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Enhancement of stochastic resonance: the role of non Gaussian noises

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Abstract

We have analyzed the phenomenon of stochastic resonance in a double well potential driven by a colored non Gaussian noise. Using a path-integral approach we have obtained a consistent Markovian approximation that enables us to get, through the two state theory, analytical expressions for the signal-to-noise ratio, finding an enhancement of this quantity when the system departs from Gaussian behavior. This finding is supported by extensive numerical simulations. We discuss the relation of these results to some experiments in sensory systems.

1. Introduction

Stochastic resonance (SR) has attracted considerable interest in the last decade due, among other aspects, to its potential technological applications for optimizing the output signal-to-noise ratio (SNR) in nonlinear dynamical systems, as well as to its connection with some biological mechanisms. The phenomenon shows the counterintuitive role played by noise in nonlinear systems as it enhances the response of a system subject to a weak external signal. There is a wealth of papers, conference proceedings and reviews on this subject [1,2], Ref. [3] being the most complete one, which show the large number of applications in science and technology, ranging from paleoclimatology [4,5], to electronic circuits [6], lasers [7], chemical systems [8,9], and the connection

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with some situations of biological interest (noise-induced information flow in sensory neurons in living systems, influence in ion-channel gating or in visual perception) [10–12]. A tendency shown in recent papers, and determined by the possible technological applications, points towards achieving an enhancement of the system response (that is: obtaining a larger output SNR) by means of the coupling of several SR units in what conforms an *extended medium* [13–18].

Majority of the studies on SR have been made analyzing a paradigmatic system: a bistable one-dimensional double-well system. Among bistable models there is one that stands out: *the two-state model* (TST) [5,19]. Such a model has proved extremely useful for the understanding of the SR phenomenon, offering also a simple framework able to provide analytical results. In almost all descriptions, and particularly within the TST, the transition rates between the two wells are estimated as the inverse of the *mean first-passage-time*, which is evaluated using standard techniques [20], and most specifically through the Kramers approximation [21]. In all cases the noises are assumed to be Gaussian [20]. However, some experimental results in sensory systems, particularly for one kind of crayfish [22] as well as recent results for rat skin [23], offer strong indications that the noise source in these systems could be non Gaussian. This point of view is supported by the results obtained in a recent contribution [24], where the study of a particular class of Langevin equations having non Gaussian stationary distribution functions [25,26] was made use of. The work in [25,26] is based on the generalized thermostatistics proposed by Tsallis [27–29] which has been successfully applied to a wide variety of physical systems [30–34]. Another noteworthy aspect is that the SNR obtained within this framework, seems to be less dependent on the precise value of the noise intensity, an aspect of great relevance from both technological and biological points of view. However, the analysis done in Ref. [24] still is not completely satisfactory.

In this contribution we analyze the case of SR when the noise source is non Gaussian. We consider the following problem

$$\dot{x} = f(x,t) + g(x)\eta(t) , \tag{1}$$

$$\dot{\eta} = -\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) + \frac{1}{\tau}\xi(t) , \tag{2}$$

where $\xi(t)$ is a Gaussian white noise of zero mean and correlation $\langle \xi(t)\xi(t') \rangle = D\delta(t-t')$, and $V_q(\eta)$ is given by [25]

$$V_q(\eta) = \frac{1}{\beta(q-1)} \ln \left[1 + \beta(q-1)\frac{\eta^2}{2} \right] , \tag{3}$$

where $\beta = \tau/D$. The function $f(x,t)$ is derived from a potential $U(x,t)$, consisting of a double well potential and a linear term modulated by $S(t) \sim F\cos(\omega t)$ ($f(x,t) = -\mathrm{d}U/\mathrm{d}x = -U_0' + S(t)$). This problem corresponds (for $\omega = 0$) to the case of diffusion in a potential $U_0(x)$, induced by η, a colored non-Gaussian noise. Clearly, when $q \to 1$ we recover the limit of η being a Gaussian colored noise (Ornstein–Uhlenbeck process [20]).

In what follows we show the form of the effective Markovian Fokker–Planck equation, obtained using an approximation based on a path integral treatment of Gaussian colored noises [35–37], we calculate its stationary probability, and we derive the expression for the first passage time (MFPT). We use the TST approach [5,19] in order to obtain the power spectral density (psd) and the SNR. These results are compared with exhaustive Monte Carlo simulations. Finally we draw some conclusions.

