Equilibration time scales of physically relevant observables

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Joint work with Noah Linden, Artur Malabarba, Tony Short, and Andreas Winter.

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  does some system approach a particular steady state, and remain close to it, when probed by an observable A?

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- As "distance" between states: *weak-distinguishability*:

$$\widetilde{\mathcal{D}}_A(\rho,\sigma) \equiv \frac{1}{4\|A\|^2}\Big|\mathrm{Tr}[\rho A] - \mathrm{Tr}[\sigma A]\Big|^2$$

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- if $\widetilde{\mathcal{D}}_A(\rho,\sigma) \ll 1 \implies$ hard to distinguish between ρ and σ via expectation values of A

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- Hence, we look for *equilibration in average*:
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  (as measured by $\widetilde{\mathcal{D}}_A$)
- So, our criterion for equilibration:

$$\left\langle \widetilde{\mathcal{D}}_A(\rho_t, \omega) \right\rangle_\infty \ll 1$$

  where $\langle f(t) \rangle_T = \frac{1}{T} \int_0^T f(t) dt$ denotes time averaging
- In such a case $\widetilde{\mathcal{D}}_A(\rho_t, \omega)$ is small for most times t

Conditions for equilibration

- It has been proven that the average distance is bounded[1,2]

$$\left\langle \widetilde{\mathcal{D}}_A(\rho_t, \omega) \right\rangle_\infty \leq \frac{1}{4 d_{\text{eff}}}$$

- where $\omega \equiv \langle \rho_t \rangle_\infty$

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- and the effective dimension of the system is

$$d_{\mathrm{eff}} \equiv \frac{1}{\sum_j \left(\mathrm{Tr}\left[P_j \rho_0\right]\right)^2} \qquad \left(d_{\mathrm{eff}} \sim \text{number levels populated}\right)$$

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- Assumption: non-degenerate energy gaps

$$(E_j - E_k) = (E_n - E_m) \neq 0 \Longleftrightarrow j = n \text{ and } k = m$$

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Conditions for equilibration

- Therefore, assuming *non degenerate energy gaps* and *high effective dimension*

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What can we say, in these general terms, about the time scales of equilibration?

Finite time equilibration

- We now look for equilibration in a finite time T, i.e. focus on

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- we want to find T_{eq} such that

$$\left\langle \widetilde{\mathcal{D}}_A(\rho_t,\omega) \right\rangle_{T_{\text{eq}}} \ll 1$$

$\Rightarrow \quad \rho_t$ close to ω for most times in $[0, T_{\text{eq}}]$

Extremely slow observables

- It turns out that one can find examples of observables with extremely slow equilibration time scales[3]
- For any pure initial state there exist observables with[4]

$$T_{\text{eq}}^{slow} \gtrsim \frac{d_{\text{eff}}}{\sigma_E},$$

σ_E – energy standard deviation

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- Typically, if N is the number of constituents of the system, $\sigma_E \sim \mathcal{O}\left(\sqrt{N}\right)$ while $d_{\text{eff}} \sim \mathcal{O}(\exp N)$
  $\implies$ time scales exponentially long in system's size

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Therefore conditions are needed on the trio $\{\rho_0, A, H\}$ in order to prove physically reasonable time scales

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The search for realistic time scales

- We can prove

$$\left\langle \widetilde{\mathcal{D}}_A(\rho_t,\omega)\right\rangle_T \leq c\, d\, \mathrm{Tr}\left[\rho_0^2\right]\xi_p\left(\tfrac{1}{T}\right); \qquad c < 5, \quad d = \dim(\mathcal{H}).$$

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- For $H = \sum_j E_j |j\rangle\langle j|$, consider the normalized distribution

$$p_\alpha \equiv p_{\{j,k\}} = \frac{|\rho_{jk} A_{kj}|}{\sum_{jk} |\rho_{jk} A_{kj}|}$$

