Analytical and numerical studies of noise-induced synchronization of chaotic systems

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We study the effect that the injection of a common source of noise has on the trajectories of chaotic systems, addressing some contradictory results present in the literature. We present particular examples of one-dimensional maps and the Lorenz system, both in the chaotic region, and give numerical evidence showing that the addition of a common noise to different trajectories, which start from different initial conditions, leads eventually to their perfect synchronization. When synchronization occurs, the largest Lyapunov exponent becomes negative. For a simple map we are able to show this phenomenon analytically. Finally, we analyze the structural stability of the phenomenon. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1386397]

The synchronization of chaotic systems has been the subject of intensive research in recent years. Besides its fundamental interest, the study of the synchronization of chaotic oscillators has a potential application in the field of chaos communications. The main idea resides in the hiding of a message within a chaotic carrier generated by a suitable emitter. The encoded message can be extracted if an appropriate receiver, one which synchronizes to the emitter, is used. One of the conditions to be fulfilled in order to achieve synchronization is that the receiver and the emitter have very similar device parameters, hence making it very difficult to intercept the encoded message. Although the usual way of synchronizing two chaotic systems is by injecting part of the emitted signal into the receiver, the possibility of synchronization using a common random forcing has also been suggested. However, there have been some contradictory results in the literature on whether chaotic systems can indeed be synchronized using such a common source of noise and the issue has begun to be clarified only very recently. In this paper we give explicit examples of chaotic systems that become synchronized by the addition of Gaussian white noise of zero mean. We also analyze the structural stability of the phenomenon, namely, the robustness of the synchronization against a small mismatch in the parameters of the chaotic sender and receiver.

I. INTRODUCTION

One of the most surprising results of the last few decades in the field of stochastic processes has been the discovery that fluctuation terms (loosely called *noise*) can actually induce some degree of order in a large variety of nonlinear systems. The first example of such an effect is that of *stochastic resonance*[1,2] by which a bistable system responds better to an external signal (not necessarily periodic) under the presence of fluctuations, either in the intrinsic dynamics or in the external input. This phenomenon has been shown to be relevant for some physical and biological systems described by nonlinear dynamical equations.[3–5] Other examples in purely temporal dynamical systems include phenomena such as noise-induced transitions,[6] noise-induced transport,[7] coherence resonance,[8–11] etc. In extended systems, noise is known to induce a large variety or ordering effects,[12] such as pattern formation,[13,14] phase transitions,[15–18] phase separation,[19,20] spatiotemporal stochastic resonance,[21,22] noise-sustained structures,[23,24] doubly stochastic resonance,[25] amongst many others. All these examples have in common that some sort of *order* appears only in the presence of the right amount of noise.

There has also been some recent interest on the interplay between chaotic and random dynamics. Some counterintuitive effects such as coherence resonance, or the appearance of a quasiperiodic behavior, in a chaotic system in the presence of noise, have been found recently.[26] The role of noise in standard synchronization of chaotic systems has been considered in Refs. 27 and 28, as well as the role of noise in synchronizing nonchaotic systems.[29,30] In this paper we address the different issue of synchronization of chaotic systems by a common random noise source, a topic that has attracted much attention recently. The accepted result is that, for some chaotic systems, the introduction of the same noise in independent copies of the systems could lead (for large enough noise intensity) to a common collapse onto the same

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trajectory, independently of the initial condition assigned to each of the copies. This synchronization of chaotic systems by the addition of random terms is a remarkable and counterintuitive effect of noise and although some clarifying papers have appeared recently, still some contradictory results exist for the existence of this phenomenon of noise-induced synchronization. It is the purpose of this paper to give further analytical and numerical evidence that chaotic systems can synchronize under such circumstances and to analyze the structural stability of the phenomenon. Moreover, the results presented here clarify the issue, thus opening directions to obtain such a synchronization in electronic circuits, e.g., for encryption purposes. Common random noise codes have been used in spread spectrum communication for a long time ago.[31] The main idea is to mix information data within a noisy code. At the receiver, the information is recovered using a synchronized replica of the noise code. More recently, the use of common noise source has also been proposed as a useful technique to improve the encryption of a key in a communication channel.[32]

The issue of ordering effect of noise in chaotic systems was already considered at the beginning of the 1980s by Matsumoto and Tsuda,[33] who concluded that the introduction of noise could actually make a system less chaotic. Later, Yu, Ott, and Chen[34] studied the transition from chaos to nonchaos induced by noise. Synchronization induced by noise was considered by Fahy and Hamman,[35] who showed that particles in an external potential, when driven by the same random forces, tend to collapse onto the same trajectory, a behavior interpreted as a transition from chaotic to nonchaotic behaviors. The same system has been studied numerically and analytically.[36–38] Pikovsky[39] analyzed the statistics of deviations from this noise-induced synchronization. A paper that generated a lot of controversy was that of Maritan and Banavar.[40] These authors analyzed the logistic map in the presence of noise:

$$x_{n+1} = 4x_n(1-x_n) + \xi_n, \qquad (1)$$

where ξ_n is the noise term, considered to be uniformly distributed in a symmetric interval $[-W, +W]$. They showed that, if W were large enough (i.e., for a large noise intensity) two different trajectories which started with different initial conditions but otherwise used the same sequence of random numbers, would eventually coincide into the same trajectory. The authors showed a similar result for the Lorenz system (see Sec. III). This result was heavily criticized by Pikovsky,[41] who proved that two systems can synchronize only if the largest Lyapunov exponent is negative. He then argued that the largest Lyapunov exponent of the logistic map in the presence of noise is always positive and concluded that the synchronization was, in fact, a numerical effect of lack of precision of the calculation. The analysis of Pikovsky was confirmed by Longa *et al.*,[42] who studied the logistic map with arbitrary numerical precision. The criterion of negative Lyapunov exponent has also been shown to hold for other types of synchronization of chaotic systems and Zhou and Lai[43] noticed that previous results by Shuai, Wong,

and Cheng[44] showing synchronization with a positive Lyapunov exponent were again an artifact of the limited precision of the calculation.

