Generation of non-uniform random numbers

General method Function $ran\_f()$

Theorem.- Let $\hat{\mathbf{x}}$ be a r.v. with probability distribution function $F_{\hat{\mathbf{x}}}(x)$

$\hat{\mathbf{u}} = F_{\hat{\mathbf{x}}}(\hat{\mathbf{x}})$ is a r.v. uniformly distributed in the interval $(0, 1)$ ($\hat{U}(0, 1)$ variable).

If $\hat{\mathbf{u}}$ is a $\hat{U}(0, 1)$ r.v., then the r.v. $\hat{\mathbf{x}} = F^{-1}_{\hat{\mathbf{x}}}(\hat{\mathbf{u}})$ has $F_{\hat{\mathbf{x}}}(x)$ as probability distribution function

Exponential distribution, i.e.

$$f_{\hat{\mathbf{x}}}(x) = \begin{cases} a \exp(-ax) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The distribution function is:

$$F_{\hat{\mathbf{x}}}(x) = \int_{-\infty} x f_{\hat{\mathbf{x}}}(x)\, dx = 1 - \exp(-ax)$$

$$x = F^{-1}_{\hat{\mathbf{x}}}(u) = \frac{-1}{a} \log(1 - u) \equiv \frac{-1}{a} \log(u)$$

```
function ran_e(a)

ran_e=-log(ran_u())/a

return

end
```

Cauchy distribution:

$$f_{\hat{\mathbf{x}}}(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

$$F_{\hat{\mathbf{x}}}(x) = \int_{-\infty}^{x} f_{\hat{\mathbf{x}}}(x)\,dx = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+x^2} = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$$

$$x = \tan(\pi(u - \frac{1}{2}))$$

Example:

$$f_{\hat{\mathbf{r}}}(r) = re^{-\frac{1}{2}r^2}, \quad 0 \leq r \leq \infty$$

$$F_{\hat{\mathbf{r}}}(r) = \int_{-\infty}^{r} f_{\hat{\mathbf{r}}}(r)\,dr = 1 - e^{-\frac{1}{2}r^2}$$

$$r = \sqrt{-2\log(1-u)} \equiv \sqrt{-2\log(u)}$$

Bernouilli distribution. The distribution probability function is:

$$F_{\hat{\mathbf{x}}}(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

The inverse function takes only two values:

$$F_{\hat{\mathbf{x}}}^{-1}(u) = \begin{cases} 0 & u < 1-p \\ 1 & u \geq 1-p \end{cases}$$

```
function ran_b(p)
if (ran_u().lt.1-p) then
  ran_b=0
else
   ran_b=1
endif
end
```

An equivalent program is:

```
function ran_b(p)
if (ran_u().lt.p) then
  ran_b=1
else
   ran_b=0
endif
end
```

Generation of a vector $(x_1, x_2, ..., x_n)$ of random variables with a given joint probability density function $f_{\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n}(x_1, ..., x_n)$

If $(u_1, ..., u_n)$ is a vector of n $\hat{U}(0,1)$ independent random variables, $(x_1, ..., x_n)$ is the solution of:

$$F_{x_1}(x_1) = u_1$$

$$F_{x_2}(x_2|x_1) = u_2$$

$$F_{x_n}(x_n|x_1, ..., x_{n-1}) = u_n$$

There are $n!$ ways of ordering the n variables $x_1, ..., x_n$

Not very useful in practice

Numerical Inversion

Divide $[0,1]$ in M subintervals $[\frac{i}{M}, \frac{i+1}{M}]_{i=0,1,...,M-1}$

Compute and store $x_i = F^{-1}(i/M)$ for $i = 0, ..., M$.

Substitute $F_{\hat{\mathbf{x}}}(x)$ by its piecewise linear approximation between points $[\frac{i}{M}, \frac{i+1}{M}]$.

Generate u according to $\hat{U}(0,1)$

$x = F^{-1}_{\hat{\mathbf{x}}}(u)$ is approximated by:

$$x = (Mu - i)x_{i+1} + (i + 1 - Mu)x_i$$

i is such that $u \in [\frac{i}{M}, \frac{i+1}{M})$, or $i = [Mu]$, (integer part of Mu).

