A Markov chain analysis of truels

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Abstract

In this work we present a detailed analysis using Markov chain theory of some versions of the *truel* game in which 3 players with different markmanships try to eliminate each other in a series of one-to-one competitions, using the rules of the game. The paradoxical result in this game is that under certain circumstances the player with the highest markmanship does not necessarily have the highest survival probability. Besides reproducing some known expressions for the winning probability of each player, including the equilibrium points, we provide with the expressions for the actual distribution of winners in a truel competition.

1 Introduction

A *truel* game can be considered as the extension of a duel played by three individuals. These players, which will be named as A, B and C, possess different markmanships, i.e. the probability of hitting a chosen target. Markmanships will be denoted as a, b and c for players A, B and C respectively. Without loss of generality we will assume throughout this paper that the players are labeled such that $a > b > c$. In this game all players share the same goal: to eliminate all the opponents so that eventually the game stops when there is only one survivor left, the winner of the game. The mechanics of the truel can be described by the following steps:

- (1) Each round – or time-step –, one of the truelists is chosen for playing.
- (2) He then decides who will be his target and, with a certain probability – the markmanship – he does achieve the goal of eliminating that opponent from the game.
- (3) Whatever the result obtained by the player, steps one and two are repeated again until there is only one survivor.

Based on the rules used for selecting the players, we can distinguish between three main types of truels:

- **Random truel.** At each round one of the remaining players is chosen randomly with equal probability.

Proceedings of the 8th Granada Seminar on Computational Physics: Modeling cooperative behavior in the social sciences, J. Marro and P.L. Garrido, eds.
AIP proceedings 779, 128 (2005).

- **Sequential truel.** In this case there exists an established firing order, which will be followed throughout the whole game. We allow players with worst markmanship to shoot firstly, followed then by players with better markmanship. According to the notation introduced earlier, the firing order in the sequential truel is C–B–A.
- **Simultaneous truel.** In this version all players shoot at the same time.

A paradoxical or counter–intuitive result appears in this game, as the "truelist" with the highest markmanship does not necessarily possess the highest survival probability. This paradoxical result was already mentioned in the early literature on truels [1]. These games were formally introduced for the first time by Kilgour in 1946 [2], although the name *truel* was coined later by Shubik [3] in the 1960s. We may also find in the literature other models similar to the truel game presenting also the same kind of counter-intuitive results. A better known example is that of the like *rock–scissors–paper* game, a game that has been applied to population dynamics [4, 5] and to some convective instabilities in rotaring fluids [**?**]. It consists in a system with three species that interact with each other creating a competitive loop: rock beats a pair of scissors, scissors beat a sheet of paper and paper beats a rock. However, as will be shown later, the truel corresponds to a more general game where players do not necessarily interact through fixed strategies.

Different versions of the truels vary on the number of tries (or "bullets") available to each player, on whether they are allowed to "pass", i.e. missing the shoot on purpose ("shooting into the air"), on the number of rounds being finite or infinite, etc. All these modifications lead to games with different outcomes [6, 7, 8]. Besides, they can be further extended through the introduction of coalitions between the truelists, that is, the appearance of cooperations between different players so that they can set a common target (these games re known as *cooperative truels* [9]), in such a way that they can obtain greater benefits from that coalition improving their own survival probability. We will restrict ourselves to the case of unlimited ammunition, and the game will continue until there is only one player left (so that there is no upper limit in the number of rounds); besides, players are also allowed to lose their turn by shooting into the air, a possibility that turns out to be useful in some particular cases.

The strategy of each player consists in choosing the appropriate target when it is his turn to shoot. Rational players will use the strategy that maximizes their own probability of winning and hence the ensemble of players will chose the strategy given by the Nash equilibrium point. In a series of seminal papers [6, 7, 8], Kilgour has analyzed the games and determined the equilibrium points under a variety of conditions.

In this paper, we analyze the games from the point of view of Markov chain theory. Besides being able to reproduce some of the results by Kilgour, we obtain the probability distribution for the winners of the games. We restrict our study to the case in which there is an infinite number of bullets and consider two different versions of the truel: random and fixed sequential chosing of the shooting player.

Furthermore, we consider a variation of the game in which instead of eliminating the competitors from the game, the objective is to convince them on a topic, making the truel suitable for a model of opinion formation.

The paper is organized as follows: in Sec. 2, and in order to introduce the general methods in an simpler context, we present a detailed analysis for the case of duels; Sec. 3 is devoted to the analysis of the random and sequential versions of the truels; we present in Sec. 4 an analysis of the opinion model; in Sec. 5 we present the distribution of winners corresponding to the truel games as well as the opinion model; the truels are generalized to more than three players in Sec. 6 and, finally, in Sec. 7 we draw the conclusions.

2 The duels

In this simpler game we consider two players, A and B, with markmanships a and b respectively, such that $a > b$. We will consider the random duel in which the person to shoot next is randomly selected with equal probility between the two players, as well as the sequential version in which the bad player, B, starts shooting and then they alternate fires. In any case, the game continues until there is only one survivor. If we take the model as an opinion model, the game continues until one player han convinced the other and both share the same opinion. Clearly, in a duel it makes no sense for a player to lose his opportunity to eliminate the opponent by shooting into the air and the only meaningful strategy is to shoot into the other player.

An analytical study done with Markov chains for both the random duel and the opinion model shows that both models can be described through the same Markov chain with three states (see Appendix 8.1 for further details). If we denote the survival (or convincing) probabilities of players A and B as π_A and π_B respectively we have

$$\pi_A = \frac{a}{a+b}\,, \qquad \pi_B = \frac{b}{a+b}\,, \tag{1}$$

a result that indicates that the higher the markmanship of a given player, the higher the survival (convincing) probability in the random duel (opinion model).

Turning to the case of the sequential duel, this game can be described with a Markov chain with four states. The analytical expressions obtained for the survival probabilities are

$$\pi_A = \frac{a}{1-(1-a)(1-b)}\,, \qquad \pi_B = \frac{b(1-a)}{1-(1-a)(1-b)}\,, \tag{2}$$

A closer study of Eqs. (2) shows that even though the worst player B starts shooting first, he achieves a higher survival probability than A only when $b > \frac{a}{1+a}$. Thus, in the sequential duel the unfavorable situation of player B having a lower markmanship than A is partially compensated by being the one shooting in first place.