In addition to the above-mentioned criticisms, Herzel and Freund[45] and Malescio[46] pointed out that the noise used to simulate Eq. (1) and the Lorenz system in Ref. 40 is not really symmetric. While the noise in the Lorenz system is nonsymmetric by construction, in the case of the map, the nonzero mean arises because the requirement $x_n \in (0,1)$, $\forall n$, actually leads one to discard the values of the random number ξ_n which would induce a violation of such condition. The average value of the random numbers which have been accepted is different from zero, hence producing an effective *biased* noise, i.e., one which does not have zero mean. The introduction of a nonzero mean noise means that the authors of Ref. 40 were altering essentially the properties of the deterministic map. Furthermore, Gade and Bassu[47] argued that the synchronization observed by Maritan and Banavar is due to the fact that the bias of the noise leads the system to a nonchaotic fixed point. With only this basis, they concluded that a zero-mean noise can never lead to synchronization in the Lorenz system. The same conclusion was reached by Sánchez *et al.*,[48] who studied experimentally a Chua circuit and concluded that synchronization by noise only occurs if the noise does not have a zero mean. The same conclusion is obtained in Ref. 49 by studying numerically a single and an array of Lorenz models, and in Ref. 50 from experiments in an array of Chua circuits with multiplicative colored noise. Therefore, from these last works, a widespread belief has emerged according to which it is not possible to synchronize two chaotic systems by injecting the same noisy unbiased, zero-mean, signal to both of them.

Contrary to these last results (but in agreement with the previously mentioned results[32–39]), Lai and Zhou[51] have shown that some chaotic maps can indeed become synchronized by additive zero-mean noise. A similar result has been obtained by Loreto *et al.*,[52] and by Minai and Anand,[32,53,54] in the case where the noise appears parametrically in the map. The implications to secure digital communications have been considered in Refs. 32, 53, and 55, and an application to ecological dynamics in fluid flows is presented in Ref. 56. An equivalent result about the synchronization of Lorenz systems using a common additive noise has been shown by the authors of the present paper in Ref. 57. The actual mechanism that leads to synchronization has been explained by Lai and Zhou,[51] see also Ref. 58. As Pikovsky[41] required, synchronization can only be achieved if the Lyapunov exponent is negative. The presence of noise allows the system to spend more time in the "convergence region" where the local Lyapunov exponent is negative, hence yielding a global negative Lyapunov exponent. This argument will be developed in more detail in Sec. II, where an explicit calculation in a simple map will confirm the analysis. The results of Lai and Zhou have been extended to the case of coupled map lattices[59] where Pikovsky's criterion has been extended for spatially extended systems.

In this paper we give further evidence that it is possible to synchronize two chaotic systems by the addition of a common noise which is Gaussianly distributed and not biased.
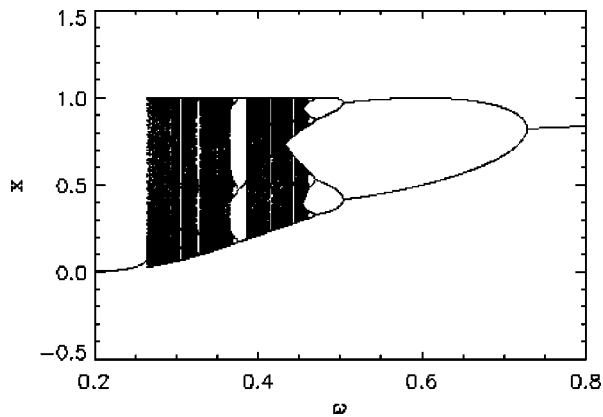
FIG. 1. Bifurcation diagram of the map given by Eqs. (2) and (3) in the absence of noise terms.

FIG. 2. Lyapunov exponent for the noiseless map ($\epsilon=0$, continuous line) and the map with a noise intensity $\epsilon=0.1$ (dotted line) and $\epsilon=0.2$ (dot-dashed line).

We analyze specifically some one-dimensional (1D) maps and the Lorenz system, all in the chaotic region. The necessary criterion introduced in Ref. 41 and the general arguments of Ref. 51 are fully confirmed and some heuristic arguments are given about the general validity of our results.

The organization of the paper is as follows. In Sec. II we present numerical and analytical results for some 1D maps, while Sec. III studies numerically the Lorenz system. In Sec. IV we analyze the structural stability of the phenomenon, i.e., the dependence of the synchronization time on the parameter mismatch. Finally, in Sec. V we present the conclusions as well as some open questions relating the general validity of our results.

II. RESULTS ON MAPS

The first example is that of the map

$$x_{n+1}=F(x_n)=f(x_n)+\epsilon\xi_n,\tag{2}$$

where ξ_n is a set of uncorrelated Gaussian variables of zero mean and variance 1. As an example, we use explicitly

$$f(x)=\exp\left[-\left(\frac{x-0.5}{\omega}\right)^2\right].\tag{3}$$

Studying the convergence or divergence of trajectories of Eq. (2) starting from different initial conditions under the same noise ξ_n is equivalent to analyzing the converge or divergence of trajectories from two identical systems of the form (2) driven by the same noise. We plot in Fig. 1 the bifurcation diagram of this map in the noiseless case. We can see the typical windows in which the system behaves chaotically. The associated Lyapunov exponent, λ, is positive in these regions. For instance, for $\omega=0.3$ (the case we will be considering throughout the paper) it is $\lambda\approx0.53$. In Fig. 2 we observe that the Lyapunov exponent becomes negative for most values of ω for large enough noise level ϵ. Again for $\omega=0.3$ and now for $\epsilon=0.2$ it is $\lambda=-0.17$. A positive Lyapunov exponent in the noiseless case implies that trajectories starting with different initial conditions, but using the same sequence of random numbers $\{\xi_n\}$, remain different for all the iteration steps. In this case, the corresponding synchronization diagram shows a spread distribution of points [see Fig. 3(a)]. However, when moderate levels of noise
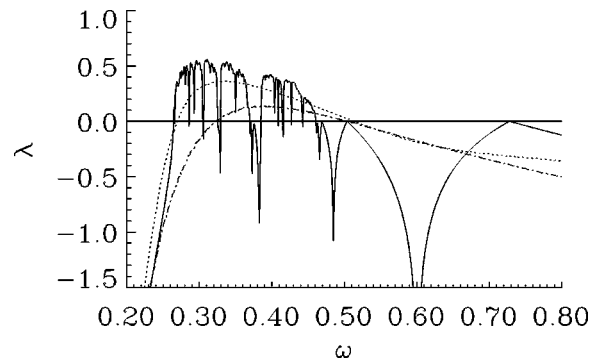
($\epsilon\gtrsim0.2$) are used, λ becomes negative and trajectories starting with different initial conditions, but using the same sequence of random numbers, synchronize perfectly, see the synchronization diagram in Fig. 3(b). Obviously, the noise intensity in the cases shown is not large enough such as to be able to neglect completely the deterministic part of the map.
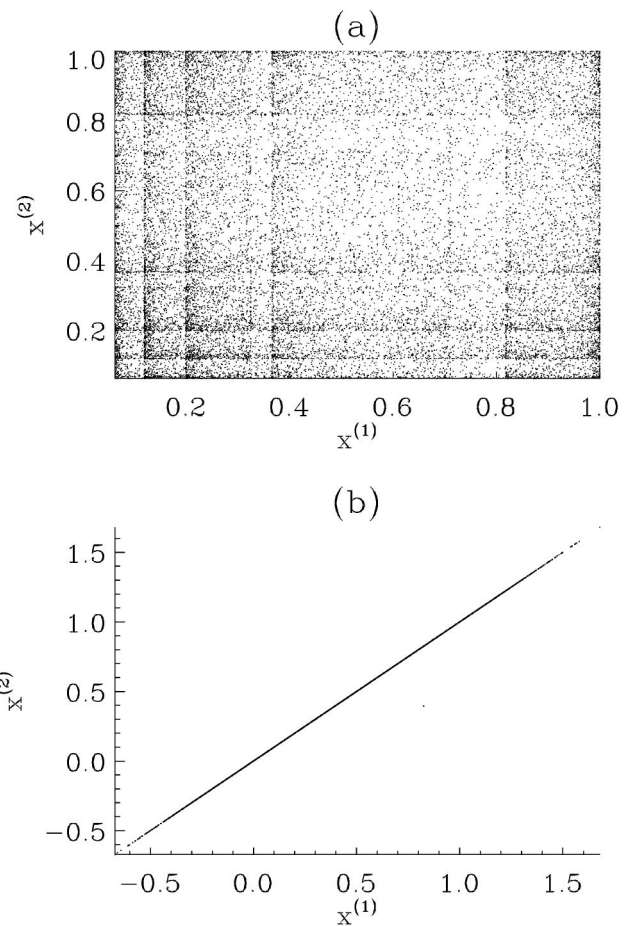

