Minimising the heat dissipation of information erasure

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MARIE CURIE ACTIONS

Overview

- The optimal unitary operator for probabilistic information erasure
- Examples: Maximally erasing a qubit with no a priori information
- Self-consistent Information erasure “beyond Landauer”

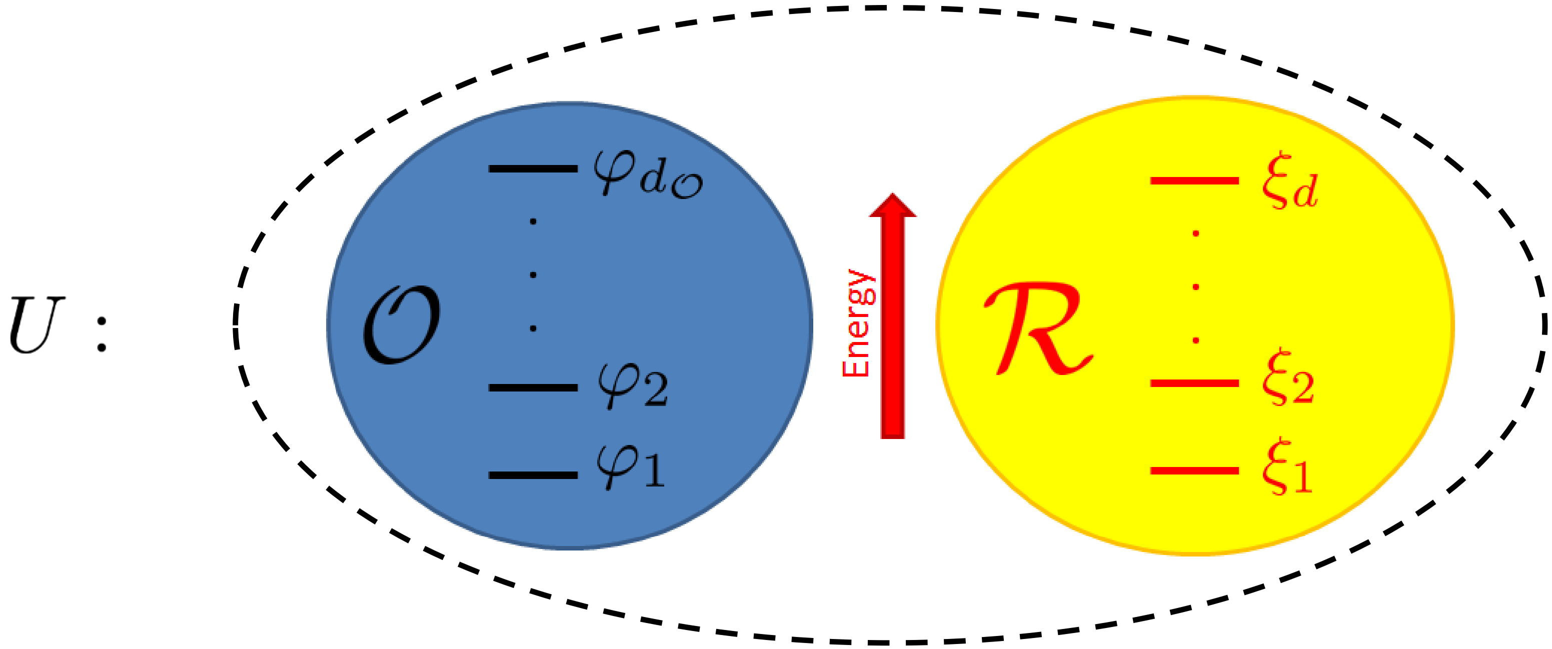
THE OPTIMAL UNITARY OPERATOR FOR PROBABILISTIC INFORMATION ERASURE

Information erasure as purification

	Classical Physics	Quantum Physics
Information erasure	Many-to-one mapping on configuration space $\Omega \mapsto \omega_1$	Many-to-one mapping on Hilbert space $\mathcal{H} \mapsto \|\varphi_1\rangle$
Landauer's limit	$\beta \Delta Q \geqslant \Delta S$	$\beta \Delta Q \geqslant \Delta S + \frac{2(\Delta S)^2}{\log^2(d-1)+4}$ NJP vol. 16, no. 10, p. 103011, 2014