If a third individual comes into play, the previous situation of a duel is no longer simple. Now every player in the truel must consider all possible actions that other opponents may take and their corresponding outcomes. In the next section we analyze the different truel games.

3 Truels

3.1 Random truel

Let us first fix the notation. We denote by P_{AB}, P_{AC} and P_{A0} the probability of player A shooting into player B, C, or into the air, respectively, with equivalent definitions for players B and C. These probabilities verify $P_{AB} + P_{AC} + P_{A0} = 1$. We will consider only "pure" strategies, namely, only one of these three probabilities is taken equal to 1 and the other two equal to 0. Finally, we denote by $\pi(a; b, c)$ the probability that player with marksmanship a wins the game when playing against other two players with marksmanships b and c. This definition implies $\pi(a; b, c) = \pi(a; c, b)$ and $\pi(a; b, c) + \pi(b; a, c) + \pi(c; a, b) = 1$. Recall that we use the convention $a > b > c$.

The corresponding Markov chain for this game is composed of 7 different states labeled as ABC, AB, AC, BC, A, B, C according to the players remaining in the game. Three of these states, A, B and C are absorbent states. The details of the calculation for the winning probabilities as well as a diagram of the allowed transitions between states are shown in Appendix 8.2. We now discuss the results in different cases.

As explained previously, completely rational players will choose strategies that are best responses (i.e. strategies that are utility–maximizing) to the strategies used by the other players. This defines an equilibrium point when all players are better off keeping their actual strategy than changing to another one. Accordingly, this equilibrium point can be defined as the set of probabilities $P_{\alpha\beta}$ (with α =A,B,C and β =A,B,C,0) such that the winning probabilities have a local maximum. This set can be found from the expressions in Appendix 8.2, with the result that the equilibrium point in the case $a > b > c$ is given by $P_{AB} = P_{CA} = P_{BA} = 1$ and $P_{AC} = P_{A0} = P_{BC} = P_{B0} = P_{CB} = P_{C0} = 0$. This strategy set is known as "strongest opponent strategy", as each player aims at the strongest of his opponents [1]. For this strategy and considering that $a > b > c$, the winning probabilities are given by

$$\pi(a; b, c) = \frac{a^2}{(a + c)(a + b + c)}, \quad \pi(b; a, c) = \frac{b}{a + b + c}, \quad \pi(c; a, b) = \frac{c(c + 2a)}{(a + c)(a + b + c)}. \tag{3}$$

An analysis of these probabilities leads to the paradoxical result that when all players use their 'best' strategy, the player with the worst marksmanship can become the player

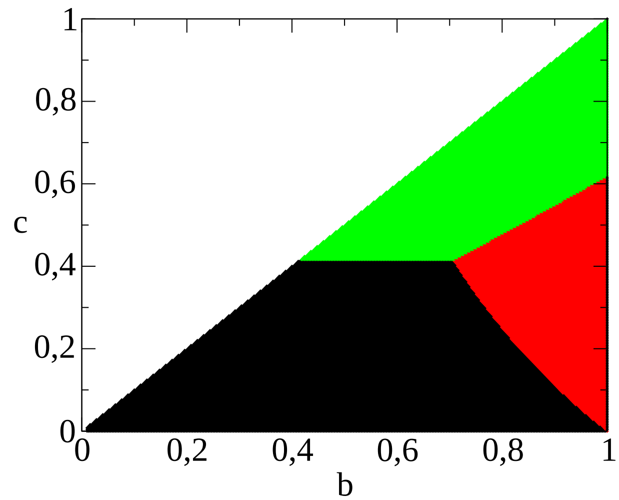


Figure 1: Diagram b vs c setting $a = 1$ where it is plotted with color codes which is the player with the highest survival probability for the case of the random truel. The black color corresponds to the region where player A has the highest winning probability, red color corresponds to player B having the highest winning probability and finally the red color corresponds to player C being the player with the highest survival probability.

with the highest winning probability. For example, when $a = 1.0$, $b = 0.8$, $c = 0.5$ the probabilities of A, B and C winning the game are 0.290, 0.348 and 0.362 respectively, precisely in inverse order of their marksmanship. This somewhat surprising result can be easily understood if one realizes that players set as primary target either player A or player B, leaving player C as the last option and therefore increasing his winning expectation. In Fig. 1 we indicate y a color code the region in parameter space in which each player possesses the highest survival probability when playing the random truel, varying markmanships b and c and keeping a fixed and equal to 1. Note that the region of player A is larger than the ones for B and C. In this figure, markmanship a has been set to its highest possible value 1, because other values $a \neq 1$ can be related through the scaling relations $\pi(a; b, c) = \pi(1; b/a, c/a)$, $\pi(b; a, c) = \pi(b/a; 1, c/a)$, $\pi(c; a, b) = \pi(c/a; 1, b/a)$.

3.2 Sequential truel

In this version of the truel there is an established order of firing. The players shoot in increasing value of their marksmanship. i.e. if $a > b > c$ the firing order is $C - B - A$. The sequence repeats until only one player remains. Again, we have left for Appendix 8.3

the details of the calculation of the winning probabilities. Our analysis of the optimal strategies reproduces that obtained by the detailed study of Kilgour [7]. The result is that there are two equilibrium points depending on the value of the function $g(a,b,c) = a^2(1-b)^2(1-c) - b^2c - a\,b\,(1-b\,c)$: if $g(a,b,c) > 0$ the equilibrium point is the strongest opponent strategy $P_{AB} = P_{BA} = P_{CA} = 1$, whereas for $g(a,b,c) < 0$ it turns out that the equilibrium point strategy is $P_{AB} = P_{BA} = P_{C0} = 1$ where the worst player C is better off by shooting into the air and hoping that the second best player B succeeds in eliminating the best player A from the game.