FIG. 3. Plot of two realizations $x^{(1)}$, $x^{(2)}$ of the map given by Eqs. (2) and (3) with $\omega=0.3$. Each realization consists of 10 000 points which have been obtained by iteration of the map starting in each case from a different initial condition (100 000 initial iterations have been discarded and are not shown). In (a) there is no noise, $\epsilon=0$, and the trajectories are independent of each other. In (b) we have used a level of noise $\epsilon=0.2$, producing a perfect synchronization (after discarding some initial iterations).

Therefore, the synchronization observed does not trivially appear as a consequence of both variables becoming themselves identical to the noise term.

According to Ref. 41, convergence of trajectories to the same one, or lack of sensitivity to the initial condition, can be stated as *negativity of the Lyapunov exponent*. The Lyapunov exponent of the map (2) is defined as

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln|F'(x_i)|. \tag{4}$$

It is the average of (the logarithm of the absolute value of) the successive slopes F' found by the trajectory. Slopes in $[-1,1]$ contribute to λ with negative values, indicating trajectory convergence. Larger or smaller slopes contribute with positive values, indicating trajectory divergence. Since the deterministic and noisy maps satisfy $F'=f'$ one is tempted to conclude that the Lyapunov exponent is not modified by the presence of noise. However, there is noise dependence through the trajectory values x_i, $i=1,\ldots,N$. In the absence of noise, λ is positive, indicating trajectory separation. When synchronization is observed, the Lyapunov exponent is negative, as required by the argument in Ref. 41.

Notice that this definition of the Lyapunov exponent assumes a fixed realization of the noise terms, and it is the relevant one to study the synchronization phenomena addressed in this paper. One could use alternative definitions.[52] For instance, if one considers the coupled system of both the x variable and the noise generator producing ξ, then the largest Lyapunov exponent of the composed system is indeed positive (and very large for a good random number generator). This simply tells us that there is a large sensitivity to the initial condition of the composed system (x,ξ) as shown by the fact that a change of the seed of the random number generator completely changes the sequence of values of both ξ and x. We consider in this paper the situation described by definition (4) with fixed noise realization.

By using the definition of the *invariant measure on the attractor*, or *stationary probability distribution* $P_{st}(x)$, the Lyapunov exponent can be calculated also as

$$\lambda = \langle \log|F'(x)| \rangle = \langle \log|f'(x)| \rangle \equiv \int P_{st}(x)\log|f'(x)|dx. \tag{5}$$

Here we see clearly the two contributions to the Lyapunov exponent: although the derivative $f'(x)$ does not change when including noise in the trajectory, the stationary probability does change (see Fig. 4), thus producing the observed change in the Lyapunov exponents. Synchronization, then, can be a general feature in maps, such as (3), which have a large region in which the derivative $|f'(x)|$ is smaller than one. Noise will be able to explore that region and yield, on average, a negative Lyapunov exponent. This is, basically, the argument developed in Ref. 51.

In order to make some analytical calculation that can obtain in a rigorous way the transition from a positive to negative Lyapunov exponent, let us consider the map given by Eq. (2) and
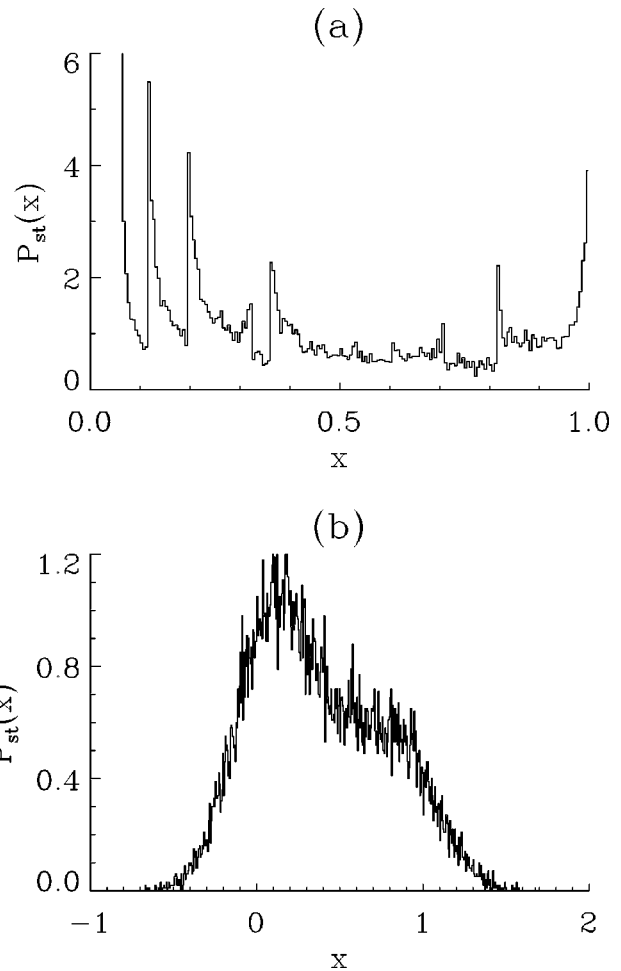

FIG. 4. Plot of the stationary distribution for the map given by Eqs. (2) and (3) with $\omega = 0.3$ in the (a) deterministic case $\epsilon = 0$, and (b) the case with noise along the trajectory, $\epsilon = 0.2$.