- These lower bounds are reachable for *some* physical setting, but not all

The physical setting



- Reservoir has the Hamiltonian $H_\mathcal{R}$
- Object and reservoir initially uncorrelated $\rho = \rho_\mathcal{O} \otimes \rho_\mathcal{R}(\beta)$
- Global unitary on system $\rho \mapsto U\rho U^\dagger =: \rho'$
- Heat dissipation in reservoir: $\Delta Q := \mathrm{tr}[H_\mathcal{R}(\rho'_\mathcal{R} - \rho_\mathcal{R}(\beta))]$

Maximising the probability of information erasure

$$\rho_{\mathcal{O}} = \sum_{l=1}^{d_{\mathcal{O}}} q_l |\varphi_l\rangle\langle\varphi_l| \ , \ \rho_{\mathcal{R}}(\beta) = \sum_{m=1}^{d} p_m |\xi_m\rangle\langle\xi_m|$$

$$\rho = \sum_{l,m} q_l p_m |\varphi_l\rangle\langle\varphi_l| \otimes |\xi_m\rangle\langle\xi_m| \equiv \sum_{n=1}^{d_{\mathcal{O}} d} \tilde{p}_n |\psi_n\rangle\langle\psi_n|$$

- All probability sets, such as $\{\tilde{p}_n\}_n$, are decreasing sets

$$p(\varphi_1|\rho'_{\mathcal{O}}) := \sum_{n=1}^{d_{\mathcal{O}} d} \tilde{p}_n \langle\psi_n| U^\dagger (|\varphi_1\rangle\langle\varphi_1| \otimes \mathbb{1}_{\mathcal{R}}) U |\psi_n\rangle \leqslant \sum_{m=1}^{d} \tilde{p}_m$$

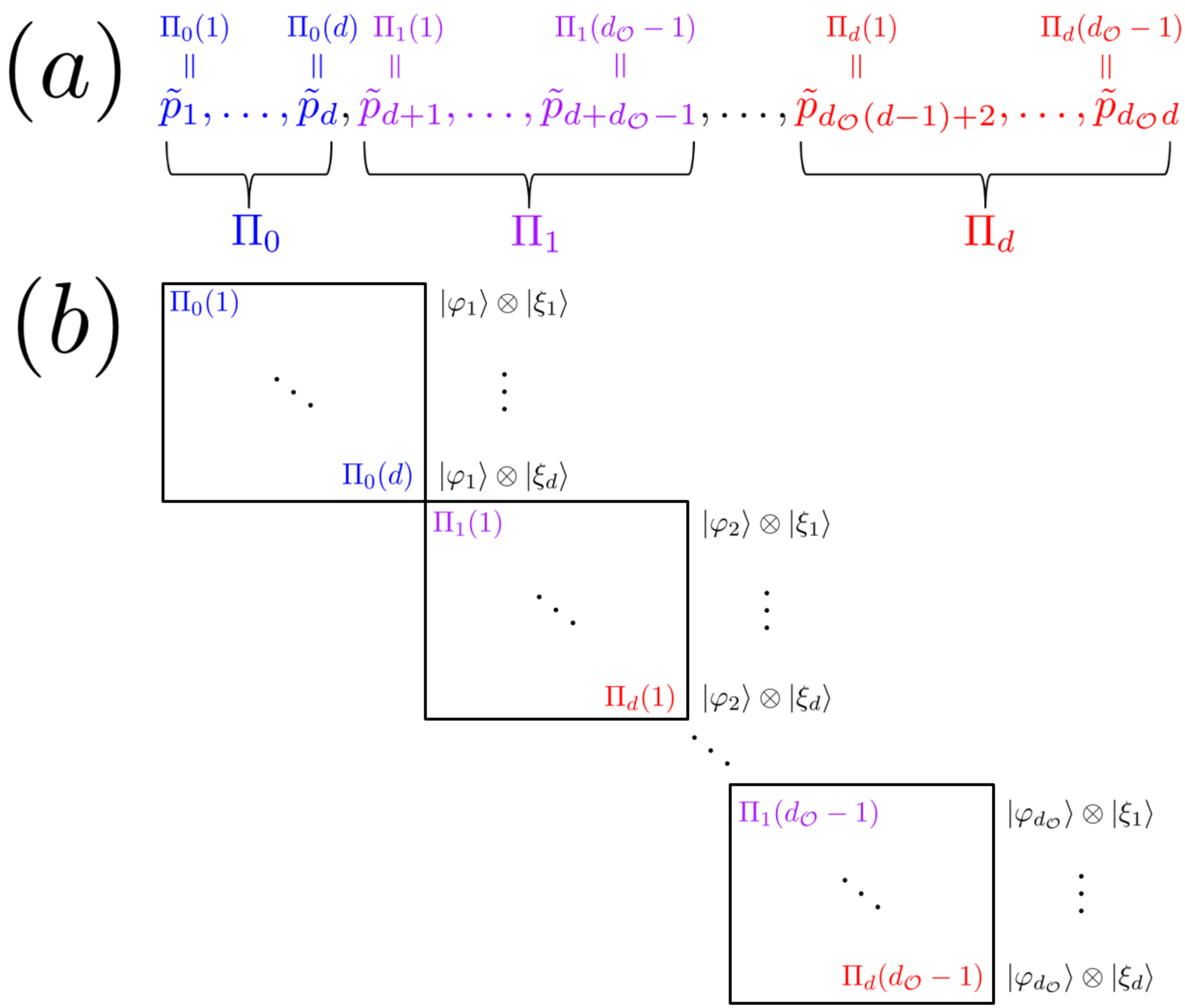
- $p_{\varphi_1}^{\max}$ when for all m $\in \{1, \ldots, d\}$, $U|\psi_m\rangle = |\varphi_1\rangle \otimes |\xi'_m\rangle$