The winning probabilities for this case, assuming $a > b > c$, are:

$$
\begin{aligned}
\pi(a;b,c) &= \frac{(1-c)(1-b)a^2}{[c(1-a)+a][b(1-a)+a]}, \\
\pi(b;a,c) &= \frac{(1-c)b^2}{(c(1-b)+b)(b(1-a)+a)}, \\
\pi(c;a,b) &= \frac{c[bc+a[b(2+b(-1+c)-3c)+c]]}{[c+a(1-c)][b+a(1-b)][a+b(1-a)]},
\end{aligned} \tag{4}
$$

if $g(a,b,c) > 0$, and

$$
\begin{aligned}
\pi(a;b,c) &= \frac{a^2(1-b)(1-c)^2}{[a+(1-a)c][a+b(1-a)+c(1-a)(1-b)]}, \\
\pi(b;a,c) &= \frac{b\left(b(1-c)^2+c\right)}{[b+(1-b)c][a+b(1-a)+c(1-a)(1-b)]}, \\
\pi(c;a,b) &= \frac{\frac{ac(1-b)(1-c)}{a+c(1-a)} + \frac{c(b+c(1-2b))}{b+c(1-b)}}{[a+b(1-a)+c(1-a)(1-b)]},
\end{aligned} \tag{5}
$$

if $g(a,b,c) < 0$. Again, as in the case of random firing, the paradoxical result appears that the player with the smallest marksmanship has the largest probability to win the game. In Fig. 2 we summarize the results showing the regions in parameter space (b,c) (setting $a = 1$) where each player has the highest probability of winning. In this case the area where player A wins the truel decreases considerably respect to the previous truel game. Basically the reason is that in this case player A is the last one to shoot, and the advantageous situation given by its high markmanship is partially lost due to the imposed firing order.

4 Opinion model

The opinion model is based upon the random firing truel though this case is composed of three people holding different opinions, A, B and C, on a topic. Their goal is to

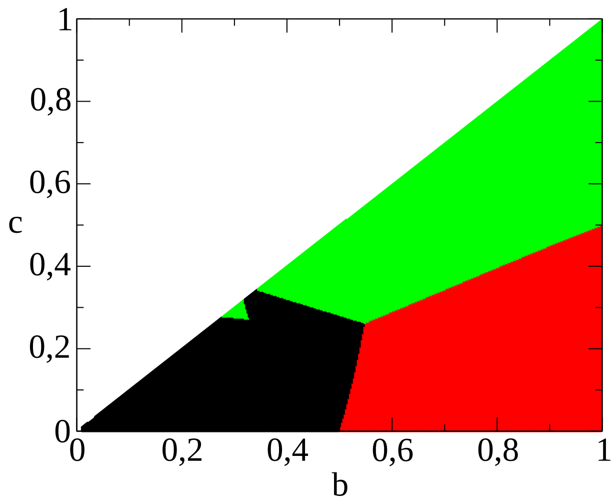


Figure 2: Same as Fig. 1 in the case that players play sequentially in increasing order of their marksmanship.

convince each other in a series of one-to-one discussions. Thus, marksmanship a (resp. b, c) is interpreted as the probability that player holding opinion A (resp. B or C) has of convincing another player of adopting this opinion. The main difference with the previous games is that the number of players present is always constant and equal to three, a fact that will strongly conditionate the results.

The analysis with Markov chains can be found in Appendix 8.4. As in the case of the random truel, there exist only one equilibrium point that corresponds to the strongest opponent strategy. The probabilities of a final consensus opinion being A, B or C, assuming $a > b > c$ are

$$\begin{aligned}
\pi(a;b,c) &= \frac{a^2\left[2cb^2 + a\left((a+b)^2 + 2(a+2b)c\right)\right]}{(a+b)^2(a+c)^2(a+b+c)}, \\
\pi(b;a,c) &= \frac{b^2(b+3c)}{(b+c)^2(a+b+c)}, \\
\pi(c;a,b) &= \frac{c^2\left[c^3 + 3(a+b)c^2 + a(a+8b)c + ab(3a+b)\right]}{(a+c)^2(b+c)^2(a+b+c)}, \qquad (6)
\end{aligned}$$

Notice that, as before, thye satisfy the scaling relations $\pi(a;b,c) = \pi(1;b/a,c/a)$, $\pi(b;a,c) = \pi(b/a;1,c/a)$, $\pi(c;a,b) = \pi(c/a;1,b/a)$. As in previous cases, we have plotted in Fig. 3 in colour code the opinion with the highest probability of becoming majoritarian. In this

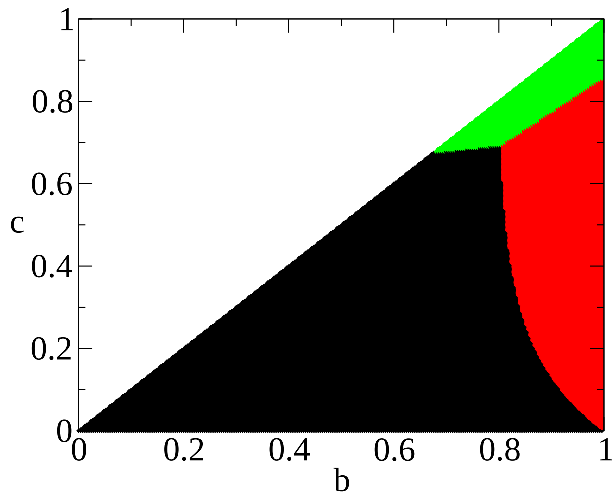


Figure 3: Same as Fig. 1 for the convincing opinion model.

case opinion A becomes majoritarian nearly for all values of b and c. Only for a tiny region opinion C can become the majority opinion. This overwhelming dominion of A can be understood if we recall that the total number of players always remains the same throughout the game. Only the opinions held by the players change. So, once opinion A convinces either a player with opinion B or a player with opinion C, it is very likely that it will eventually become the majority opinion due to its high *convincing* probability.

5 Distribution of winners

Now imagine that we set up a truel competition. Sets of three players are chosen randomly amongst a population whose marksmanship are uniformly distributed in the interval $(0, 1)$. The distribution of winners is characterized by a probability density function, $f(x)$, such that $f(x)dx$ is the proportion of winners whose marksmanship lies in the interval $(x, x + dx)$. This distribution is obtained as:

$$f(x) = \int da\,db\,dc\;[\pi(a; b, c)\delta(x - a) + \pi(b; a, c)\delta(x - b) + \pi(c; a, b)\delta(x - c)] \qquad (7)$$

or

$$f(x) = 3\int_0^1 db \int_0^1 dc\,\pi(x; b, c) \qquad (8)$$

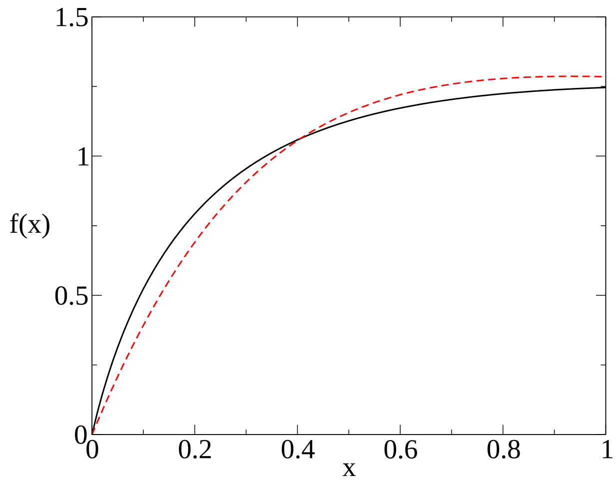


Figure 4: Distribution of winners for the random truel when the players use the *strongest opponent strategy*.