2. Effective Markovian approximation

Applying the path-integral formalism to the Langevin equations given by Eqs. (1) and (2), and applying an adiabatic elimination procedure [35–37] it is possible to arrive at an *effective Markovian approximation*. The specific details are shown elsewhere [38]. Such an approximation yields the following FPE for the evolution of the probability $P(x,t)$

$$\partial_t P(x,t) = -\partial_x[A(x)P(x,t)] + \tfrac{1}{2}\partial_x^2[B(x)P(x,t)]\,, \tag{4}$$

where

$$A(x) = \frac{U'}{1+\tau U_0''[1+\frac{\tau}{2D}(q-1)U_0'^2]} \tag{5}$$

and

$$B(x) = D\left[\frac{1+\frac{\tau}{2D}(q-1)U_0'^2}{1+\tau U_0''[1+\frac{\tau}{2D}(q-1)U_0'^2]}\right]^2 . \tag{6}$$

The stationary distribution of the FPE in Eq. (4) is thus

$$P^{st}(x) = \frac{\aleph}{B}\exp[-\Phi(x)]\,, \tag{7}$$

where $\aleph$ is the normalization factor, and

$$\Phi(x) = \frac{2}{D}\int\frac{A}{B}\mathrm{d}y\,. \tag{8}$$

The indicated FPE and its associated stationary distribution enable us to obtain the MFPT through a Kramers-like approximation. This quantity is the necessary ingredient to work with the TST approach.

2.1. Mean first passage time

The MFPT can be obtained, in a Kramers-like approximation [20] from

$$T(x_0) = \int_a^{x_0}\frac{\mathrm{d}y}{\Psi}\int_{-\infty}^{y}\frac{\mathrm{d}z\,\Psi}{B}\,, \tag{9}$$

where

$$\Psi(x) = \exp\left(-2\int\mathrm{d}y\frac{A}{B}\right)\,. \tag{10}$$

In order to obtain analytical results, we will focus on polynomial-like forms for the potential. We adopt

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2} + S(t)x\,, \tag{11}$$

setting $S(t) = F\cos(\omega t)$, and assuming that ω^{-1} is large compared to the characteristic relaxation times in both wells. For this kind of potential the normalization constant $\aleph$ diverges for any value of $\tau > 0$. This can be seen from the asymptotic behavior of $\Phi(x)$, for $x \to \infty : \Phi(x) \to 0$, while $B^{-1} \to \infty$, resulting in an ill defined stationary probability density in Eq. (7). In order to find approximate relations for the MFPT, and other related quantities, we assume that Eq. (4) is valid only for values of x near the wells and when the dispersion of the η process is finite, that is $\langle \eta^2 \rangle < \infty$ (or $q \in [5/3, 3)$) [38]. Such a cutoff is justified a posteriori, analyzing the probability distribution obtained from the simulations.

With the potential given by Eq. (11), $A(x)$ and $B(x)$ have the form

$$A(x) = \frac{(x^3 - x)}{1 + \tau(3x^2 - 1)[1 + \frac{\tau}{2D}(q-1)(x^3 - x)^2]}\,,$$

$$B(x) = \left\{ \frac{[1 + \frac{\tau}{2D}(q-1)(x^3 - x)^2]}{1 + \tau(3x^2 - 1)[1 + \frac{\tau}{2D}(q-1)(x^3 - x)^2]} \right\}^2 .$$

In order to obtain the MFPT and related quantities, we have integrated Eq. (9) numerically.

2.2. Two-state-theory

We consider a system described by a discrete random dynamical variable x that adopts two possible values: c_1 and $c_2 = -c_1$, with probabilities $n_{1,2}(t)$ respectively. Such probabilities satisfy the condition $n_1(t) + n_2(t) = 1$. The master equation [20] governing the evolution of $n_1(t)$ (and similarly for $n_2(t) = 1 - n_1(t)$) is

$$\begin{aligned}\frac{dn_1}{dt} = -\frac{dn_2}{dt} &= W_2(t)n_2(t) - W_1(t)n_1(t)\,,\\ &= W_2(t) - [W_2(t) + W_1(t)]n_1\,,\end{aligned} \tag{12}$$

where the $W_{1,2}(t)$ are the transition rates *out of* the $x = c_{1,2}$ states.

If the system is subject (through one of its parameters) to a time dependent signal of the form $F\cos(\omega_s t)$, up to first order in the amplitude F (assumed to be small), the transition rates may be expanded as

$$\begin{aligned} W_1(t) &= \mu_1 - \alpha_1 F\cos(\omega_s t)\,,\\ W_2(t) &= \mu_2 + \alpha_2 F\cos(\omega_s t)\,,\end{aligned} \tag{13}$$

where the constants $\mu_{1,2}$ and $\alpha_{1,2}$ depend on the detailed structure of the system under study. For the symmetric case we have $\mu_1 = \mu_2 = \mu$ and $\alpha_1 = \alpha_2 = \alpha$ [18].