Minimising the heat dissipation

$$\rho'_{\mathcal{R}} = \sum_{m=1}^{d} p'_m |\xi'_m\rangle\langle\xi'_m|, \quad \mathrm{tr}[H_{\mathcal{R}}\rho'_{\mathcal{R}}] = \sum_{m=1}^{d} p'_m \langle\xi'_m|H_{\mathcal{R}}|\xi'_m\rangle$$

- For arbitrary decreasing set $\{p'_m\}_m$, $\mathrm{tr}[H_{\mathcal{R}}\rho'_{\mathcal{R}}]$ is minimised when, for all m, $|\xi'_m\rangle = |\xi_m\rangle$
- To minimise $\mathrm{tr}[H_{\mathcal{R}}\rho'_{\mathcal{R}}]$, we must majorise $\{p'_m\}_m$
- For all $m \in \{1, \ldots, d\}$, $n \in \{(m-1)d_{\mathcal{O}} + 1, \ldots, md_{\mathcal{O}}\}$, $U|\psi_n\rangle = |\varphi_l^m\rangle \otimes |\xi_m\rangle$

Minimising the heat dissipation for maximal information erasure



Trade-off between probability of information erasure and heat dissipation

- Require that $p(\varphi_1|\rho'_{\mathcal{O}}) \geqslant p^{\max}_{\varphi_1} - \delta$ for $\delta \in [0, p^{\max}_{\varphi_1} - q_1]$
- Optimal case: $U|\psi_n\rangle = \sum_i \alpha_i^n |\phi_i\rangle \otimes |\xi_i\rangle$
- Entanglement increases entropy, so best when $U|\psi_n\rangle$ are separable
- For an increasing set $\{\delta_j\}_j$, with decreasing set $\{\Delta Q_j\}_j$, swap subset of Π_0 with those of $\Pi_{m\geqslant 1}$, and permute them to preserve ordering structure
- To allow for continuous δ, replace SWAP with SWAP_γ

$$\mathrm{SWAP}_\gamma : \begin{cases} |\varphi_1\rangle \otimes |\xi_{d-i}\rangle \mapsto \sqrt{1-\gamma}\,|\varphi_1\rangle \otimes |\xi_{d-i}\rangle + \sqrt{\gamma}\,|\varphi_{l+1}\rangle \otimes |\xi_m\rangle\,, \\ |\varphi_{l+1}\rangle \otimes |\xi_m\rangle \mapsto \sqrt{\gamma}\,|\varphi_1\rangle \otimes |\xi_{d-i}\rangle - \sqrt{1-\gamma}\,|\varphi_{l+1}\rangle \otimes |\xi_m\rangle\,, \end{cases}$$