We may also consider a variation of the competition in which the winner of one game keeps on playing against other two randomly chosen players. The resulting distribution of players, $\bar{f}(x)$, can be computed as the steady state solution of the recursion equation:

$$\bar{f}(x,t+1) = \int da\, db\, dc\, \left[\pi(a;b,c)\delta(x-a) + \pi(b;a,c)\delta(x-b) + \pi(c;a,b)\delta(x-c)\right] \bar{f}(a,t) \tag{9}$$

or

$$\bar{f}(x) = \frac{1}{3}\bar{f}(x)f(x) + 2\int_0^1 db \int_0^1 dc\, \pi(x;b,c)\bar{f}(b) \tag{10}$$

In Fig. 4 we have plotted both distributions $f(x)$ and $\bar{f}(x)$[1] when players adopt the set of strategies corresponding to the equilibrium point given by Eq. (3), i.e., the *strongest opponent strategy*. Notice that, despite the paradoxical result mentioned before, the distribution of winners still has it maximum at $x = 1$, indicating that the best marksmanship players are nevertheless the ones who win in more occasions.

In Fig. 5 we plot the distribution of winners $f(x)$ and $\bar{f}(x)$ in a competition where players play the sequential truel. As before, the solid line corresponds to the former truel competition and the discontinuous line corresponds to the competition where the winner of the truel goes on playing. Notice that now the distribution of winners $f(x)$ has a maximum at $x \approx 0.57$. This result reflects the counter–intuitive result obtained earlier,

[1] In this case, the integral relation Eq. (10) has been solved numerically.

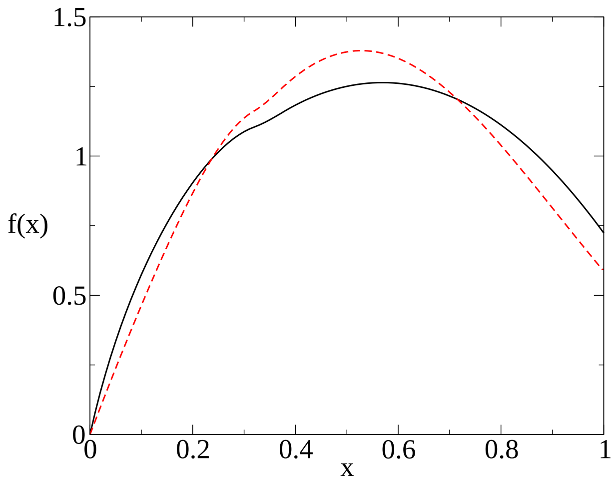


Figure 5: Same as Fig.4 in the case that players play sequentially in increasing order of their marksmanship. Notice that now both distributions of winners present maxima for $x < 1$ indicating that the best a priori players do not win the game in the majority of the cases.

and is that players who perform better on average are *not* those with higher markmanship, instead, are the ones with *intermediate* values.

Finally, we plot in Fig. 6 both distributions of winning opinions, $f(x)$ and $\bar{f}(x)$. As in the case of the random truel, we can observe how the player most favored on average is the one with the highest markmanship available.

6 Generalization to N players : N–uels

We have shown for three players the existence of an interesting and *a priori* counter–intuitive result where the player with the highest markmanship does not win the truel in all cases. But, what happens if there are more than three players? For a general case of N players, it is rather difficult to obtain exact analytic expressions. Already for a low number of individuals the expressions obtained increase very rapidly in complexity. However, by means of computer simulations we are able to obtain the distribution of winners for a number of players $N > 3$.

In the upper panel of Fig. 7 we show a histogram corresponding to the ranking obtained when the random truel is played by 4 players. The fourth classified would correspond to the distribution of players eliminated from the game in first place, the third classified would be the one eliminated in second place and so on. The distribution of the fourth classified shows that individuals eliminated firstly in the game are those with higher markmanships. Indeed, the maximum is located at $x = 1$, indicating then that the better

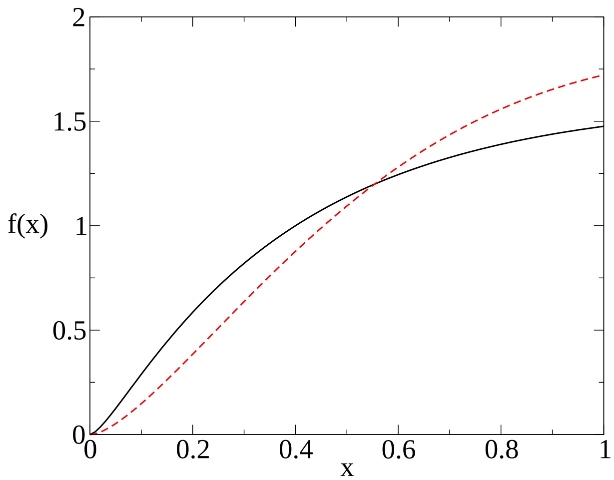


Figure 6: Same as Fig.1 for the convincing opinion model.

you are the higher the probability of being eliminated. Another aspect we can extract from this figure deals with the distribution of the first and second classified: these curves correspond to the case where there are only two players left in the game, i.e., to a duel. Therefore, as we have seen previously, it is more likely in this situation that players with lower markmanships are eliminated firstly rather than those with higher markmanships (that is the reason why the curve for the second classified presents a maximum in the origin). It is worth mentioning that already for 4 players the histogram associated to the first classified – i.e., the winner of the 4–uel – presents a maximum for a value of $x < 1$. This result implies that the best performing player does not correspond anymore to the player with the highest markmanship, as it happened for $N = 3$. Indeed, the optimum value is located in between $(0.35, 0.45)$.

Our next step would be a survey for different values of N. In the lower panel of Fig. 7 we see how the histogram of the winners of a N–uel evolves when varying N. It can be clearly seen that for values of $N \geq 4$ the optimum/maximum value of the distribution is indeed progressively enhanced and shifted towards zero when N is increased.