α denotes pairs of levels with energy gap $G_\alpha = (E_j - E_k)$

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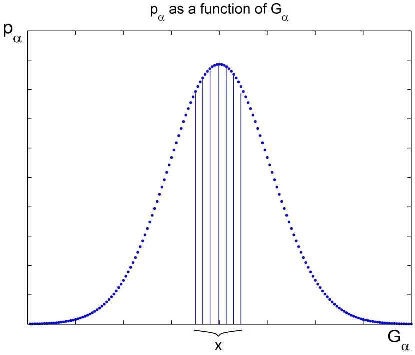
- Notice that p_α involves all the operators that define time scales: $\{\rho_0, A, H\}$
- p_α gives us information about the relevant frequencies (energy gaps)
- For example, its standard deviation σ_G roughly gives the maximum relevant frequency

- Out of the distritubion p_α, we construct the function $\xi_p(x)$:

$$\xi_p(x) \equiv \max_G \sum_{G_\alpha \in [G, G+x]} p_\alpha$$

- $\xi_p(x)$ quantifies *the maximum probability that fits* x:



- We have distribution $p_\alpha \propto |\rho_{jk} A_{jk}|$, the function $\xi_p(x)$ which quantifies the probability that fits an interval x, and

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For example, T_{eq} such that

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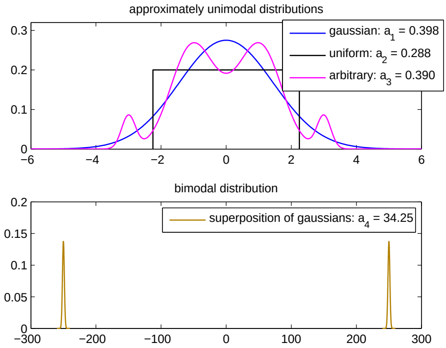
- Estimate for $\xi_p\left(\frac{1}{T}\right)$: if the standard deviation σ_G quantifies the width of distribution p_α, then

$$\xi_p\left(\tfrac{1}{T}\right) \approx \frac{a}{\sigma_G} \frac{1}{T}, \qquad a \sim 1.$$

- A few illustrating examples, for continuous distributions:



$$\left(\xi_p\left(\frac{1}{T}\right) = \frac{a}{\sigma_G}\frac{1}{T}\right)$$

$$(a \gg 1)$$

- Putting the equations together

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gives

$$T_{\mathrm{eq}} \approx \frac{c\,a}{10^{-4}} \frac{d\,\mathrm{Tr}\left[\rho_0^2\right]}{\sigma_G}, \qquad c < 5,\ a \sim 1.$$

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- We've found a case in which T_{eq} *does not grow with bath* $\Rightarrow$ *a lot better than the slow time scale scenario of* $T_{\text{eq}} \propto d$
- Moreover, up to constant prefactors, for small systems T_{eq} scales like the fastest relevant time scale $\frac{1}{\sigma_G}$

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- So prefactor $\left(d\,\mathrm{Tr}\left[\rho_0^2\right]\right)$ does not scale with d for a bath in the maximally mixed state – i.e. infinite temperature case – What about the finite temperature case?

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- By truncating the Hilbert space, we prove that for a bath with *exponential density of states*

$$T_{\mathrm{eq}} \approx \frac{c\,a}{10^{-4}} \frac{d_{\mathrm{trunc}}\,\mathrm{Tr}\left[\rho_0^2\right]}{\sigma_G}$$

$$d_{\mathrm{trunc}}\,\mathrm{Tr}\left[\rho_0^2\right] \lesssim d_S\,\mathrm{Tr}\left[\rho_S^2\right]\mathcal{O}\left(\exp\left[\beta\|H_S\| + (1+\sqrt{2d_s})K\beta\|H_I\|\right]\right),$$

where K controls error in truncating the space

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T_{eq} does not scale with size of the bath!