$$\frac{1}{3}\begin{pmatrix} p_1 \\ p_1 \\ p_1 \\ \hline p_2 \\ p_2 \\ p_3 \\ \hline p_2 \\ p_3 \\ p_3 \end{pmatrix} \Rightarrow \frac{1}{3}\begin{pmatrix} p_1 \\ p_1 \\ p_2 \\ \hline p_1 \\ p_2 \\ p_3 \\ \hline p_2 \\ p_3 \\ p_3 \end{pmatrix} \Rightarrow \frac{1}{3}\begin{pmatrix} p_1 \\ p_1 \\ p_3 \\ \hline p_1 \\ p_2 \\ p_3 \\ \hline p_2 \\ p_2 \\ p_3 \end{pmatrix} \Rightarrow \frac{1}{3}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \hline p_1 \\ p_2 \\ p_3 \\ \hline p_1 \\ p_2 \\ p_3 \end{pmatrix} \begin{matrix} |\varphi_1\rangle \otimes |\xi_1\rangle \\ |\varphi_1\rangle \otimes |\xi_2\rangle \\ |\varphi_1\rangle \otimes |\xi_3\rangle \\ |\varphi_2\rangle \otimes |\xi_1\rangle \\ |\varphi_2\rangle \otimes |\xi_2\rangle \\ |\varphi_2\rangle \otimes |\xi_3\rangle \\ |\varphi_3\rangle \otimes |\xi_1\rangle \\ |\varphi_3\rangle \otimes |\xi_2\rangle \\ |\varphi_3\rangle \otimes |\xi_3\rangle \end{matrix}$$

$$\delta_1 = 0 \qquad \delta_2 = \frac{p_1 - p_2}{3} \qquad \delta_3 = \frac{p_1 - p_3}{3} \qquad \delta_4 = \frac{2p_1 - p_2 - p_3}{3}$$

$$\mathcal{H}_{\mathcal{O}} \simeq \mathcal{H}_{\mathcal{R}} \simeq \mathbb{C}^3, \quad \rho = \frac{1}{3}\mathbb{1}_{\mathcal{O}} \otimes \rho_{\mathcal{R}}(\beta)$$

Effect of energy conserving, Markovian dephasing

- Hamiltonian cycle $H_{\mathcal{O}} + H_{\mathcal{R}} \Rightarrow H_1 \Rightarrow H_{\mathcal{O}} + H_{\mathcal{R}}$

$$U = e^{-i\tau H_1}$$

- ΔQ is the energy lost from a battery as a result of U
- System with dephasing with respect to the eigenbasis of H_1:

$$\mathcal{V} = e^{\tau \mathscr{L}_1}$$

$$\mathscr{L}_1 : \rho \mapsto i[\rho, H_1]_- + \Gamma \sum_{n=1}^{d_{\mathcal{O}} d} \left(|\phi_n^1\rangle\langle\phi_n^1|\rho|\phi_n^1\rangle\langle\phi_n^1| - \frac{1}{2}[\rho, |\phi_n^1\rangle\langle\phi_n^1|]_+ \right)$$

- Environment does not exchange energy with system $\implies$ ΔQ can still be interpreted as energy lost from a battery

EXAMPLES: MAXIMALLY ERASING A QUBIT WITH NO A PRIORI INFORMATION

The set-up

- $\mathcal{H}_{\mathcal{O}} \simeq \mathbb{C}^2 \quad , \quad \rho_{\mathcal{O}} = \frac{1}{2}\mathbb{1}_{\mathcal{O}}$

- Consider two reservoir types: a spin chain of length N, and a d-dimensional subspace of a harmonic oscillator