7 Summary

We have presented a detailed analysis of the random truel, the sequential truel and an opinion model using the methods of Markov chain theory. We are able to reproduce in a language which is more familiar to the Physics community most of the results of the alternative analysis by Kilgour [7]. Besides computing the optimal rational strategy, we have focused on computing the distribution of winners in a truel competition. We have

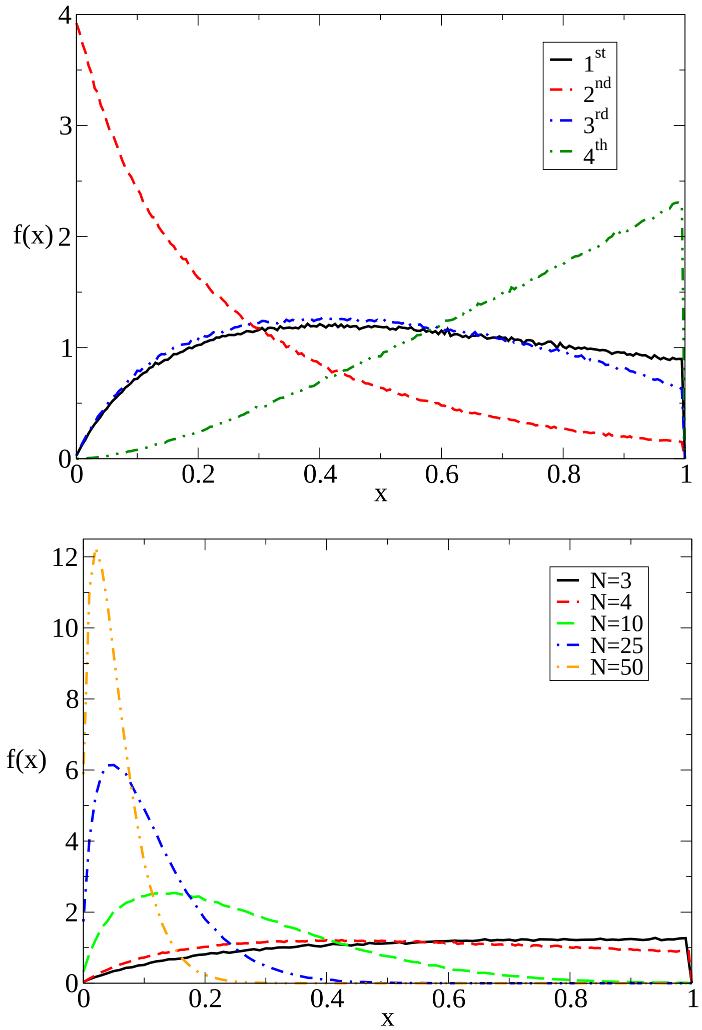


Figure 7: Upper panel: Histogram of the classified corresponding to the random truel for $N = 4$ players. Lower panel: Different histograms of the first classifieds when playing the random N–uel corresponding to different values of $N = 3$, 4, 10, 25 and 50.

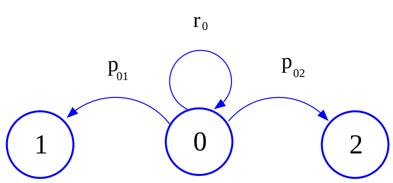
shown that in the random case (as well as in the opinion model), the distribution of winners still has its maximum at the highest possible marksmanship, $x = 1$, despite the fact that sometimes players with a lower marksmanship have a higher probability of winning the game. In the sequential firing case, the paradox is more present as the distribution of winners has a maximum at $x < 1$. Finally we have presented some numerical results concerning the distribution of winners when $N > 3$ and shwon that already for $N = 4$ the distribution presents a maximum value located at $x < 1$. Furthermore, as N increases this optimal value tends to zero.

We acknowledge financial support by the Ministerio de Educación y Ciencia (Spain) and FEDER projects FIS2004-5073, FIS2004-953. P.A. is supported by grant from the government of the Balearic Islands.

8 Appendix

8.1 Duels

In Fig. 8 we show a Markov chain with three states $0, 1, 2$ corresponding to the random duel and also the opinion model. The Table in Fig. 8 shows the correspondence between the players remaining on the game and their corresponding state for both the random duel and the opinion model.



	Random Duel	*Opinion Duel*
States	*Remaining players*	*Remaining opinions*
0	A B	A B
1	A	A A
2	B	B B

Figure 8: Table: description of the different states for the random duel and opinion model. Diagram: Markov chain corresponding to both the random duel and opinion model with two opinions.

From Markov chain theory[8] we can calculate the probability u_i^j that starting from state i we eventually end up in state j after a sufficiently large number of steps. We are interested in calculating the probability that starting from state 0 we end up either in state 1 or state 2. The set of equations to be solved are

$$u_0^1 = p_{01}u_1^1 + p_{00}u_1^1 \quad (11)$$
$$u_0^2 = p_{02}u_2^2 + p_{00}u_0^2 \quad (12)$$
$$(13)$$

where the transition probabilities p_{ij} between states are given by :

$$p_{00} = \frac{1}{2}[2 - a - b]\,, \;\; p_{01} = \frac{1}{2}a\,, \;\; p_{02} = \frac{1}{2}b \quad (14)$$

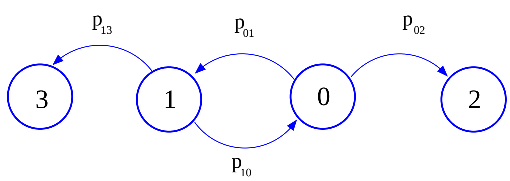
Recalling that by definition $u_j^j = 1$ we may solve Eqs. (11,11) obtaining

$$u_0^1 = \frac{p_{01}}{1 - p_{00}}\,, \qquad u_0^2 = \frac{p02}{1 - p_{00}}\,, \tag{15}$$

Substituting the transition probabilities in the previous set of equations we obtain the survival probabilities for player A (u_0^1) and player B (u_0^2)

$$\pi_A = \frac{a}{a+b}\,, \qquad \pi_B = \frac{b}{a+b}\,, \tag{16}$$

We now consider the Markov chain describing the sequential duel. It is composed of four states $0,1,2,3$ and is depicted in Fig. 9. The table from Fig. 9 shows the relation between the states and the players that are still on the game.