Change of variables $y = y(x)$

$$f_{\hat{\mathbf{y}}}(y) = \frac{f_{\hat{\mathbf{x}}}(x)}{\mid \frac{dy}{dx} \mid}$$

Linear transformation $y = ax + b$.

$$f_{\hat{\mathbf{y}}}(y) = \frac{1}{\mid a \mid} f_{\hat{\mathbf{x}}}(\frac{y-b}{a})$$

If $\hat{\mathbf{x}}$ is $\hat{U}(0,1)$, then $\hat{\mathbf{y}}$ is $\hat{U}(b, a+b)$ for $a > 0$ or $\hat{U}(a+b, b)$

for $a < 0$

If $\hat{\mathbf{x}}$ is a Gaussian $\hat{G}(0,1)$ r.v. then y is a $\hat{G}(b,a)$ r.v.

The change $y = x^a$ leads to:

$$f_{\hat{\mathbf{y}}}(y) = \frac{1}{\mid a \mid} y^{\frac{1}{a}-1}$$

valid for $0 < y < 1$ if $a > 0$ or for $1 < y < \infty$ if $a < 0$.

$$f_{\hat{\mathbf{x}}}(x) = (1+\alpha)x^{\alpha}$$

$\hat{\mathbf{x}} = \hat{\mathbf{u}}^a$ with $a = 1/(\alpha+1)$, being $\hat{\mathbf{u}}$ a $\hat{U}(0,1)$ random variable.

Gaussian distribucion

Gaussian random variable of mean 0 and variance 1, $\hat{\mathbf{z}}$

$\hat{\mathbf{x}} = \sigma\hat{\mathbf{z}} + \mu$

$$f_{\hat{\mathbf{z}}}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$F_{\hat{\mathbf{z}}}(z) = \int_{-\infty}^{z} f_{\hat{\mathbf{z}}}(z)\, dz = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = \frac{1}{2}(1+\mathrm{erf}(\frac{z}{\sqrt{2}})) = u$$

$$z = \sqrt{2}\mathrm{erf}^{-1}(2u-1)$$

Implemented in NAG

```
function ran_g()

ran_g = g05cef(ran_u(),ifail)

return

end
```

Aproximations to inverse error function

$$z \approx t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3}$$

$$t = \sqrt{-2\log(1-u)}$$

c_0=2.515517, c_1=0.802853, c_2=0.010328, $d1$=1.432788,

d_2=0.189269 y d_3= 0.001308

error less than 4.5×10^{-4} if $0.5 \leq u \leq 1.0$.

```
function ran_g()
data c0,c1,c2/2.515517,0.802853,0.010328/
data d1,d2,d3/1.432788,0.189269,0.001308/
u=ran_u()
if (u.gt.0.5) then
t=sqrt(-2.0*log(1.-u))
ran_g=t-(c0+t*(c1+c2*t))/(1.0+t*(d1+t*(d2+t*d3)))
else
t=sqrt(-2.0*log(u))
ran_g=-t+(c0+t*(c1+c2*t))/(1.0+t*(d1+t*(d2+t*d3)))
endif
return
end
```

Box-Muller-Wiener algorithm (BMW)

Two Gaussian independent random variables $\hat{\mathbf{x}}_1,\ \hat{\mathbf{x}}_2$ (mean zero and variance one).

$$f_{\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2}(x_1,x_2) = f_{\hat{\mathbf{x}}_1}(x_1) f_{\hat{\mathbf{x}}_2}(x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$$

Polar coordinates:

$$\hat{\mathbf{x}}_1 = \hat{\mathbf{r}}\cos(\hat{\theta})$$

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{r}}\sin(\hat{\theta}) \tag{1}$$

Joint pdf of $\hat{\mathbf{r}}$ and $\hat{\theta}$ is :

$$f_{\hat{\mathbf{r}},\hat{\theta}}(r,\theta) = J\left(\frac{x_1, x_2}{r, \theta}\right) f_{\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2}(x_1, x_2)$$

$$f_{\hat{\mathbf{r}},\hat{\theta}}(r,\theta) = \frac{1}{2\pi} r e^{-\frac{r^2}{2}}$$

$$f_{\hat{\mathbf{r}},\hat{\theta}}(r,\theta) = f_{\hat{\mathbf{r}}}(r) f_{\hat{\theta}}(\theta)$$

with

$$f_{\hat{\mathbf{r}}}(r) = r e^{-\frac{r^2}{2}}$$

and

$$f_{\hat{\theta}}(\theta) = \frac{1}{2\pi}$$

$\hat{\mathbf{r}}$ y $\hat{\theta}$ independent random variables.