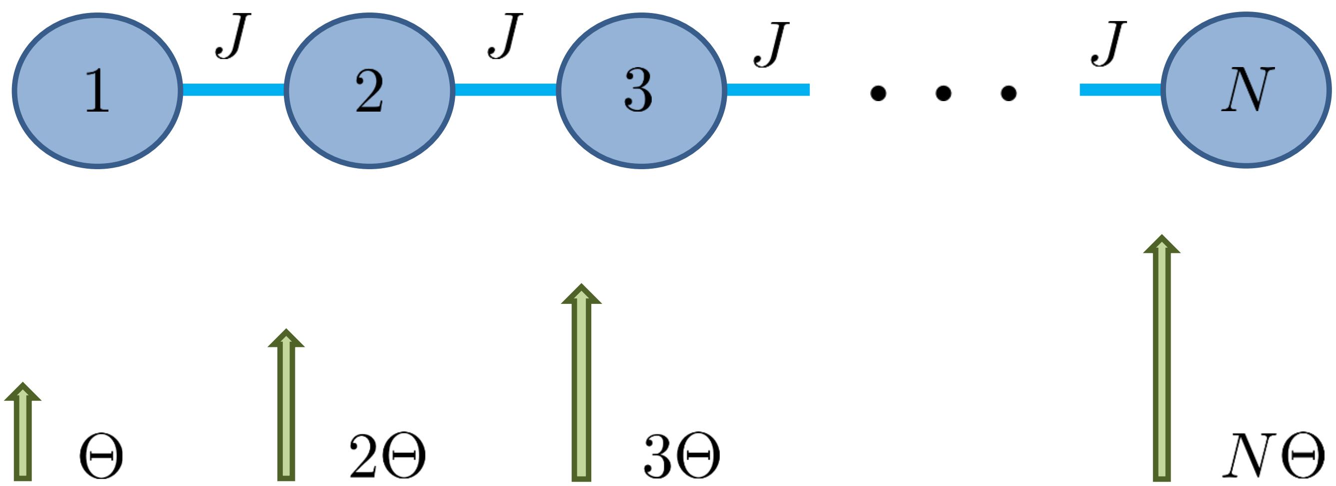
- In each case we evaluate $p_{\varphi_1}^{\max}$ and

$$\Delta L := \Delta Q - \frac{1}{\beta}\left(\Delta S + \frac{2(\Delta S)^2}{\log^2(d-1)+4}\right)$$

- Also, we consider the effect of energy-conserving, Markovian dephasing

Example 1: Reservoir as a spin chain

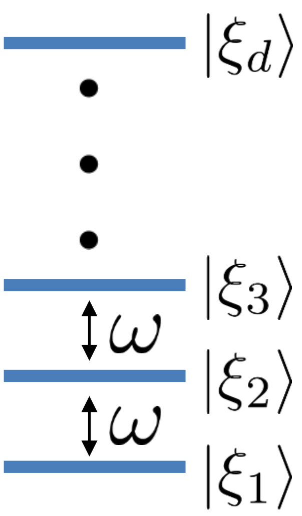
$$H_{\mathcal{R}} = \sum_{k=1}^{N} (k\Theta)\sigma_z^k + J \sum_{k=1}^{N-1} \sum_{a\in\{x,y,z\}} \sigma_a^k \otimes \sigma_a^{k+1}$$



Example 2: Reservoir as d lowest energy levels of a single-mode harmonic oscillator of frequency ω

$$\lim_{d\to\infty} p_{\varphi_1}^{\max} = 1$$

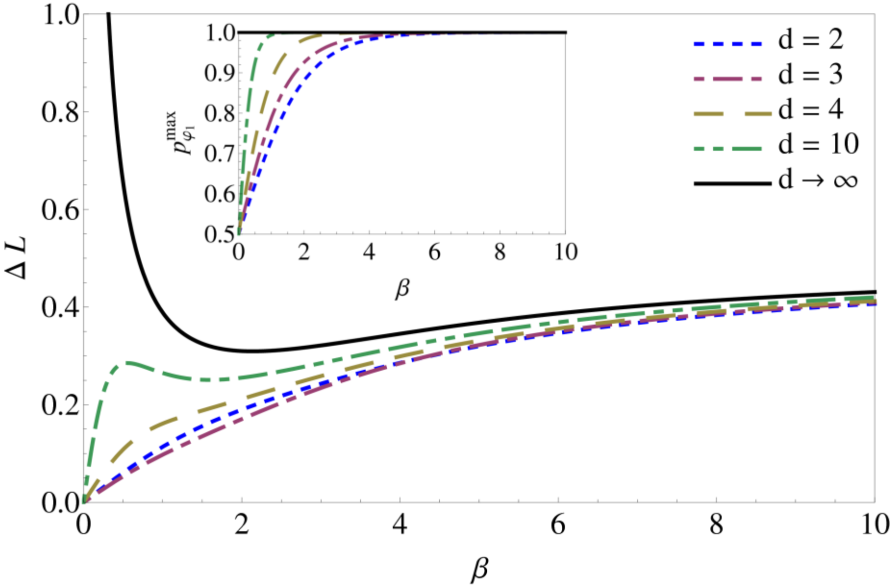
$$\lim_{d\to\infty} \Delta Q = \frac{\omega}{2}\coth\left(\frac{\beta\omega}{2}\right) = k_B T + \epsilon$$



