $y = x/L$, $q = vL/D$ and especially

$$\varphi(y,\tau) = \sqrt{\frac{D}{\Gamma^2 L}}\phi(x,t) \tag{3}$$

Periodic Boundaries

The **propagator** of this problem is a Gaussian wrapped around the unit circle, i.e. essentially Jacobi's ϑ_3 function:

$$\varphi_0(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+q\tau+n)^2}{4\tau}} = \sum_{n=-\infty}^{\infty} e^{ik_n(y+q\tau)} e^{-k_n^2\tau} \ . \tag{4}$$

The **dimensionless correlator** is then simply

$$\langle \varphi(y_1,\tau_1)\varphi(y_2,\tau_2)\rangle = \tau_1 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{1}{2k_n^2}\left(e^{-k_n^2(\tau_2-\tau_1)} - e^{-k_n^2(\tau_1+\tau_2)}\right) \times e^{ik_n(y_1-y_2+q(\tau_1-\tau_2))} \ , \tag{5}$$

where $\tau_2 \geq \tau_1$. This time-order enters the exponentials when the integration runs over the $\delta(\tau-\tau')$-function of the correlator.

The **exponents** usually derived for interfaces are the **roughness** exponent α, the **growth** exponent β and the **dynamical** exponent z. They are all based on the dimensionful (physical) **equal-time correlation function** in the thermodynamic limit:

$$\lim_{L\to\infty} \langle (\phi(x_1,t) - \phi(x_2,t))^2\rangle = (x_1-x_2)^{2\alpha}\mathcal{G}\left(\frac{t}{(x_1-x_2)^z}\right) \tag{6}$$

At equal times (5) becomes **independent of the velocity**.

Without velocity and with divergent L, dimensional analysis is sufficient for the determination of the exponents, because they are a **physical necessity**

$$\boxed{\alpha = 1/2,\ z = 2,\ \beta = \alpha/z = 1/4 \quad \text{PBC independent of } v} \tag{7}$$

Of course, the calculation can be done explicitly in order to determine the proper behaviour of the scaling function $\mathcal{G}$ in (6).

Exponents from roughness

Alternatively, the exponents can be derived from the **width of the interface**:

$$w^2(L,t) \equiv \langle \overline{\phi(x,t)^2}\rangle - \langle \overline{\phi(x,t)}^2\rangle \ , \tag{8}$$

where $\overline{A}$ denotes the spatial average. Again,

$$w^2(L,t) = L^{2\alpha}\mathcal{G}\left(\frac{t}{L^z}\right) \ . \tag{9}$$

An equal-time average destroys all velocity dependence. One finds

$$\lim_{t\to\infty} w^2 = \frac{\Gamma^2 L}{24D} \tag{10}$$

and

$$\lim_{L\to\infty} w^2 = \frac{\Gamma^2}{D}\sqrt{\frac{tD}{2\pi}} \tag{11}$$

and therefore the **same set of exponents** as in (7). The exponents can also be derived from dimensional analysis.

The drift is irrelevant in case of periodic boundary conditions.

Fixed Boundaries

The picture changes completely when the **boundaries are fixed** to $\phi(x=0,t) = \phi(x=L,t) = 0$. The **propagator** is now "almost" the mirror-charge version of (4):

$$\varphi_0(y,\tau;y_0) = \frac{1}{\sqrt{4\pi\tau}}\sum_{n=-\infty}^{\infty}\left(e^{-\frac{(y-y_0+2n)^2}{4\tau}} - e^{-\frac{(y+y_0+2n)^2}{4\tau}}\right) \times e^{-\frac{1}{2}(y-y_0)q - \frac{1}{4}\tau q^2} \tag{12}$$

This time, the velocity does not dissappear in the correlator. It is not possible to simply gauge it away. This velocity gives rise to a **second length scale**, namely D/v, which in turn can give rise to **anomalous dimensions**.

$v = 0$

From the outset it is clear that the exponents are those derived above. Explicit calculation gives again

$$\lim_{t\to\infty} w^2 = \frac{\Gamma^2 L}{24D} \tag{13}$$

$$\lim_{L\to\infty} w^2 = \frac{\Gamma^2}{D}\sqrt{\frac{tD}{2\pi}} \tag{14}$$

and therefore

$$\boxed{\alpha = 1/2,\ z = 2,\ \beta = \alpha/z = 1/4 \quad \text{FB with of } v = 0} \tag{15}$$

$v \neq 0$

Now an anomalous dimension could occur — and so it does. After some algebra one finds by a saddle-point approximation

$$\lim_{t\to\infty} w^2 = \frac{2\Gamma^2}{3}\sqrt{\frac{L}{2\pi v D}} + \mathcal{O}(L^0) \ . \tag{16}$$

Moreover

$$\lim_{L\to\infty} w^2 = \Gamma^2\sqrt{\frac{t}{2\pi}} + \mathcal{O}(t^{3/2}) \tag{17}$$

and therefore the **anomalous dimensions** are

$$\boxed{\alpha = 1/4,\ z = 1,\ \beta = \alpha/z = 1/4 \quad \text{FB with of } v \neq 0} \tag{18}$$