$\hat{\theta}$ is a random variable uniformly distributed in the interval $[0, 2\pi]$

$$\hat{\theta} = 2\pi\hat{\mathbf{v}}$$

$\hat{\mathbf{u}}$ is a $\hat{U}(0,1)$ variable. We saw that:

$$\hat{\mathbf{r}} = \sqrt{-2\log(\hat{\mathbf{u}})}$$

Hence we arrive at:

$$\hat{\mathbf{x}}_1 = \sqrt{-2\log(\hat{\mathbf{u}})}\cos(2\pi\hat{\mathbf{v}})$$

$$\hat{\mathbf{x}}_2 = \sqrt{-2\log(\hat{\mathbf{u}})}\sin(2\pi\hat{\mathbf{v}})$$

Box-Muller-Wiener algorithm.

Exact, produces 2 independent Gaussian random variables starting from 2 independent uniform random variables

Slow

Rejection methods improve somewhat (20%), but not in vector computers.

Use numerical inversion.

Central limit theorem:

$$\hat{\mathbf{z}} = \sqrt{\frac{12}{N}}\left(\sum_{k=1}^{N}\hat{\mathbf{u}}_k - \frac{N}{2}\right)$$

where $\hat{\mathbf{u}}_k$ are $\hat{U}(0,1)$ variables, tends to a Gaussian in the limit

$N \to \infty$.

The algorithm is used typically with $N = 12$

Joint Gaussian variables.

Generate d-dimensional real Gaussian field $h(\vec{r})$

$$\begin{aligned} \langle h(\vec{r}) \rangle &= 0 \\ \langle h(\vec{r}) h(\vec{r}\,') \rangle &= C(\vec{r}, \vec{r}\,') = C(\vec{r} - \vec{r}\,') \end{aligned}$$

Numerically

$$C^{(s)}(\vec{r}) = \frac{1}{N} \sum_{\vec{r}\,'} h^{(s)}(\vec{r}\,') h^{(s)}(\vec{r} + \vec{r}\,')$$

Perform an ensemble average over M (preferably a large number) realizations:

$$C(\vec{r}) \equiv \langle C^{(s)}(\vec{r}) \rangle \equiv \frac{1}{M} \sum_{s=1}^{M} C^{(s)}(\vec{r})$$

Constructive sequential algorithm (Schmidt's ortonormalization process).

Generate a set $\{u_j\}$, $j = 1, \ldots, N$, of independent Gaussian variables of zero mean and variance one.

$$h_1 = \beta_{11} u_1$$

$\beta_{11} = (C_{1,1})^{1/2}$

$$h_2 = \beta_{21} u_1 + \beta_{22} u_2$$

$$\begin{aligned} \langle h_2^2 \rangle &= C_{2,2} \\ \langle h_1 h_2 \rangle &= C_{1,2} \end{aligned}$$

In general, we write

$$h_j = \sum_{i=1}^{j} \beta_{ji} u_i$$

$$\langle h_j h_i \rangle = C_{i,j}, \qquad i = 1, \ldots, j$$

Useful when the number N of variables is small.

For large N, use Fourier transform. Fourier Filtering Method.

$$S(\vec{k}) = L^{-d/2} \sum_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{r}\vec{k}} C(\vec{r})$$

satisfies exactly:

$$S(\vec{k}) = L^{-d} |\hat{h}^{(s)}(\vec{k})|^2$$

To generate realizations $h^{(s)}(\vec{r})$, $s = 1, \ldots, M$ of the field $h(\vec{r}\,)$, the FFM proceeds in the following way:

(i) Given $C(\vec{r}\,)$, compute $S(\vec{k}\,)$

(ii) Generate a set of independent Gaussian random variables

$u^{(s)}(\vec{r})$ of mean zero and variance one.

(iii) Compute the Fourier transform $\hat{u}^{(s)}(\vec{k})$ of $u^{(s)}(\vec{r})$.