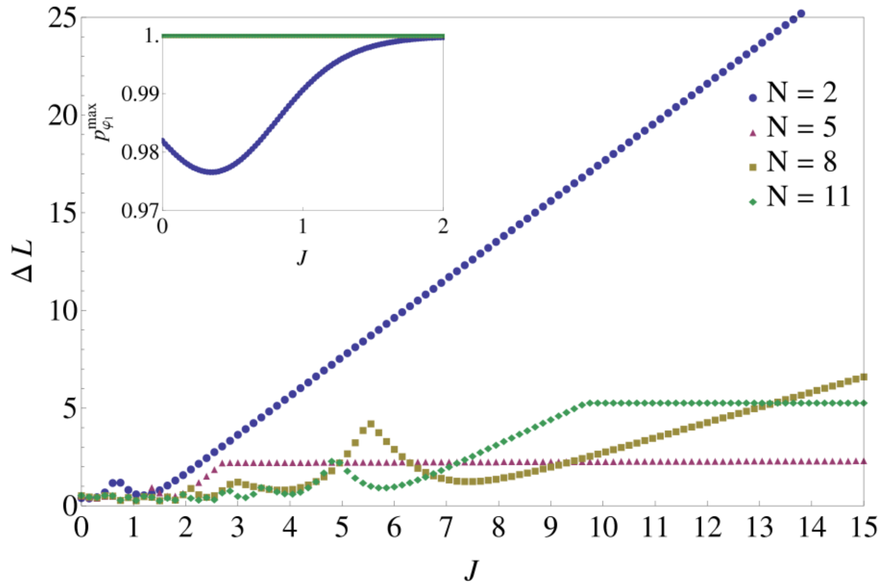
- Optimal case when spectrum of harmonic oscillator is approximately continuous

Comparison between two models: Unitary case

harmonic oscillator



spin chain



$\Theta = \beta = 1$

- In the unitary case, spin chain out-performs harmonic oscillator:

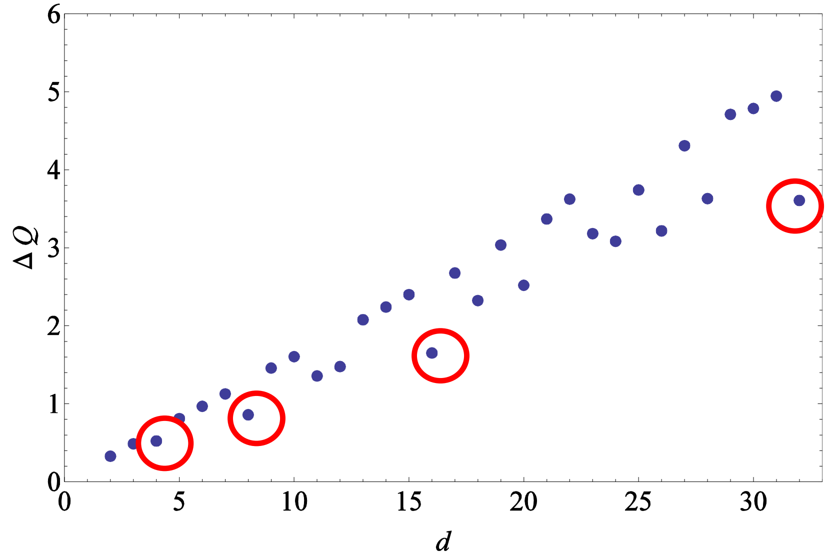
$(N = 11, J = \beta = 1, \Theta = 0.25) \implies p_{\varphi_1}^{\max} \approx 1$ and $\Delta L \approx 0.12$