States	*Remaining players*
0	**A** B
1	A **B**
2	A
3	B

Figure 9: Table: description of the different states for the sequential duel where it is highlighted which player is shooting. Diagram: Markov chain with four states corresponding to the sequential duel.

The set of equations to be solved are

$$\begin{aligned}
u_0^2 &= p_{02}u_2^2 + p_{01}u_1^2 && (17)\\
u_0^3 &= p_{01}u_1^3 && (18)\\
u_1^2 &= p_{10}u_0^2 && (19)\\
u_1^3 &= p_{13}u_3^3 + p_{10}u_0^3 && (20)\\
& && (21)
\end{aligned}$$

where

$$p_{01} = 1 - a\,, \;\; p_{02} = a\,, \;\; p_{10} = 1 - b\,, \;\; p_{13} = b \tag{22}$$

The general solutions for Eqs. (17),(18),(19),(20) are

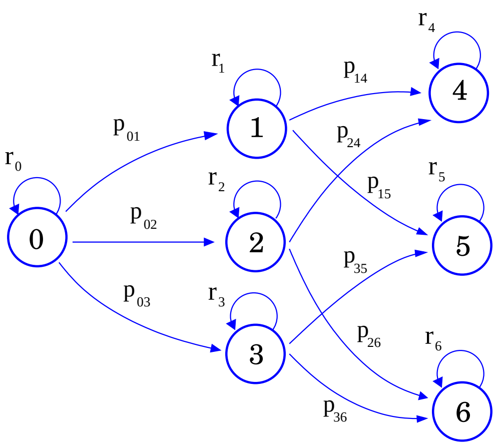
$$u_0^2 = \frac{p_{02}}{1 - p_{01}p_{10}}\,, \qquad u_0^3 = \frac{p01p_{13}}{1 - p_{01}p_{10}}\,, \tag{23}$$

which, after substituting the transition probabilities give as a result

$$\pi_A = u_0^2 = \frac{a}{1-(1-a)(1-b)}\,, \qquad \pi_B = u_0^3 = \frac{b(1-a)}{1-(1-a)(1-b)}\,, \tag{24}$$

8.2 Random firing truel

In this case we may distinguish seven possible states according to the remaining players labeled as $0, 1, \ldots, 6$. The allowed transitions between those states are shown in the diagram in Fig. 10, where p_{ij} denotes the transition probability from state i to state j (the self–transition probability p_{ii} is denoted by r_i).



States	*Remaining players*
0	ABC
1	AB
2	AC
3	BC
4	A
5	B
6	C

Figure 10: Table with the description of all the possible states for the random firing game, and diagram representing the allowed transitions between the states shown in the table.

We are particularly interested in the calculation of u_0^4, u_0^5 and u_0^6, corresponding to the winning of the game by player A, B and C respectively. The relevant set of equations is

$$\begin{aligned}
u_0^4 &= p_{01}\ u_1^4 + p_{02}\ u_2^4 + p_{03}\ u_3^4 + p_{00}\ u_0^4, & u_0^5 &= p_{01}\ u_1^5 + p_{02}\ u_2^5 + p_{03}\ u_3^5 + p_{00}\ u_0^5,\\
u_1^4 &= p_{14}\ u_4^4 + r_1\ u_1^4, & u_1^5 &= p_{15}\ u_5^5 + r_1\ u_1^5,\\
u_2^4 &= p_{24}\ u_4^4 + r_2\ u_2^4, & u_2^5 &= r_2\ u_2^5,\\
u_3^4 &= r_3\ u_3^4, & u_3^5 &= r_3\ u_3^5 + p_{35}\ u_5^5.
\end{aligned}$$

We can then solve the previous set of equations for u_0^4 and u_0^5, and then obtain u_0^6 through the relation $u_0^4 + u_0^5 + u_0^6 = 1$, obtaining

$$
\begin{aligned}
u_0^4 &= \frac{p_{01}\ p_{14}}{(1-p_{00})(1-r_1)} + \frac{p_{02}\ p_{24}}{(1-p_{00})(1-r_2)},\\
u_0^5 &= \frac{p_{01}\ p_{15}}{(1-p_{00})(1-r_1)} + \frac{p_{03}\ p_{35}}{(1-p_{00})(1-r_3)},\\
u_0^6 &= \frac{p_{02}\ p_{26}}{(1-p_{00})(1-r_2)} + \frac{p_{03}\ p_{36}}{(1-p_{00})(1-r_3)}.
\end{aligned}
\tag{25}
$$

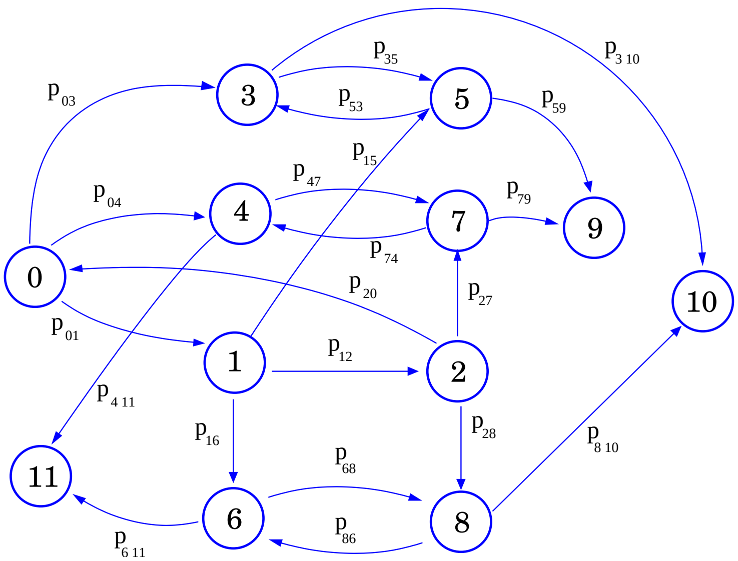
Where the transition probabilities p_{ij} are given by

$$
\begin{array}{ll}
r_0 = 1 - \frac{1}{3}(a(1-P_{A0}) + b(1-P_{B0}) + c(1-P_{C0})), & p_{01} = \frac{1}{3}(aP_{AC} + bP_{BC}),\\
p_{02} = \frac{1}{3}(aP_{AB} + cP_{CB}), & p_{03} = \frac{1}{3}(bP_{BA} + cP_{CA}),\\
p_{14} = p_{24} = \frac{1}{2}a, & p_{15} = p_{35} = \frac{1}{2}b,\\
p_{26} = p_{36} = \frac{1}{2}c, & r_1 = 1 - \frac{1}{2}(a+b),\\
r_2 = 1 - \frac{1}{2}(a+c), & r_3 = 1 - \frac{1}{2}(b+c).
\end{array}
\tag{26}
$$