(iv) Generate the Fourier transform of the field as:

$$\hat{h}^{(s)}(\vec{k}) = L^{d/2} S(\vec{k})^{1/2} \hat{u}^{(s)}(\vec{k})$$

(v) Compute the required field $h^{(s)}(\vec{r})$ as the inverse Fourier transform of $\hat{h}^{(s)}(\vec{k})$.

Step (i) needs to be done only once for each function $C(\vec{r}\,)$

Step (iii) can be avoided by generating directly the random field $u^{(s)}(\vec{k})$ in Fourier space

Respect the symmetries of the field $u^{(s)}(\vec{k})$, namely:

$u^{(s)}(-\vec{k}) = [u^{(s)}(\vec{k})]^*$.

Power-law correlations:

$$C(\vec{r}) \sim r^{-\gamma} \quad \text{for } r \to \infty$$

$S(k) \sim k^{\gamma - d}$ for $k \to 0$.

To be precise ($d = 1$)

$$C_i = \begin{cases} i^{-\gamma} & \text{if} \quad 0 < i \leq L/2 - 1 \\ (L-i)^{-\gamma} & \text{if} \quad L/2 \leq i \leq L-1 \\ 1 & \text{if} \quad i = 0, \end{cases}$$

$S_{min}(k) < 0$!!

minimal substraction procedure.

Substracting the minimum value $S_0(k) = S(k) - S_{min}$,

$S_0(k)$ is used instead of $S(k)$

$C(0)$ is no longer equal to 1 .

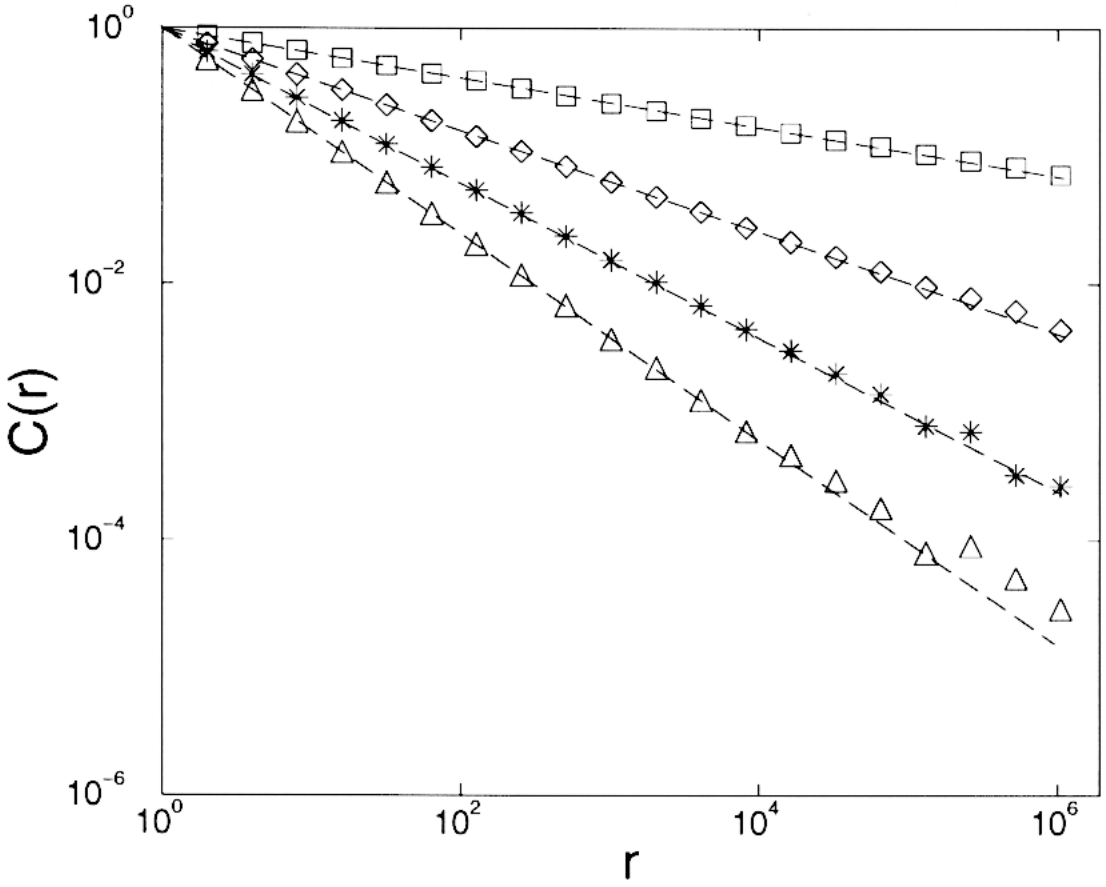
$C(0, L = 2^{21}) \sim 1.01$ $C(0, L = 2^6) \sim 1.17$