$$f(x) = \begin{cases} a(1 - \exp(1 + x)) & \text{if } x < -1 \\ -2 - 2x & \text{if } x \in (-1, -0.5) \\ 2x & \text{if } x \in (-0.5, 0.5) \\ 2 - 2x & \text{if } x \in (0.5, 1) \\ a(-1 + \exp(1 - x)) & \text{if } x > 1 \end{cases} \tag{6}$$

with $0 < a < 1$. This particular map, based in the tent map,[60] has been chosen just for convenience. The following arguments would apply to any other map that in the absence of noise takes most frequently values in the region with the highest slopes, but which visits regions of smaller slope when noise is introduced. This is the case, for example, of the map (3). In the case of (6), the values given by the deterministic part of the map, after one iteration from arbitrary initial conditions, fall always in the interval $(-1,1)$. This is the region with the highest slope $|F'|=2$. In the presence of noise the map can take values outside this interval and, since the slopes encountered are smaller, the Lyapunov exponent can only be reduced from the deterministic value. To formally substantiate this point, it is enough to recall the definition of Lyapunov exponent (4): an upper bound for $|F'(x)|$ is 2, so that a bound for λ is immediately obtained: $\lambda \leq \ln 2$. Equality is obtained for zero noise.

The interesting point about the map (6) and similar ones is that one can demonstrate analytically that λ can be made negative. The intuitive idea is that it is enough to decrease a in order to give arbitrarily small values to the slopes encountered outside $(-1,1)$, a region accessible only thanks to noise. To begin with, let us note that $|F'(x)|=2$ if $x \in (-1,1)$, and $|F'(x)| < a$ if $|x| > 1$, so that an upper bound to (4) can be written as

$$\lambda \leq \lim_{N \to \infty} \left(\frac{N_I}{N} \ln 2 + \frac{N_O}{N} \ln a \right)$$

$$= p_I \ln 2 + p_O \ln a = \ln 2 - p_O \ln(2/a). \quad (7)$$

N_I/N and N_O/N are the proportion of values of the map inside $I=(0,1)$ and outside this interval, respectively, and we have used that as $N \to \infty$ they converge with p_I and p_O, the invariant measure associated with I and to the rest of the real line, respectively ($p_I + p_O = 1$). A sufficient condition for $x_{n+1} = f(x_n) + \epsilon \xi_n$ to fall outside I is that $|\xi_n| > 2/\epsilon$. Thus, $p_O = \text{probability}(|x_{n+1}| > 1) > \text{probability}(|\xi_n| > 2/\epsilon) = \text{erfc}(\sqrt{2}/\epsilon) \equiv T$, where we have used the Gaussian character of the noise. In consequence, one finds from (7)

$$\lambda \leq \ln 2 - T \ln(2/a). \quad (8)$$

The important point is that $T = \text{erfc}(\sqrt{2}/\epsilon)$ is independent of the map parameters, in particular of a. Thus, (8) implies that by decreasing a the value of λ can be made as low as desired. By increasing ϵ such that $T > \ln 2/\ln(2/a)$, λ will be certainly negative. Thus we have shown analytically that strong enough noise will always make negative the Lyapunov exponent of the map (6) and, accordingly, it will induce yield "noise-induced synchronization" in that map.

III. THE LORENZ SYSTEM

In this section we give yet another example of noise-induced synchronization. We consider the well-known Lorenz[61] model with additional random terms of the following form:[40]

$$\dot{x} = p(y-x),$$

$$\dot{y} = -xz + rx - y + \epsilon \xi, \quad (9)$$

$$\dot{z} = xy - bz.$$

ξ is white noise: a Gaussian random process of mean zero, $\langle \xi(t) \rangle = 0$ and delta correlated, $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$. We have used $p=10$, $b=8/3$, and $r=28$ which, in the deterministic case, $\epsilon=0$ are known to lead to a chaotic behavior (the largest Lyapunov exponent is $\lambda \approx 0.9 > 0$). As stated in Sec. I, previous results seem to imply that synchronization is only observed for a noise with a nonzero mean. However, our results show otherwise.

We have integrated numerically the above-mentioned equations using the stochastic Euler method.[62] Specifically, the evolution algorithm reads
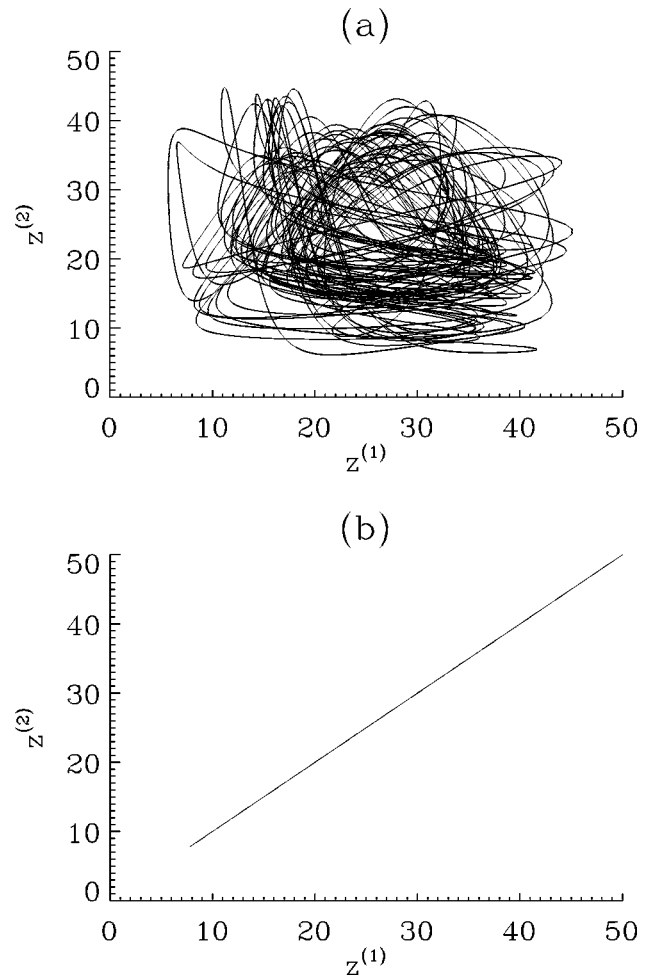
$$x(t+\Delta t) = x(t) + \Delta t[p(y(t)-x(t))],$$

FIG. 5. Plot of two realizations $z^{(1)}, z^{(2)}$ of the Lorenz system Eq. (9) with $p=10$, $b=8/3$, and $r=28$. Each plotted realization starts from a different initial condition and consists of an initial warming up time of $t=12\,000$ (not shown) and runs for a time $t=600$ in the dimensionless units of the Lorenz system of equations. (a) The deterministic case ($\epsilon=0$) and (b) the results for $\epsilon=40$. Notice the perfect synchronization in (b).