Following the procedure of Ref. [19] to compute the SNR, integrating Eq. (12), we can calculate the correlation function $\langle x(t+\tau)x(t)|x_0,t_0\rangle$. From this correlation function we obtain the t-averaged correlation function $C(\tau) = \langle \lim_{t_0\to-\infty}\langle x(t+\tau)x(t)|x_0,t_0\rangle\,\rangle_t$, given by

$$C(\tau) = R_1 \exp(-\mu|\tau|) + R_2 \cos(\omega_s\tau)\,, \tag{14}$$

with

$$R_1 = 4c_1^2 + O(F^2)\,,$$

$$R_2 = \frac{2A^2c_1^2\alpha^2}{(\mu^2+\omega^2)}\,. \tag{15}$$

We compute the *t-averaged* psd ($\langle\tilde{S}(\omega)\rangle_t$) as the Fourier transform of $C(\tau)$. Next, we compute the *one-sided* t-averaged psd ($S(\omega)$), defined for $\omega > 0$, as $S(\omega) = \langle\tilde{S}(\omega)\rangle_t + \langle\tilde{S}(-\omega)\rangle_t$. We finally get

$$S(\omega) = 4R_1\frac{\mu}{(\mu^2+\omega^2)} + 2\pi R_2\delta(\omega-\omega_s)\,. \tag{16}$$

In the one-sided t-averaged psd (Eq. (16)), two contributions can be distinguished: the signal output which is given by the δ function centered at the signal frequency and the broadband noise output, given by a dominant ($O(A^0)$) Lorentzian term plus some less important ($O(A^2)$) terms that have been neglected.

We define R, the SNR, as the ratio of the strength of the output signal and the broadband noise output evaluated at the signal frequency, obtaining

$$R = \frac{\pi R_2}{R_1(2\mu/(\mu^2+\omega_s^2))} = \frac{F^2\pi\alpha^2}{2\mu}\,. \tag{17}$$

According to Eqs. (9) and (13), the relevant quantities μ and α are given by

$$\mu = \frac{1}{T}_{|S(t)=0} \tag{18}$$

$$\alpha = \left[\frac{1}{T^2}\frac{dT}{dS(t)}\right]_{S(t)=0}\,, \tag{19}$$

with $S(t)$ the applied signal.

2.3. Theoretical results

Here, we present the results for the SNR obtained evaluating Eq. (17). We show results for R, the SNR, as a function of D, the noise intensity, for two different situations: fixed q and several τ, fixed τ and various q.

In Fig. 1 we depict R vs. D, for a fixed value of the time correlation τ ($\tau = 0.1$) and various q. The general trend is that the maximum of the SNR curve increases when $q < 1$, this is when the system departs from the Gaussian behavior. Fig. 2 again shows R vs. D, but for a fixed value of q ($q = 0.75$) and several values of τ. The general trend agrees with the results for colored Gaussian noises [39], where it was shown that the

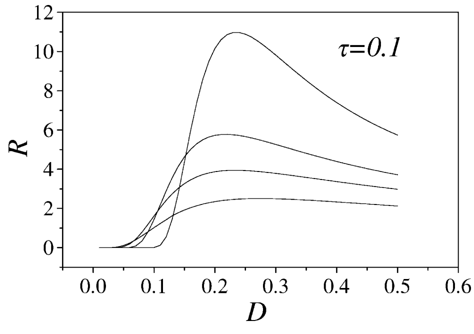

Fig. 1. Theoretical value of SNR vs. D, for $\tau = 0.1$ and the following values of $q = 0.25, 0.75, 1.0, 1.25$ (from top to bottom).

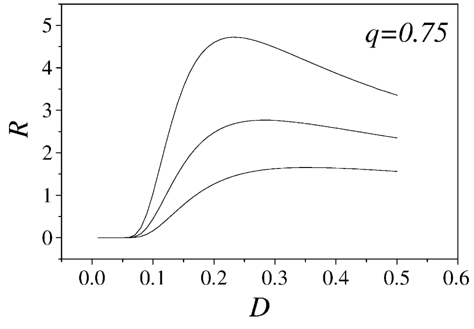

Fig. 2. Theoretical value of SNR vs. D, for $q = 0.75$ and the following values of $\tau = 0.25, 0.75, 1.5$ (from top to bottom).

increase of the correlation time induces a decrease of the maximum of SNR as well as its shift towards larger values of the noise intensity. The latter fact is a consequence of the suppression of the switching rate with increasing τ. Both qualitative trends are confirmed by Monte Carlo simulations of the system in Eq. (1).