Another possibility:

$$C_j = (1 + j^2)^{-\gamma/2} \quad \text{if} \quad -L/2 + 1 \leq j \leq L/2$$

$$S(k) = \frac{2\pi^{1/2}}{\Gamma(\beta + 1)} \left(\frac{k}{2}\right)^{\beta} K_\beta(k)$$

$K_\beta(k)$ is the modified Bessel function of order $\beta = (\gamma - 1)/2$.



Variable composition

$\hat{\mathbf{x}}_1, \ldots \hat{\mathbf{x}}_n$ independent random variables with distribution function $F(x)$.

$$\hat{\mathbf{z}} = \max(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_n)$$

$$F_{\hat{\mathbf{z}}}(z) = F(z)^n$$

If $\hat{\mathbf{x}}_i$ is $\hat{U}(0,1)$, $F(z) = z$, then:

$$F_{\hat{\mathbf{z}}}(z) = z^n, \quad z \in [0,1]$$

$$f(z) = nz^{n-1}$$

In many cases:

$$f_{\hat{\mathbf{x}}}(x) = \sum_i \alpha_i f_i(x)$$

$$\sum_i \alpha_i = 1$$

We can interpret α_i as the probabilities of a discrete random variable $\hat{\mathbf{z}}$ taking integer values:

$$f_{\hat{\mathbf{z}}}(z) = \sum_i \alpha_i \delta(z - i)$$

(a) choose an integer value i according to the r.v. $\hat{\mathbf{z}}$

(b) choose a value of the r.v. defined by $f_i(x)$.

$$f_{\hat{\mathbf{x}}}(x) = \sum_i \text{Prob}(\hat{\mathbf{z}} = i) f(x|\hat{\mathbf{z}} = i) = \sum_i \alpha_i f_i(x)$$

Example

$$f_{\hat{\mathbf{x}}}(x) = \frac{5}{6}(1 + x^4), \quad 0 \leq x \leq 1$$

Can be written as:

$$f_{\hat{\mathbf{x}}}(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) = \frac{5}{6}1 + \frac{1}{6}(5x^4)$$

$f_1(x)$ is a $\hat{U}(0,1)$ r.v.

$f_2(x)$ can be sampled by $x = u^{1/5}$ (with u a $\hat{U}(0,1)$ variable).

```
if (ran_u().gt.1./6.) then
x=ran_u()
else
x=ran_u()**(1./5.)
endif
```

Equivalent algorithm:

```
x=ran_u()
if (ran_u().lt.1./6.) x=x**(1./5.)
```

Rejection Methods

One proposes a value for the random variable $\hat{\mathbf{x}}$ which is accepted (otherwise rejected) with a given probability.

Example:

$$f_{\hat{\mathbf{x}}}(x) = C \exp(-\frac{x^2}{2}) \quad -1 \leq x \leq 1$$

No need to know the normalization constant C

It is more probable to obtain values of $\hat{\mathbf{x}}$ for which $f_{\hat{\mathbf{x}}}(x)$ is larger.

1 - Propose x from a uniform distribution in (-1,1)

2 - Accept x with probability proportional to $f_{\hat{\mathbf{x}}}(x) = \exp(-\frac{x^2}{2})$

Accept with probability $\alpha f_{\hat{\mathbf{x}}}(x) \in (0, 1)$.

$\alpha^{-1} \geq \max f_{\hat{\mathbf{x}}}(x)$. $\alpha = 1/C$.

Step (2) is a Bernouilli process (either we accept or not) with probability $\alpha f_{\hat{\mathbf{x}}}(x) = \exp(-\frac{x^2}{2})$.

```
1     x=2*ran_u()-1
      if(ran_u().gt.exp(-0.5*x*x)) goto 1
```

General case:

1 - Propose a value for $\hat{\mathbf{x}}$ according to a pdf $g_{\hat{\mathbf{x}}}(x)$

2 - Accept the proposed value with probability $h(x)$,

$0 \leq h(x) \leq 1, \ \forall x.$

Example above: $g_{\hat{\mathbf{x}}}(x)$ is a uniforme distribution in $(-1, 1))$ and $h(x) = \exp(-\frac{x^2}{2}))$.