$$y(t+\Delta t) = y(t) + \Delta t[-x(t)z(t) + rx(t) - y(t)]$$

$$+ \epsilon \sqrt{\Delta t} g(t), \quad (10)$$

$$z(t+\Delta t) = z(t) + \Delta t[x(t)y(t) - bz(t)].$$

The values of $g(t)$ are drawn at each time step from an independent Gaussian distribution of zero mean and variance one and they have been generated by a particularly efficient algorithm using a numerical inversion technique.[63] The time step used is $\Delta t = 0.001$ and simulations range typically for a total time of the order of $t=10^4$ (in the dimensionless units of the Lorenz system of equations). The largest Lyapunov exponent has been computed using a simultaneous integration of the linearized equations.[64] For the deterministic case, trajectories starting with different initial conditions are completely uncorrelated, see Fig. 5(a). This is also the situation for small values of ϵ. However, when using a noise intensity $\epsilon=40$ the noise is strong enough to induce synchronization of the trajectories. Again, the presence of the noise terms forces the largest Lyapunov exponent to become negative (for $\epsilon=40$ it is $\lambda \approx -0.2$). As in the examples of the maps,
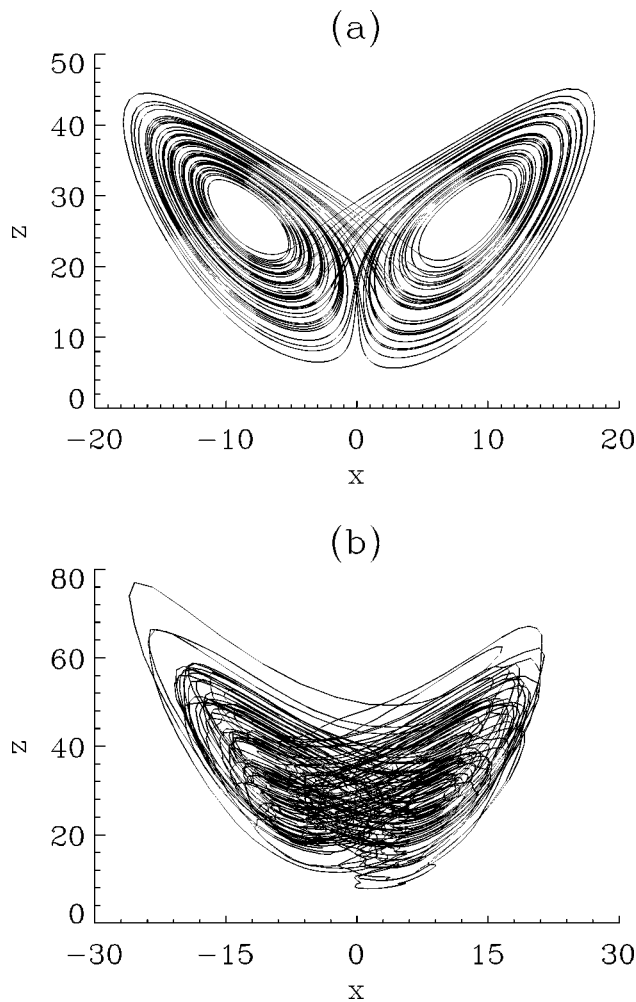
FIG. 6. "Butterfly" attractor of the Lorenz system in the cases (a) of no noise $\epsilon=0$, and (b) $\epsilon=40$ using the same time series as in Fig. 5.

after some transient time, two different evolutions which have started in completely different initial conditions synchronize toward the same value of the three variables [see Fig. 5(b) for the z coordinate]. Therefore, these results prove that synchronization by common noise in the chaotic Lorenz system does occur for sufficiently large noise intensity. This result contradicts previous ones in the literature.[46,48] The main difference with these papers is in the intensity of the noise: it has to be taken sufficiently large, as here, in order to observe synchronization. Notice that although the noise intensity is large, the basic structure of the "butterfly" Lorenz attractor remains present as shown in Fig. 6. Again, this result shows that, although the noise intensity used could be considered large, the synchronization is rather different from what would be obtained from a trivial common synchronization of both systems to the noise variable by neglecting the deterministic terms.

IV. STRUCTURAL STABILITY

An important issue concerns the structural stability of this phenomenon, in particular how robust is noise synchronization to small differences between the two systems one is trying to synchronize. Whether or not the synchronization of

two trajectories of the same noisy Lorenz system (or of any other chaotic system) observed here, equivalent to the synchronization of two identical systems driven by a common noise, could be observed in the laboratory, depends on whether the phenomenon is robust when allowing the two Lorenz systems to be not exactly equal (as they cannot be in a real experiment). If one wants to use this kind of stochastic synchronization in electronic emitters and receivers (for instance, as a means of encryption) one should be able to determine the allowed discrepancy between circuits before the lack of synchronization becomes unacceptable. Additional discussions on this issue may be found in Refs. 39, 32, and 65.

We consider the following two maps forced by the same noise:

$$x_{n+1}=f(x_n)+\xi_n, \tag{11}$$

$$y_{n+1}=g(y_n)+\xi_n. \tag{12}$$

Linearizing in the trajectory difference $u_n=y_n-x_n$, assumed to be small, we obtain

$$u_{n+1}=g'(x_n)u_n+g(x_n)-f(x_n)\equiv g'(x_n)u_n+\Delta(x_n). \tag{13}$$

We have defined $\Delta(x)\equiv g(x)-f(x)$, and we are interested in the situation in which the two systems are just slightly different, for example, because of a small parameter mismatch, so that Δ will be small in some sense specified below. Iteration of (13) leads to the formal solution:

$$u_n=M(n-1,0)u_0+\sum_{m=0}^{n-1}M(n-1,m+1)\Delta(x_m). \tag{14}$$

We have defined $M(j,i)=\prod_{k=i}^{j}g'(x_k)$, and $M(i-1,i)\equiv1$. An upper bound on (14) can be obtained:

$$|u_n|^2\leqslant|M(n-1,0)|^2|u_0|^2$$
$$+\sum_{m=0}^{n-1}|M(n-1,m+1)|^2|\Delta(x_m)|^2. \tag{15}$$