8.3 Sequential firing

As in the random firing case, we describe this game as a Markov chain composed of 11 different states, also with three absorbent states: 9 , 10 and 11. In Fig. 11 we can see the corresponding diagram for this game, together with a table describing all possible states. Based on this diagram, we can write down the relevant set of equations for the transition probabilities u_i^j:

$$
\begin{array}{lll}
u_0^9 = p_{03}u_3^9 + p_{01}u_1^9 + p_{04}u_4^9, & u_0^{10} = p_{03}u_3^{10} + p_{01}u_1^{10}, & u_0^{11} = p_{01}u_1^{11} + p_{04}u_4^{11},\\
u_1^{10} = p_{12}u_2^{10} + p_{15}u_5^{10} + p_{16}u_6^{10}, & u_1^9 = p_{12}u_2^9 + p_{15}u_5^9, & u_1^{11} = p_{12}u_2^{11} + p_{16}u_6^{11},\\
u_2^{11} = p_{28}u_8^{11} + p_{27}u_7^{11} + p_{20}u_0^{11}, & u_2^9 = p_{27}u_7^9 + p_{20}u_0^9, & u_2^{10} = p_{28}u_8^{10} + p_{20}u_0^{10},\\
u_3^9 = p_{35}u_5^9, & u_3^{10} = p_{35}u_5^{10} + p_{3\,10}, & \\
u_4^9 = p_{47}u_7^9, & u_4^{11} = p_{47}u_7^{11} + p_{4\,11}, & \\
u_5^9 = p_{53}u_3^9 + p_{59}, & u_5^{10} = p_{53}u_3^{10}, & \\
u_6^{10} = p_{68}u_8^{10}, & u_6^{11} = p_{68}u_8^{11} + p_{6\,11}, & \\
u_7^9 = p_{74}u_4^9 + p_{79}, & u_7^{11} = p_{74}u_4^{11}, & \\
u_8^{10} = p_{86}u_6^{10} + p_{8\,10}, & u_8^{11} = p_{86}u_6^{11}. &
\end{array}
\tag{27}
$$



States	*Remaining players*
0	A B **C**
1	A **B** C
2	**A** B C
3	**B** C
4	**A** C
5	B **C**
6	**A** B
7	A **C**
8	A **B**
9	C
10	B
11	A

Figure 11: Table: Description of the different states of the game for the case of sequential firing. The highlighted player is the one chosen for shooting in that state. Diagram: scheme representing all the allowed transitions between the states shown in the table for the case of a truel with sequential firing in the order C$\longrightarrow$ B $\longrightarrow$ A with $a > b > c$.

The general solutions for the probabilities u_0^9, u_0^{10} and u_0^{11} are given by

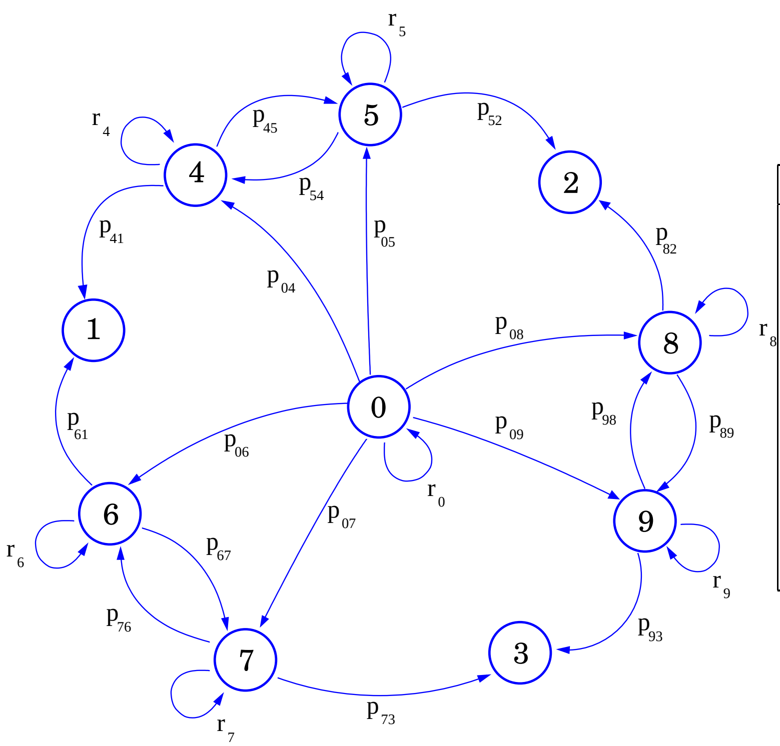
$$\begin{aligned}
u_0^9 &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{59}(p_{03}p_{35}+p_{01}p_{15})}{1-p_{35}p_{53}}+\frac{p_{79}(p_{04}p_{47}+p_{01}p_{12}p_{27})}{1-p_{47}p_{74}}\right],\\
u_0^{10} &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{3\,10}(p_{03}+p_{01}p_{15}p_{53})}{1-p_{35}p_{53}}+\frac{p_{01}p_{8\,10}(p_{16}p_{68}+p_{12}p_{28})}{1-p_{68}p_{86}}\right],\\
u_0^{11} &= \frac{1}{1-p_{01}p_{12}p_{20}}\left[\frac{p_{4\,11}(p_{04}+p_{01}p_{12}p_{27}p_{74})}{1-p_{47}p_{74}}+\frac{p_{01}p_{6\,11}(p_{16}+p_{12}p_{28}p_{86})}{1-p_{68}p_{86}}\right],
\end{aligned} \tag{28}$$

with transition probabilities given by

$$\begin{array}{lll}
p_{01}=(1-c)+cP_{C0}, & p_{03}=cP_{CA}, & p_{04}=cP_{CB},\\
p_{12}=(1-b)+bP_{B0}, & p_{15}=bP_{BA}, & p_{16}=bP_{CA},\\
p_{20}=(1-a)+aP_{A0}, & p_{27}=aP_{AB}, & p_{28}=aP_{AC},\\
p_{35}=p_{86}=1-b, & p_{3\,10}=p_{8\,10}=b, & \\
p_{47}=p_{68}=1-a, & p_{4\,11}=p_{6\,11}=a, & \\
p_{53}=p_{74}=1-c, & p_{59}=p_{79}=c. &
\end{array}$$

8.4 Convincing opinion

For this model we show in Fig. 12 the diagram of all the allowed states and transitions, together with a table describing the possible states.