The second step can be interpreted as a Bernouilli variable,$\hat{\mathbf{y}}$, taking the value 1 with probability $h(x)$.

In the rejection process, we obtain values of the r.v. $\hat{\mathbf{x}}$ distributed according to $f_{\hat{\mathbf{x}}}(x|\hat{\mathbf{y}} = 1)$. Bayes theorem allows us

to write:

$$f_{\hat{\mathbf{x}}}(x|\hat{\mathbf{y}}=1) = \frac{f_{\hat{\mathbf{x}}\hat{\mathbf{y}}}(x,1)}{\int_{-\infty}^{\infty} f_{\hat{\mathbf{x}}\hat{\mathbf{y}}}(x,1)\,dx}$$

$$f_{\hat{\mathbf{x}}\hat{\mathbf{y}}}(x,1) = g_{\hat{\mathbf{x}}}(x)\text{prob}(\hat{\mathbf{y}}=1|x) = g_{\hat{\mathbf{x}}}(x)h(x)$$

$$f_{\hat{\mathbf{x}}}(x) = \frac{h(x)g_{\hat{\mathbf{x}}}(x)}{\int_{-\infty}^{\infty} h(x)g_{\hat{\mathbf{x}}}(x)\,dx}$$

Properly normalized.

Acceptance probability, ϵ, method efficiency

$$\epsilon = \int_{-\infty}^{\infty} P(\hat{\mathbf{y}}=1|x)g_{\hat{\mathbf{x}}}(x)\,dx = \int_{-\infty}^{\infty} h(x)g_{\hat{\mathbf{x}}}(x)\,dx$$

It is convenient to take $h(x)$ as big as possible but still keeping the condition $0 \leq h(x) \leq 1$.

Eample above, we can easily relate:

$$\epsilon = \int_{-1}^{1} \frac{1}{2}\exp(-\frac{x^2}{2})\,dx = \frac{1}{2C}$$

$C = 1/(2\epsilon)$.

$C = (\text{erf}(1/\sqrt{2})\sqrt{2\pi})^{-1} = 0.5844$, $\epsilon = 0.8556$.

Another way:

$$g_{\hat{\mathbf{x}}}(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) \quad x \in (-\infty, \infty)$$

$$h(x) = 1 \quad \text{if } x \in [-1,1]$$

$$h(x) = 0 \quad \text{otherwise} \tag{2}$$

Generate x according to a Gaussian distribution as usual and accept with probability 1 (i.e. accept) if the x value lies in the interval $[-1, 1]$.

```
1       x=ran_g()
        if(abs(x).gt.1.) goto 1
```

the efficiency of this method is:

$$\epsilon = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})\, dx = \text{erf}(1/\sqrt{2}) = 0.68269$$

Another example:

$$f_{\hat{\mathbf{x}}}(x) = C \exp(-\frac{x^2}{2} - x^4) \;\; x \in (-\infty, \infty)$$

We choose the functions:

$$g_{\hat{\mathbf{x}}}(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \;\; x \in (-\infty, \infty)$$

$$h(x) = \exp(-x^4)$$

satisfying $0 \leq h(x) \leq 1$.

```
1       x=ran_g()
        if(ran_u().gt.exp(-x**4) goto 1
```

Generate a point randomly and uniformly distributed in the disc of center $(0,0)$ and radius 1:

$$f_{\hat{\mathbf{x}}\hat{\mathbf{y}}}(x,y) = \begin{cases} C & \text{if } (x,y) \in D(0,1) \\ 0 & \text{otherwise} \end{cases}$$

($C = 1/\pi$, is irrelevant).