The first term on the right-hand side is what would be obtained for identical dynamical systems. We know that $M(n-1,0)\rightarrow e^{\lambda n}$ as $n\rightarrow\infty$, where λ is the largest Lyapunov exponent associated with (12). We are interested in the situation in which $\lambda<0$, for which this term vanishes at long times. Further analysis is done first for the case in which $\Delta(x)$ is a bounded function (or x is a bounded trajectory with Δ continuous). In this situation, there is a real number μ such that $|\Delta(x_m)|<\mu$. We then get

$$|u_n|^2\leqslant\mu^2\sum_{m=0}^{n-1}|M(n-1,m+1)|^2 \tag{16}$$

an inequality valid for large n. Let us now define $K=\max_x|g(x)|$, the maximum slope of the function $g(x)$. A trivial bound is now obtained as

$$|u_n|^2\leqslant\mu^2\frac{1-K^{2n}}{1-K^2}. \tag{17}$$

This can be further improved in the case $K<1$, where we can write

$$|u_n|^2 \le \mu^2 \frac{1}{1-K^2}. \qquad (18)$$

As a consequence, differences in the trajectories remain bounded at all iteration steps n. Since, according to the definition (4), $\ln K$ is also an upper bound for the Lyapunov exponent for all values of ϵ and, in particular, for the noiseless map, $\epsilon=0$, this simply tells us that if the deterministic map is nonchaotic, then the addition of a common noise to two imperfect but close replicas of the map will still keep the trajectory difference within well-defined bounds. The situation of interest here, however, concerns the case in which a negative Lyapunov exponent arises only as the influence of a sufficiently large noise term, i.e., the deterministic map is chaotic and $K>1$. In this case, the sum in Eq. (16) contains products of slopes which are larger or smaller than 1. It is still true that the terms in the sum for large value of $n-m$ can be approximated by $M(n-1,m+1) \approx e^{(n-m-1)\lambda}$ and, considering this relation to be valid for all values of n,m, we would get

$$\sum_{m=0}^{n-1} |M(n-1,m+1)|^2 \approx e^{2\lambda n} \sum_{m=0}^{n-1} e^{-2\lambda(m+1)} = \frac{1-e^{2\lambda n}}{1-e^{2\lambda}} \qquad (19)$$

and, thus, at large n,

$$|u_n|^2 \lesssim \mu^2 (1-e^{2\lambda})^{-1}. \qquad (20)$$

It can happen, however, that the product defining $M(n-1,m+1)$ contains a large sequence of large slopes $g'(x_i)$. These terms (statistically rare) will make the values of $|u_n|$ violate the above-mentioned bound at sporadic times. Analysis of the statistics of deviations from synchronization was carried out in Ref. 39. Although for $\lambda<0$ the most probable deviation is close to zero, power-law distributions with long tails are found, and indeed its characteristics are determined by the distribution of slopes encountered by the system during finite amounts of time, or finite-time Lyapunov exponents, as the above-mentioned arguments suggest. Therefore, we expect a dynamics dominated by relatively large periods of time during which the difference between trajectories remains bounded by a small quantity, but intermittently interrupted by bursts of large excursions of the difference. This is indeed observed in the numerical simulations of the maps defined previously. This general picture is still valid even if $|\Delta(x)|$ is not explicitly bounded.

We have performed a more quantitative study for the case in which two noisy Lorenz systems with different sets of parameters, namely

$$\dot{x}_1 = p_1(y_1-x_1),$$

$$\dot{y}_1 = -x_1 z_1 + r_1 x_1 - y_1 + \epsilon\xi, \qquad (21)$$

$$\dot{z}_1 = x_1 y_1 - b_1 z_1,$$

and
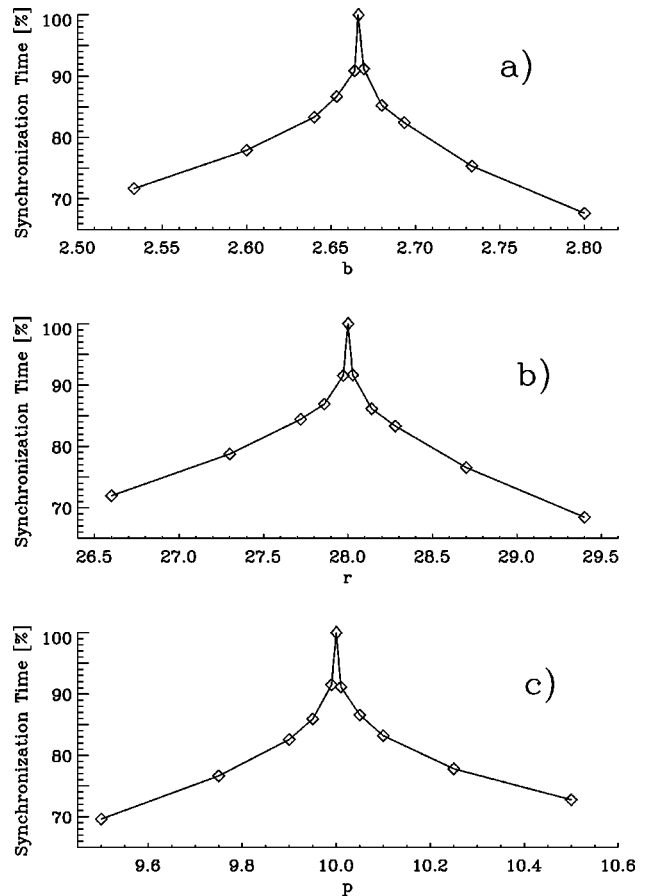
$$\dot{x}_2 = p_2(y_2-x_2),$$

FIG. 7. Percentage of time that two slightly dissimilar Lorenz systems, subjected to the same noise, remain synchronized (up to a 10% discrepancy in the z variable). For one of the two systems we fix $p_1=10$, $b_1=8/3$, and $r_1=28$ while for the other we vary systematically one of the parameters keeping the other two constant: in (a) the parameter b_2 varies, in (b) the parameter r_2 varies, and in (c) the parameter p_2 varies. Notice that the percentage of synchronization time is still higher than 85% if the relative difference between the parameters is less than 1%.