States	*Opinions*
0	A B C
1	C C C
2	B B B
3	A A A
4	B C C
5	B B C
6	A C C
7	A A C
8	A B B
9	A A B

Figure 12: Table: description of the different states of the opinion model. Diagram: scheme representing the allowed transitions between the states.

The corresponding set of equations describing this convincing opinion model, as derived from the diagram, are

$$
\begin{aligned}
&u_0^1 = p_{00}u_0^1 + p_{06}u_6^1 + p_{04}u_4^1 + p_{05}u_5^1 + p_{07}u_7^1, && \\
&u_0^2 = p_{00}u_0^2 + p_{04}u_4^2 + p_{05}u_5^2 + p_{08}u_8^2 + p_{09}u_9^2, && \\
&u_0^3 = p_{00}u_0^3 + p_{08}u_8^3 + p_{09}u_9^3 + p_{07}u_7^3 + p_{06}u_6^3, && \\
&u_4^1 = r_4u_4^1 + p_{45}u_5^1 + p_{41}, && u_4^2 = r_4u_4^2 + p_{45}u_5^2, \\
&u_5^1 = r_5u_5^1 + p_{54}u_4^1, && u_5^2 = r_5u_5^2 + p_{54}u_4^2 + p_{52}, \\
&u_6^1 = r_6u_6^1 + p_{67}u_7^1 + p_{61}, && u_6^3 = r_6u_6^3 + p_{67}u_7^3, \\
&u_7^1 = r_7u_7^1 + p_{76}u_6^1, && u_7^3 = r_7u_7^3 + p_{76}u_6^3 + p_{73}, \\
&u_8^2 = r_8u_8^2 + p_{89}u_9^2 + p_{82}, && u_8^3 = r_8u_8^3 + p_{89}u_9^3, \\
&u_9^2 = r_9u_9^2 + p_{98}u_8^2, && u_9^3 = r_9u_9^3 + p_{98}u_8^3 + p_{93}.
\end{aligned}
\tag{29}
$$

And the general solution for the probabilities u_0^1, u_0^2 and u_0^3 is

$$\begin{aligned}
u_0^1 &= \frac{1}{1-p_{00}}\left[\frac{p_{61}(p_{06}(1-r_7)+p_{07}p_{76})}{(1-r_6)(1-r_7)-p_{67}p_{76}} + \frac{p_{41}(p_{04}(1-r_5)+p_{05}p_{54})}{(1-r_4)(1-r_5)-p_{45}p_{54}}\right], \\
u_0^2 &= \frac{1}{1-p_{00}}\left[\frac{p_{52}(p_{04}p_{45}+p_{05}(1-r_4))}{(1-r_4)(1-r_5)-p_{45}p_{54}} + \frac{p_{82}(p_{08}(1-r_9)+p_{09}p_{98})}{(1-r_8)(1-r_9)-p_{89}p_{98}}\right], \\
u_0^3 &= \frac{1}{1-p_{00}}\left[\frac{p_{73}(p_{06}p_{67}+p_{07}(1-r_6))}{(1-r_6)(1-r_7)-p_{67}p_{76}} + \frac{p_{93}(p_{09}(1-r_8)+p_{08}p_{89})}{(1-r_8)(1-r_9)-p_{89}p_{98}}\right],
\end{aligned} \tag{30}$$

where the transition probabilities are given by

$$\begin{array}{lll}
p_{04} = \frac{1}{3}cP_{CA}, & p_{06} = \frac{1}{3}cP_{CB}, & p_{08} = \frac{1}{3}bP_{BC}, \\
p_{05} = \frac{1}{3}bP_{BA}, & p_{07} = \frac{1}{3}aP_{AB}, & p_{09} = \frac{1}{3}aP_{AC}, \\
p_{41} = p_{61} = \frac{2}{3}c & p_{45} = p_{98} = \frac{1}{3}b, & p_{54} = p_{76} = \frac{1}{3}c, \\
p_{52} = p_{82} = \frac{2}{3}b, & p_{67} = p_{89} = \frac{1}{3}a, & p_{73} = p_{93} = \frac{2}{3}a, \\
p_{00} = \frac{1}{3}[3-a-b-c], & r_4 = \frac{2}{3}(1-c)+\frac{1}{3}(1-b), & r_5 = \frac{1}{3}(1-c)+\frac{2}{3}(1-b), \\
r_6 = \frac{2}{3}(1-c)+\frac{1}{3}(1-a), & r_7 = \frac{1}{3}(1-c)+\frac{2}{3}(1-a), & r_8 = \frac{2}{3}(1-b)+\frac{1}{3}(1-a), \\
r_9 = \frac{1}{3}(1-b)+\frac{2}{3}(1-a). & &
\end{array} \tag{31}$$

References

[1] D.M. Kilgour and S.J. Brams. The truel. *Mathematics Magazine*, 70:315–326, 1997.

[2] C. Kinnaird. *Encyclopedia of Puzzles and Pastimes*. Citadel, Secaucus, NJ (USA), 1946.

[3] M. Shubik. *Game Theory in the Social Sciences*. MIT Press, Cambridge, MA (USA), 1982.

[4] B. Sinervo and C.M.M. Lively. The rock–scissors–paper game and the evolution of alternative male strategies. *Nature*, 340:240–243, 1996.

[5] M. Frean and E.R. Abraham. Rock–scissors–paper and the survival of the weakest. *Proc. R. Soc. London B*, 268:1323–1327, 2001.

[6] D.M. Kilgour. The simultaneous truel. *Int. Journal of Game Theory*, 1:229–242, 1972.

[7] D.M. Kilgour. The sequential truel. *Int. Journal of Game Theory*, 4:151–174, 1975.

[8] D.M. Kilgour. Equilibrium points of infinite sequential truels. *Int. Journal of Game Theory*, 6:167–180, 1977.

[9] W. Bossert, S.J. Brams, and D.M. Kilgour. Cooperative vs. non–cooperative truels: Little agreement, but does that matter? *Games and Economic Behavior*, 40:185–202, 2002.