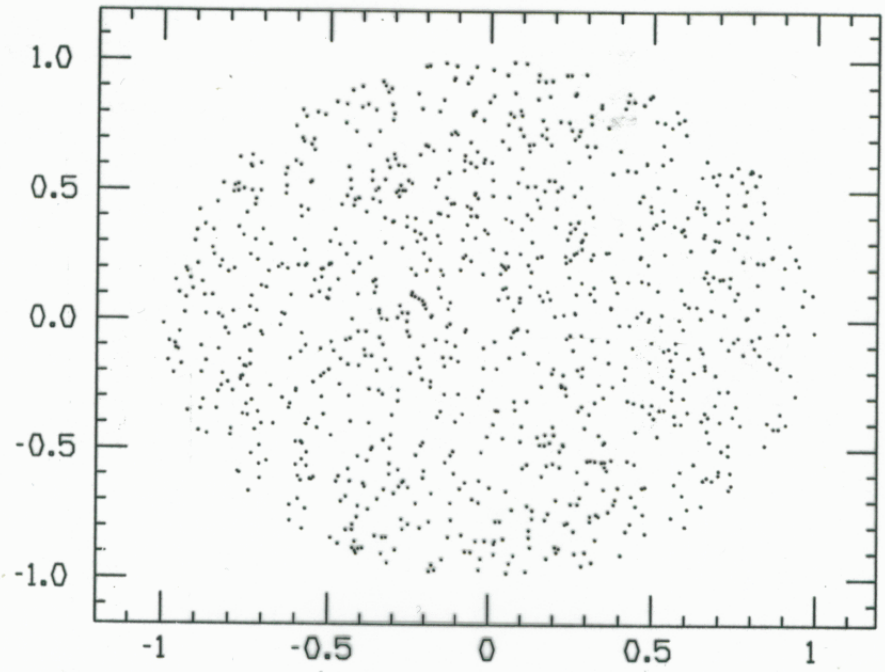
We take the following functions:

$$g(x,y) = 1/4 \quad \forall (x,y) \in [-1,1] \times [-1,1]$$

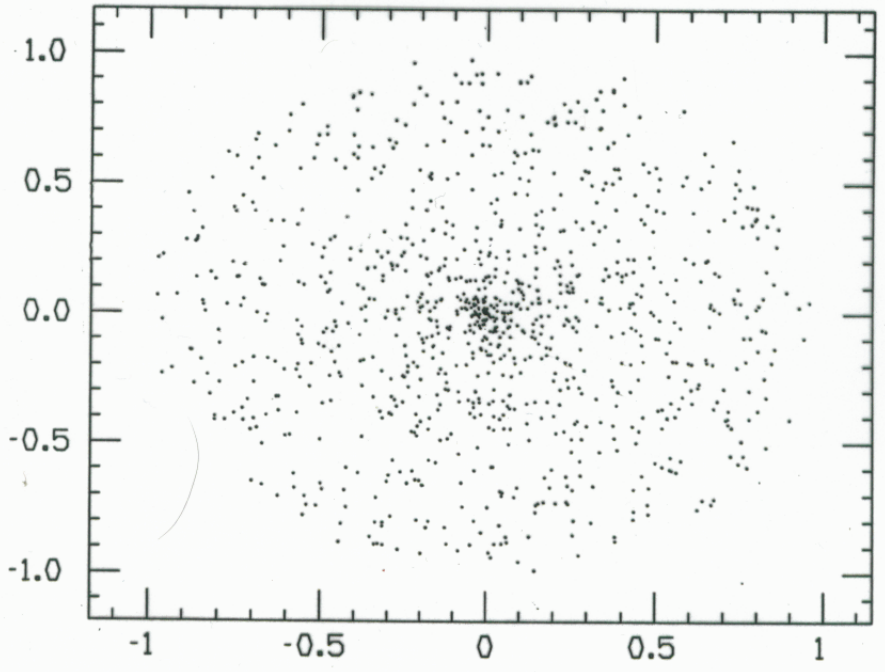
$$h(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D(0,1) \\ 0 & \text{otherwise} \end{cases}$$

```
1     x=2*ran_u()-1
      y=2*ran_u()-1
      r2=x*x+y*y
      if (r2.gt.1.) goto 1
      r=sqrt(r2)
      c=x/r
      s=y/r
```

1000 puntos método de rechazo



1000 puntos método polar



Generates sine and cosine of an angle uniformly distributed in the interval $[0, 2\pi]$ without computing the functions sin and cos

Alternate algorithm:

```
u=2*pi*ran_u()
c=cos(u)
s=sin(u)
```

which is slower (in serial computers) than the rejection method.

Variance of the rejection method:

If the proposed value is not accepted, we repeat the previous ones

Produces correlated values for the random variable

Useful in vector computers.