$$\dot{y}_2 = -x_2 z_2 + r_2 x_2 - y_2 + \epsilon\xi, \qquad (22)$$

$$\dot{z}_2 = x_2 y_2 - b_2 z_2,$$

are forced by the same noise $\xi(t)$. In order to discern the effect of each parameter separately, we have varied independently each one of the three parameters, (p,b,r), while keeping constant the other two. The results are plotted in Fig. 7. In Fig. 7 we plot the percentage of time in which the two Lorenz systems are still synchronized with a tolerance of 10%. This means that trajectories are considered synchronized if the relative difference in the z variable is less than 10%. According to the general discussion for maps, we expect departures from approximate synchronization from time to time. They are in fact observed, but from Fig. 7 we conclude that small variations (of the order of 1%) still yield a synchronization time of more than 85%. In Fig. 8 we show that the loss of synchronization between the two systems appears in the form of bursts of spikes whose amplitude is only limited by the size of the attractor in the phase space.
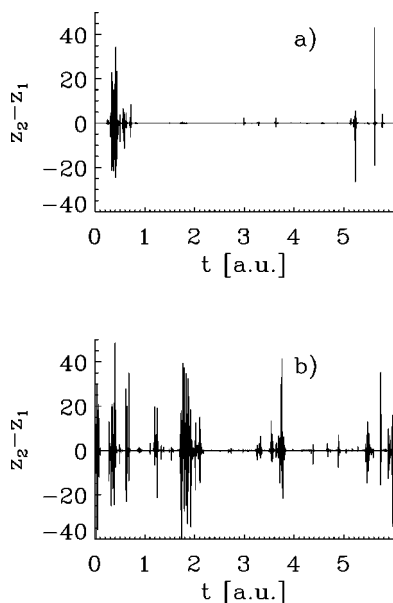
FIG. 8. Time evolution of the difference between two trajectories z_1 and z_2 corresponding to two Lorenz systems driven by the same noise but with a small mismatch in the r parameter: (a) $r_1 = 28$ and $r_2 = 0.99999 \times r_1$ and in (b) $r_2 = 0.999 \times r_1$. Notice that although the synchronization time diminishes with increasing parameter mismatch, the maximum absolute difference between the two variables does not depend on the mismatch.

Moreover, it can also be clearly seen in Fig. 8 that large (but infrequent) spike amplitudes appear for arbitrarily small mismatch.

In the realm of synchronization of chaotic oscillators, two different types of analogous intermittent behaviors have been associated also with the fluctuating character of the finite-time conditional Lyapunov exponents noted previously. One is on–off intermittency[66] where the synchronization manifold is slightly unstable on average but the finite time Lyapunov exponent is negative during relatively long periods of time. In the other one, named bubbling,[67] the synchronization is stable on average but the local conditional Lyapunov exponent becomes occasionally positive. While in the former case bursting always occurs due to the necessarily imperfect initial synchronization, in the latter it is strictly a consequence of the mismatch of the entraining systems. In this sense, the behavior reported in the preceding paragraph should be considered as a manifestation of bubbling in synchronization by common noise.

V. CONCLUSIONS AND OPEN QUESTIONS

In this paper we have addressed the issue of synchronization of chaotic systems by the addition of common random noises. We have considered three explicit examples: two 1D maps and the Lorenz system under the addition of zero-mean, Gaussian, white noise. While the map examples confirm previous results in similar maps, and we have obtained with them analytical confirmation of the phenomenon, the synchronization observed in the Lorenz system contradicts some previous results in the literature. The reason is that previous works considered noise intensities smaller than the ones we found necessary for noise synchronization in this

system. Finally, we have analyzed the structural stability of the observed synchronization. In the Lorenz system, synchronization times larger than 85% (within an accuracy of 10%) can still be achieved if the parameters of the system are allowed to change in less than 1%.

It is important to point out that noise-induced synchronization between identical systems subjected to a common noise is equivalent to noise induced order, in the sense that the Lyapunov exponent defined in (4) becomes negative in a single system subjected to noise. One can ask whether the state with negative Lyapunov exponent induced by noise may be still be called "chaotic" or not. This is just a matter of definition: If one defines chaos as exponential sensibility to initial conditions, and one considers this *for a fixed noise realization*, then the definition of Lyapunov exponent implies that trajectories are no longer chaotic in this sense. But one can also consider the extended dynamical system containing the forced one *and* the noise generator (e.g., in numerical computations, it would be the computer random number generator algorithm). For this *extended system* there is strong sensibility to initial conditions in the sense that small differences in noise generator seed leads to exponential divergence of trajectories. In fact, this divergence is at a rate given by the Lyapunov exponent of the noise generator, which approaches infinity for a true Gaussian white process. Trajectories in the noise-synchronized state are in fact more irregular than in the absence of noise, and attempts to calculate the Lyapunov exponent just from the observation of the time series will lead to a positive and very large value, since it is the *extended* dynamical system that is observed when analyzing the time series[68] (typically such attempts will fail because the high dimensionality of good noise generators, ideally infinity, would put them out of the reach of standard algorithms for Lyapunov exponent calculations). Again, whether or not to call such irregular trajectories with just partial sensibility to initial conditions "chaotic" is just a matter of definition. More detailed discussion along these lines can be found in Ref. 52.

There still remain many open questions in this field. They involve the development of a general theory, probably based in the invariant measure, that could give us a general criterion to determine the range of parameters (including noise levels) for which the Lyapunov exponent becomes negative, thus allowing synchronization. In this work and similar ones, the word synchronization is used in a very restricted sense, namely: the coincidence of asymptotic trajectories. This contrasts with the case of interacting periodic oscillations where a more general theory of synchronization exists to explain the phenomenon of nontrivial phase locking between oscillators that individually display very different dynamics. Indications of the existence of analog nontrivial phase locking have been reported for chaotic attractors.[69] There a "phase" with a chaotic trajectory defined in terms of a Hilbert transform is shown to be synchronizable by external perturbations in a similar way as happens with periodic oscillators. Whether or not this kind of generalized synchronization can be induced by noise is, however, a completely open question. Last, but not least, it would also be interesting to explore whether analogs of the recently reported synchro-

nization of spatiotemporal chaos[70,71] may be induced by noise.

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[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).

[2] C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).

[3] *Proceedings of the NATO Advanced Research Workshop: Stochastic Resonance in Physics and Biology,* edited by F. Moss, A. Bulsara, and M. F. Shlesinger [J. Stat. Phys. **70** (1993)].

[4] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).

[5] A. Longtin and D. R. Chialvo, Phys. Rev. Lett. **81**, 4012 (1998).

[6] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).

[7] P. Hänggi and R. Bartussek, in *Nonlinear Physics of Complex Systems*, edited by J. Parisi, S.C. Müller, and W. Zimmermman (Springer, New York, 1999).

[8] H. Gang, T. Ditzinger, C. Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).

[9] W. Rappel and S. Strogatz, Phys. Rev. E **50**, 3249 (1994).

[10] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).

[11] B. Lindner and L. Schimansky-Geier, Phys. Rev. E **61**, 6103 (2000).

[12] J. García-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems* (Springer, New York, 1999).

[13] J. García-Ojalvo, A. Hernändez-Machado, and J. M. Sancho, Phys. Rev. Lett. **71**, 1542 (1993).

[14] J. M. R. Parrondo, C. Van den Broeck, J. Buceta, and F. J. de la Rubia, Physica A **224**, 153 (1996).

[15] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994); C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).

[16] J. García-Ojalvo, J. M. R. Parrondo, J. M. Sancho, and C. Van den Broeck, Phys. Rev. E **54**, 6918 (1996).

[17] S. Mangioni, R. Deza, H. Wio, and R. Toral, Phys. Rev. Lett. **79**, 2389 (1997); Phys. Rev. E **61**, 223 (2000).