Truncating the Hilbert space

- Take the bath in the *microcanonical ensemble*: energy window of width Δ, centered around E_B, state

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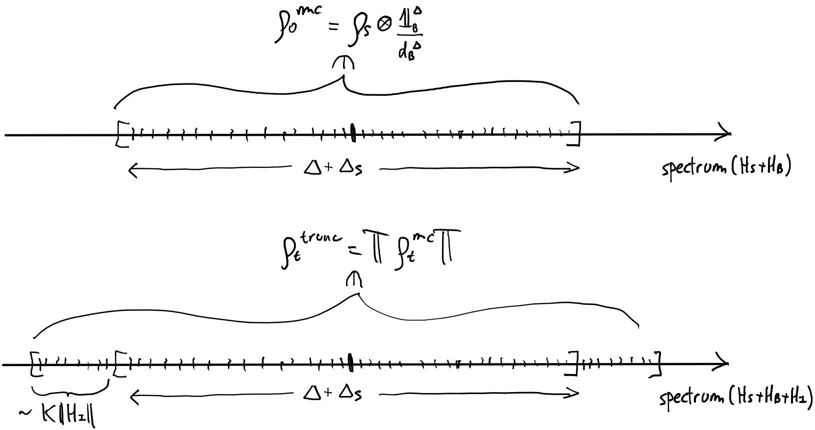
- But, one does not need to consider the full Hilbert space...

- ρ_0^{mc} is contained in an energy window of $(H_S + H_B)$.
- Including H_I can take the state out of the window
- Still, one can truncate the Hilbert space to a window of $(H_S + H_B + H_I)$, with $\|\rho_t^{\mathrm{mc}} - \Pi\rho_t^{\mathrm{mc}}\Pi\|_1 \leq \frac{2}{K}$



The full result

- We now give the full, exact result:

$$\left\langle \widetilde{\mathcal{D}}_A\left(\rho_t^{mc}, \omega^{mc}\right) \right\rangle_T \leq c\, a \frac{d_{\text{trunc}}\, \text{Tr}\left[\rho_0^2\right]}{T \sigma_G} + c\, \delta\, d_{\text{trunc}}\, \text{Tr}\left[\rho_0^2\right] + \frac{18}{K^2}$$

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- For approximately unimodal distributions p_α spread over many values one finds that $a \sim 1$ and $\delta \ll 1$
- Finally, T_{eq} can be expressed in terms of $\{\rho_0, A, H\}$ from

$$\frac{1}{\sigma_G} \leq \frac{\left(d_{\text{trunc}} \operatorname{Tr}\left[\rho_0^2\right]\right)^{1/4}}{\sqrt{\left|\operatorname{Tr}\left(\left[[\rho_0, H], H\right] A\right)\right|}}$$

Conclusion

- There exist observables with extremely long equilibration time scales $T_{\mathrm{eq}} \sim \frac{d}{\sigma_E}$
$\Longrightarrow$ further conditions are needed in order to prove physically realistic equilibration times
- We found conditions over $\{\rho_0, A, H\}$ (more specifically over $p_\alpha \propto |\rho_{jk} A_{kj}|$) which ensure equilibration time scales that do not scale with the dimension of the space
- future work – can we prove that the conditions hold for certain class of systems (e.g. spin lattices with local interactions?)

Thank you!

EXTRAS - distinguishability

- Stricter notion: measurement outcomes $\{a_1, \ldots, a_N\}$, with associated operators $\mathcal{M} = \{M_1, \ldots, M_N\}$.
  The *distinguishability* is defined

$$\mathcal{D}_{\mathcal{M}}(\rho, \sigma) = \frac{1}{2} \sum_{j}^{N} \Big| \mathrm{Tr}\big[\rho M_j\big] - \mathrm{Tr}\big[\sigma M_j\big] \Big|$$

- One can see

$$\left\langle \mathcal{D}_{\mathcal{M}}(\rho_t, \omega) \right\rangle_T \leq \sqrt{N} \sqrt{\sum_j \left\langle \widetilde{\mathcal{D}}_{M_j}(\rho_t, \omega) \right\rangle_T}.$$

Each term $\left\langle \widetilde{\mathcal{D}}_{M_j}(\rho_t, \omega) \right\rangle_T$ can be bounded as shown.