The **crossover-time** is given by L/v. This should happen *before* the interface reaches its saturation in normal roughening, i.e. the **crossover-length** for the system to show anomalous scaling of the roughness is given by

$$L/v \ll L^2/D \ . \tag{19}$$

Dimensional Analysis

The textbook or rather armchair analysis goes as follows. Assume a **self-affine solution** $\phi(bx, b^z t) = b^\alpha \phi(x,t)$ and plug it into Eq. (1), to obtain a simple relation of the exponents:

$$\underbrace{\alpha - z}_{\partial_t\phi} = \underbrace{\alpha - 2}_{\partial_x^2\phi} = \underbrace{\alpha - 1}_{\partial_x\phi} = \underbrace{-\frac{1+z}{2}}_{\eta} \ . \tag{20}$$

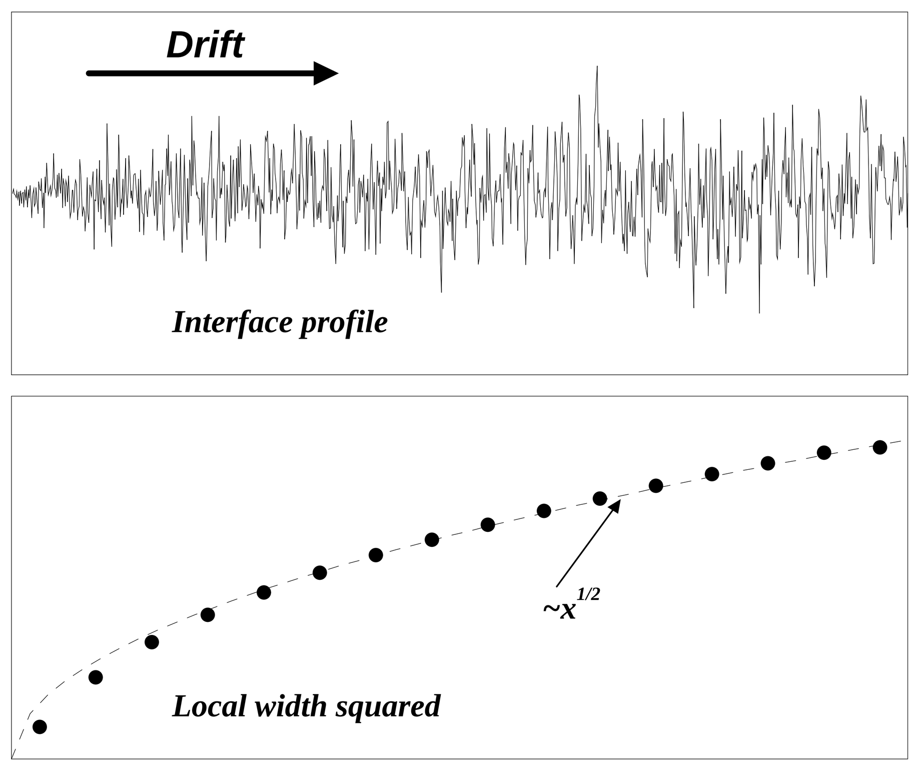
Of course they cannot be satisfied simultaneously. But satisfying all but $\alpha - 2$ renders this one, by coarse graining with parameter b, irrelevant. Thus

$$\alpha = 0,\ z = 1,\ \beta = \alpha/z = 0 \qquad \text{by coarse graining arguments} \tag{21}$$

This is wrong and there is no reason why it should work. Dimensional analysis does not fail, what fails is the coarse graining argument. It is worth noting that dropping the diffusion term leads to random deposition.

Physical Explanation

With fixed boundaries, a non-zero velocity leads to a continuous reinitialisation of the interface. Only if it manages to stay under the influence of the noise before it disappears on the other end, can it develop its full roughness. However, it stays only for L/v, while it takes L^2/D until the roughness is fully developed. Due to the velocity, the space coordinate becomes a time coordinate; **the roughness exponent of the interface with drift is the growth exponent of the interface without drift**.



Upper panel: An example of an interface profile with fixed boundaries and drift term. Lower panel: The *local* width squared (numerical data, circles) is proportional to $x^{1/2}$ (fitted, dashed line), so that $\alpha = 1/4$

Conclusion

- The real thrill may have been missed in the literature: Appearance of an anomalous dimension in a linear problem based on EW.
- Exponents are not derivable by dimensional analysis, and violate $\alpha = z - d/2$.
- Exponents are easily derivable by intuitive physical arguments.
- Coarse graining suggests the diffusion term becomes irrelevant, but it does not.
- A warning: Exponents can easily depend on boundary conditions.

Finally: Is the same mechanism present in the **quenched Edwards-Wilkinson equation** (qEW)? It does not seem so:

Boundaries	Drift	α for EW	α for qEW
Periodic boundaries	$v = 0$	1/2	$\approx 5/4$
Periodic boundaries	$v \neq 0$	1/2	1/2 ?
Fixed boundaries	$v = 0$	1/2	$\approx 5/4$
Fixed boundaries	$v \neq 0$	1/4	1/2