[18] M. Ibañes, J. García-Ojalvo, R. Toral, and J. M. Sancho, Phys. Rev. Lett. (to be published).

[19] J. García-Ojalvo, A. Lacasta, J. M. Sancho, and R. Toral, Europhys. Lett. **42**, 125 (1998).

[20] M. Ibañes, J. García-Ojalvo, R. Toral, and J. M. Sancho, Phys. Rev. E **60**, 3597 (1999).

[21] P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 2134 (1995).

[22] F. Marchesoni, L. Gammaitoni, and A. R. Bulsara, Phys. Rev. Lett. **76**, 2609 (1996).

[23] R. J. Deissler, J. Stat. Phys. **54**, 1459 (1989).

[24] M. Santagiustina, P. Colet, M. San Miguel, and D. Walgraef, Phys. Rev. Lett. **79**, 3633 (1997); Phys. Rev. E **58**, 3843 (1998).

[25] A. A. Zaikin, J. Kurths, and L. Schimasky-Greier, Phys. Rev. Lett. **85**, 227 (2000).

[26] C. Palenzuela, R. Toral, C. R. Mirasso, O. Calvo, and J. D. Gunton, preprint cond-mat/0007371.

[27] M. K. Ali, Phys. Rev. E **55**, 4804 (1997).

[28] V. Andrade, R. L. Davidchack, and Y. C. Lai, Phys. Rev. E **61**, 3230 (2000).

[29] R. V. Jensen, Phys. Rev. E **58**, R6907 (1998).

[30] P. Ashwin, Physica D **125**, 302 (1999).

[31] A. J. Viterbi, *Principles of Spread Spectrum Communication* (Addison–Wesley, Reading, MA, 1995).

[32] A. A. Minai and T. D. Pandian, Chaos **8**, 621 (1998).

[33] K. Matsumoto and I. Tsuda, J. Stat. Phys. **31**, 87 (1983).

[34] L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990).

[35] S. Fahy and D. R. Hamman, Phys. Rev. Lett. **69**, 761 (1992).

[36] B. Kaulajys and G. Vektaris, Phys. Rev. E **52**, 2091 (1995).

[37] Y. Y. Chen, Phys. Rev. Lett. **77**, 4318 (1996).

[38] B. Kaulakys, F. Ivanauskas, and T. Meskauskas, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 533 (1999).

[39] A. S. Pikovsky, Phys. Lett. A **165**, 33 (1992).

[40] A. Maritan and J. R. Banavar, Phys. Rev. Lett. **72**, 1451 (1994).

[41] A. S. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994); A. Maritan and J. R. Banavar, *ibid.* **73**, 2932 (1994).

[42] L. Longa, E. M. F. Curado, and F. A. Oliveira, Phys. Rev. E **54**, R2201 (1996).

[43] C. Zhou and C. H. Lai, Phys. Rev. E **58**, 5188 (1998).

[44] J. W. Shuai, K. W. Wong, and L. M. Cheng, Phys. Rev. E **56**, 2272 (1997).

[45] H. Herzel and J. Freund, Phys. Rev. E **52**, 3238 (1995).

[46] G. Malescio, Phys. Rev. E **53**, 6551 (1996).

[47] P. M. Gade and C. Basu, Phys. Lett. A **217**, 21 (1996).

[48] E. Sánchez, M. A. Matías, and V. Pérez-Muñuzuri, Phys. Rev. E **56**, 4068 (1997).

[49] M. N. Lorenzo and V. Pérez-Muñuzuri, Phys. Rev. E **60**, 2779 (1999).

[50] V. Pérez-Muñuzuri and M. N. Lorenzo, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 2321 (1999).

[51] C. H. Lai and C. Zhou, Europhys. Lett. **43**, 376 (1998).

[52] V. Loreto, G. Paladin, and A. Vulpiani, Phys. Rev. E **53**, 2087 (1996).

[53] A. A. Minai and T. Anand, Phys. Rev. E **57**, 1559 (1998); **59**, 312 (1999); Physica D **125**, 241 (1999).

[54] A. A. Minai, Phys. Lett. A **251**, 31 (1999).

[55] S. Sundar and A. A. Minai, Phys. Rev. Lett. **85**, 5456 (2000).

[56] C. López, E. Hernández-García, O. Piro, A. Vulpiani, and E. Zambianchi, Chaos **11**, 397 (2001).

[57] R. Toral, C. Mirasso, E. Hernández-García, and O. Piro, in *Unsolved Problems on Noise and Fluctuations, UPoN'99,* edited by D. Abbott and L. Kiss (American Institute of Physics, Melville, NY, 2000), Vol. 511, pp. 255–260.

[58] S. Rim, D. U. Hwang, I. Kim, and C. M. Kim, Phys. Rev. Lett. **85**, 2304 (2000).

[59] L. Baroni, R. Livi, and A. Torcini, preprint chao-dyn/9907005; Phys. Rev. E **63**, 036226 (2001).

[60] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).

[61] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (Springer, Berlin, 1982).

[62] M. San Miguel and R. Toral, *Stochastic Effects in Physical Systems, Instabilities and Nonequilibrium Structures VI,* edited by E. Tirapegui, J. Martínez, and R. Tiemann (Kluwer Academic, Dordrecht, 2000), pp. 35–130.

[63] R. Toral and A. Chakrabarti, Comput. Phys. Commun. **74**, 327 (1993).

[64] T. S. Parker and L. O. Chua, *Practical Numerical Algorithms for Chaotic Systems* (Springer, Berlin, 1989).

[65] K. Yoshimura, Phys. Rev. E **60**, 1648 (1999).

[66] P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994); S. C. Venkataramani, B. R. Hunt, E. Ott, D. J. Gauthier, and J. C. Bienfang, Phys. Rev. Lett. **77**, 5361 (1996); J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **52**, R1253 (1995); D. J. Gauthier and J. C. Bienfang, Phys. Rev. Lett. **77**, 1751 (1996).

[67] L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990); N. Platt, E. A. Spiegel, and C. Tresser, *ibid.* **70**, 279 (1993); N. Platt, S. M. Hammel, and J. F. Heagy, *ibid.* **72**, 3498 (1994); J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994); Y. H. Yu, K. Kwak, and T. K. Lim, Phys. Lett. A **198**, 34 (1995); H. L. Yang and E. J. Ding, Phys. Rev. E **54**, 1361 (1996).

[68] T. Bohr, M. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence* (Cambridge University Press, Cambridge, 1988).

[69] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[70] A. Amengual, E. Hernández-García, R. Montagne, and M. San Miguel, Phys. Rev. Lett. **78**, 4379 (1997).

[71] P. Grassberger, Phys. Rev. E **59**, R2520 (1999).