2.4. Simulations

We have integrated Eqs. (1) and (2) numerically using the Heun method. With only a few exceptions, in most cases the results were obtained averaging over 2000 trajectories (5000 trajectories for $\tau = 0$).

Fig. 3 shows the simulation results for the same situation and parameters indicated in Fig. 1. Here, in addition to the increase of the maximum of the SNR curve for values of $q < 1$, we see also an aspect that is not well reproduced or predicted by the effective Markovian approximation. It is the fact that the maximum of the SNR curve flattens for lower values of q, indicating that the system, when departing from Gaussian behavior, does not require a fine tuning of the noise intensity in order to maximize its response to a weak external signal. Fig. 4 shows the simulation results for the same

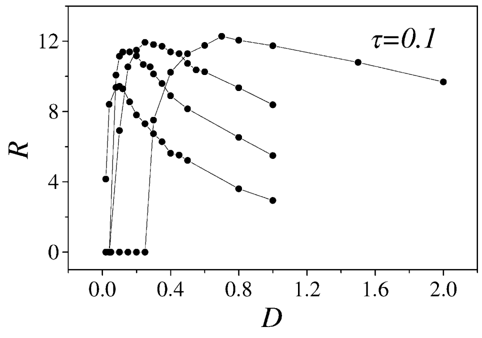

Fig. 3. Simulation results of SNR vs. D, for $\tau = 0.1$ and the following values of $q = 0.25, 0.75, 1.0, 1.25$ (from top to bottom).

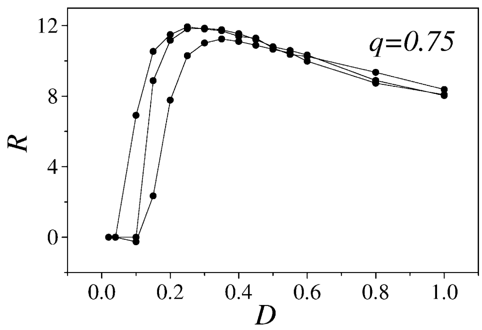

Fig. 4. Simulation results of SNR vs. D, for $q = 0.75$ and the following values of $\tau = 0.25, 0.75, 1.5$ (from top to bottom).

situation and parameters indicated in Fig. 2. Again we found an agreement with the behavior found for colored Gaussian noises [39].

We have also obtained the spectral amplification factor η [40]. The results not shown here, are in complete agreement with those for the SNR.

3. Conclusions

In this contribution, and motivated by some experimental results in sensory systems [22,23], we have analyzed the problem of SR when the noise source is non Gaussian. We have chosen a colored non Gaussian noise source with a probability distribution based on the generalized thermostatistics [27–29]. Making use of a path integral approach, we have obtained an effective Markovian approximation that allows us to get some analytical results. In addition, we have performed exhaustive Monte Carlo simulations. Even though the agreement between theory and numerical simulations is only partial and qualitative, the effective Markovian approximation turns out to be extremely useful to (qualitatively) predict general trends in the behavior of the system under study.

Our numerical and theoretical results indicate that:

- For a fixed value of τ, the maximum value of the SNR increases with decreasing q.
- For a given value of q, the optimal noise intensity (that one that maximizes SNR) decreases with q and its value is approximately independent of τ.
- For a fixed value of the noise intensity, the optimal value of q is independent of τ (except for $\tau = 0$ where the only allowed value is $q = 1$) and in general it turns out that $q \neq 1$.

In general terms we observe that the SNR, as we depart from Gaussian behavior (with $q < 1$), shows two main aspects: firstly its maximum as a function of the noise intensity increases, secondly it becomes less dependent on the precise value of the noise intensity. Both aspects are of great relevance for technological applications [3]. However, as was indicated in Ref. [23], non Gaussian noises could be an intrinsic characteristic in biological systems, particularly in sensory systems [10,22,23]. In addition to the increase in the response (SNR), the reduction in the need for *tuning* a precise value of the noise intensity is of particular relevance in order to understand how a biological system can exploit this phenomenon. The present results indicate that the noise model used here offers an adequate framework to analyze such a problem. A detailed analysis and comparison with experimental data will be the subject of further work.