$\hat{\mathbf{x}}_n$ r.v. generated in the n-essim method repetition:

$$f_{\hat{\mathbf{x}}_n}(x) = f_{\hat{\mathbf{x}}_n}(x|accep)p(accep) + f_{\hat{\mathbf{x}}_n}(x|rejec)p(rejec)$$

$$f_{\hat{\mathbf{x}}_n}(x) = \text{Prob}(accep|x)g_{\hat{\mathbf{x}}}(x) + f_{\hat{\mathbf{x}}_n}(x|rejec)(1 - p(accep))$$

$$f_{\hat{\mathbf{x}}_n}(x) = h(x)g_{\hat{\mathbf{x}}}(x) + f_{\hat{\mathbf{x}}_{n-1}}(x)\left[1 - \int_{-\infty}^{\infty} h(x)g_{\hat{\mathbf{x}}}(x)\,dx\right]$$

Recurrence equation whose solution is:

$$f_{\hat{\mathbf{x}}_n}(x) = (1-\epsilon)^n \left[f_{\hat{\mathbf{x}}_0}(x) - \frac{h(x) g_{\hat{\mathbf{x}}}(x)}{\int_{-\infty}^{\infty} h(x) g_{\hat{\mathbf{x}}}(x)\, dx} \right] + \\ + \frac{h(x) g_{\hat{\mathbf{x}}}(x)}{\int_{-\infty}^{\infty} h(x) g_{\hat{\mathbf{x}}}(x)\, dx}$$

In terms of $f_{\hat{\mathbf{x}}}(x)$:

$$f_{\hat{\mathbf{x}}_n}(x) = (1-\epsilon)^n \left[f_{\hat{\mathbf{x}}_0}(x) - f_{\hat{\mathbf{x}}}(x) \right] + f_{\hat{\mathbf{x}}}(x)$$

$0 < \epsilon \leq 1$ the solution tends to the distribution $f_{\hat{\mathbf{x}}}(x)$ in the limit $n \to \infty$ independently of the initial distribution $f_{\hat{\mathbf{x}}_0}$:

$$\lim_{n \to \infty} f_{\hat{\mathbf{x}}_n}(x) = f_{\hat{\mathbf{x}}}(x)$$

If $f_{\hat{\mathbf{x}}_0}(x) = f_{\hat{\mathbf{x}}}(x)$ then $f_{\hat{\mathbf{x}}_n}(x) = f_{\hat{\mathbf{x}}}(x),\ \forall n$

If $f_{\hat{\mathbf{x}}_0}(x) \neq f_{\hat{\mathbf{x}}}(x)$, the factor $(1-\epsilon)^n$ makes sure that the initial condition will be lost after a sufficient number of steps.

To neglect the first M_0 steps (**thermalization**).

We can write the evolution equation as:

$$f_{\hat{\mathbf{x}}_n}(x) = \int_{-\infty}^{\infty} f(x|y) f_{\hat{\mathbf{x}}_{n-1}}(y)\, dy$$

$$f(x|y) = h(x) g_{\hat{\mathbf{x}}}(x) + \left[1 - \int_{-\infty}^{\infty} h(x) g_{\hat{\mathbf{x}}}(x)\, dx \right] \delta(x-y)$$

Generation of discrete distributions

Geometric distribution. x_i can take integer values $x_i = 0, 1, 2, ...$ with probability $p_i = pq^i$ $(p + q = 1)$. The distribution function is:

$$F_{\hat{\mathbf{x}}}(m) = \sum_{i=0}^{m} p_i = \sum_{i=0}^{m} pq^i = 1 - q^{m+1}$$

$$1 - q^{m+1} \leq u \rightarrow q^{m+1} \geq 1 - u \equiv u$$

$$m = \left[\frac{\log(u)}{\log(q)}\right]$$

Rejection method: We propose a valu according to $g(x)$:

$$g(x) = \sum_i g_i \delta(x - x_i)$$

this value is accepted with probablity h_i, $0 \leq h_i \leq 1$, $\forall i$,

The resulting pdf of the distribution is:

$$f_{\hat{\mathbf{x}}}(x) = \frac{\Sigma_i \, h_i g_i \delta(x - x_i)}{\Sigma_i \, h_i g_i}$$