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References

[1] F. Moss et al. (Eds.), Proceedings of the NATO Advance Research Workshop on Stochastic Resonance in Physics and Biology, J. Stat. Phys. 70 (1/2) (1993).
[2] A. Bulsara et al. (Eds.), Proceedings of the Second International Workshop on Fluctuations in Physics and Biology, Nuovo Cim. D 17 (1995).
[3] L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. 70 (1988) 223.
[4] R. Benzi, A. Sutera, A. Vulpiani, J. Phys. A 14 (1981) L453.
[5] C. Nicolis, Tellus 34 (1982) 1.
[6] R.N. Mantegna, B. Spagnolo, Phys. Rev. E 49 (1994) R1792.
[7] J.M. Iannelli, A. Yariv, T.R. Chen, Y.H. Zhuang, Appl. Phys. Lett. 65 (1994) 1983.
[8] A. Guderian, G. Dechert, K. Zeyer, F. Schneider, J. Phys. Chem. 100 (1996) 4437.
[9] V. Petrov, Q. Ouyang, H.L. Swinney, Nature 388 (1997) 655.

[10] J.K. Douglas, L. Wilkens, E. Pantazelou, F. Moss, Nature 365 (1993) 337.
[11] S.M. Bezrukov, I. Vodyanoy, Nature 378 (1995) 362.
[12] E. Simonotto, M. Riani, C. Seife, M. Roberts, J. Twitty, F. Moss, Phys. Rev. Lett. 78 (1997) 1186.
[13] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, A. Bulsara, Phys. Rev. Lett. 75 (1995) 3.
[14] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, A. Bulsara, Phys. Rev. E 53 (1996) 2081.
[15] F. Castelpoggi, H.S. Wio, Europhysics Letters 38 (1997) 91.
[16] F. Castelpoggi, H.S. Wio, Phys. Rev. E 57 (1998) 5112.
[17] M. Kuperman, H.S. Wio, G. Izús, R. Deza, F. Castelpoggi, Physica A 257 (1998) 275.
[18] S. Bouzat, H.S. Wio, Phys. Rev. E 59 (1999) 5142.
[19] B. McNamara, K. Wiesenfeld, Phys. Rev. A 39 (1989) 4854.
[20] C.W. Gardiner, Handbook of Stochastic Methods, 2nd Edition, Springer, Berlin, 1985.
[21] P. Hänggi, P. Talkner, M. Borkovec, Rev. Mod. Phys. 62 (1990) 251.
[22] K. Wiesenfeld, D. Pierson, E. Pantazelou, Ch. Dames, F. Moss, Phys. Rev. Lett. 72 (1994) 2125.
[23] D. Nozaki, D.J. Mar, P. Grigg, J.J. Collins, Phys. Rev. Lett. 72 (1999) 2125.
[24] H.S. Wio, S. Bouzat, Brazil. J. Phys. 29 (1999) 136.
[25] L. Borland, Phys. Lett. A 245 (1998) 67.
[26] L. Borland, Phys. Rev. E 57 (1998) 6634.
[27] E.M.F. Curado, C. Tsallis, J. Phys. A 24 (1991) L69.
[28] E.M.F. Curado, C. Tsallis, J. Phys. A 24 (1991) 3187.
[29] E.M.F. Curado, C. Tsallis, J. Phys. A 25 (1992) 1019.
[30] A.R. Plastino, A. Plastino, Phys. Lett. A 174 (1993) 384.
[31] D.H. Zanette, P.A. Alemany, Phys. Rev. Lett. 75 (1995) 366.
[32] B.M. Boghosian, Phys. Rev. E 53 (1996) 4754.
[33] C. Tsallis, A.M.C. de Souza, Phys. Lett. A 235 (1997) 444.
[34] G. Drazer, H.S. Wio, C. Tsallis, Phys. Rev. E 61 (2000) 1417.
[35] H.S. Wio, P. Colet, L. Pesquera, M.A. Rodriguez, M. San Miguel, Phys. Rev. A 40 (1989) 7312.
[36] F. Castro, A. Sanchez, H.S. Wio, Phys. Rev. Lett. 75 (1995) 1691.
[37] H.S. Wio, Application of path integration to stochastic processes: an introduction, in: G. Zgrablich (Ed.), Fundamentals and Applications of Complex Systems, Nueva Ed. Univ., U.N. San Luis, 1999, p. 253.
[38] M.A. Fuentes, Raúl Toral, Horacio S. Wio, Effective Markovian approximation for non Gaussian noises: a path integral approach, submitted to Physica A.
[39] L. Gammaitoni, E. Manichella-Saetta, S. Santucci, F. Marchesoni, C. Presilla, Phys. Rev. A 40 (1989) 2114.
[40] P. Jung, P. Hänggi, Phys. Rev. A 44 (1991) 8032.