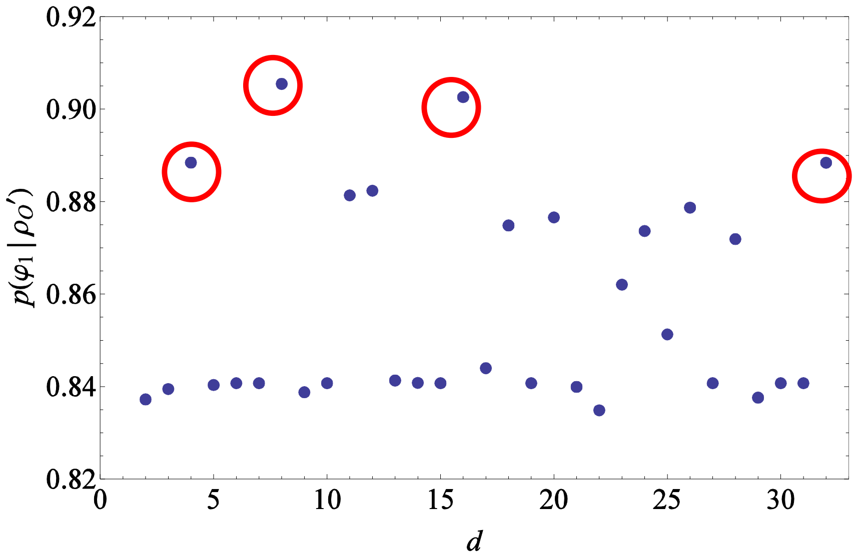
while

$(d = 2^{11}, \beta = 1, \omega = 0.1) \implies p_{\varphi_1}^{\max} \approx 1$ and $\Delta L \approx 0.29$

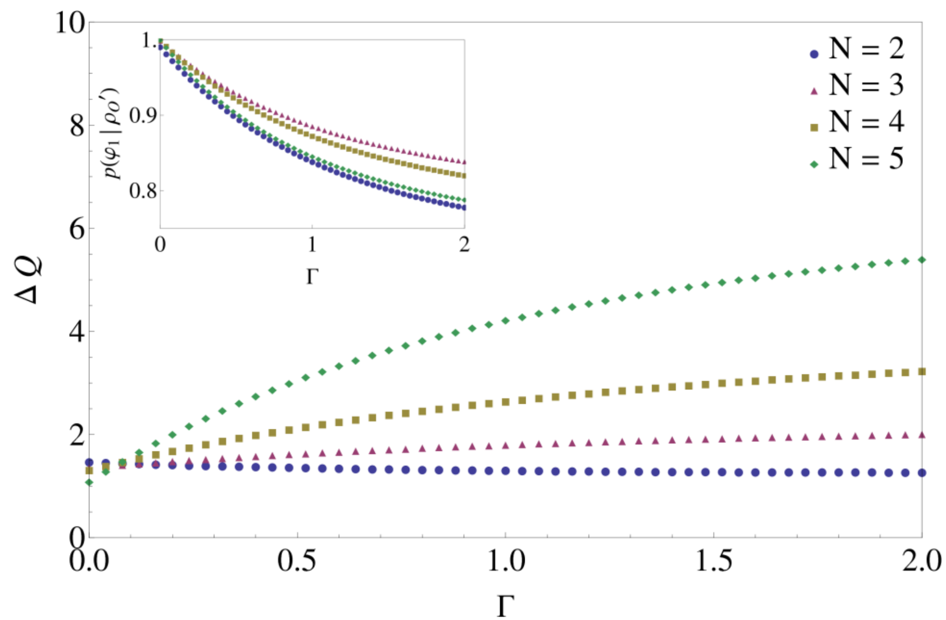
Comparison between two models: dephasing case

harmonic oscillator





spin chain



$\Theta = J = \beta = 1$

SELF-CONSISTENT INFORMATION ERASURE “BEYOND LANDAUER”

Change the conceptual framework

- Concepts to retain: Hamiltonian cycles. Temperature and hence thermal states

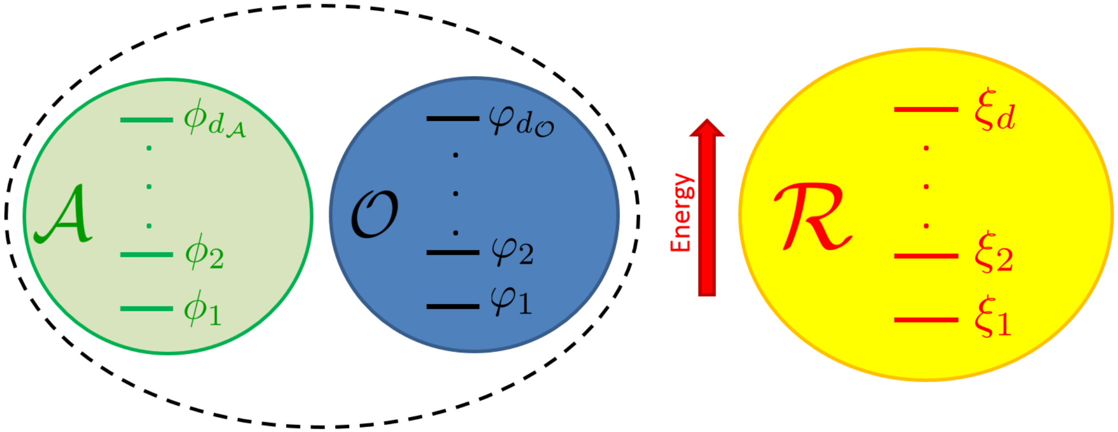
- Concepts to abandon:

    - Unitary evolution
      $\rightarrow$ Generalised evolution

      and/ or

    - Object initially uncorrelated with thermal reservoir
      $\rightarrow$ Object initially a subsystem of a thermal state

Object, auxiliary and reservoir



$$\rho = \rho_{\mathcal{A}+\mathcal{O}} \otimes \rho_{\mathcal{R}}(\beta)$$

$$U \in \mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{O} \otimes \mathcal{H}_\mathcal{R})$$

- If rank of $\rho_{\mathcal{A}+\mathcal{O}} \leqslant d_\mathcal{A}$, full erasure without using reservoir
- This is achieved for states with no correlations, classical correlations, quantum discord, and pure entanglement
- Classical correlation $\implies$ $\rho'_\mathcal{A}$ left the same, but this cannot be used as a catalyst using same U each time
- Pure entanglement $\implies$ $\mathcal{A}$ also purified which can be used to cool $\mathcal{R}$, as shown in Nature 474, 61-63 (2011)

Object as subsystem of reservoir

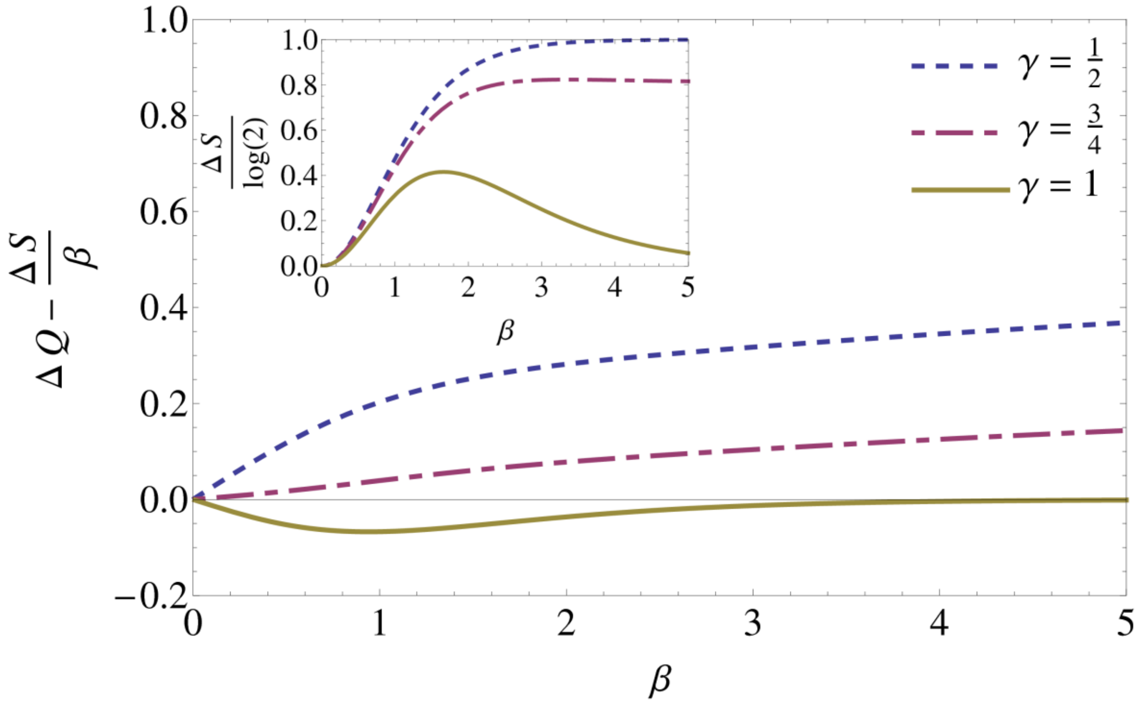
$$H \in \mathcal{L}(\mathcal{H}_{\mathcal{O}} \otimes \mathcal{H}_{\mathcal{K}}) = \sum_{n=1}^{d_{\mathcal{O}} d_{\mathcal{K}}} \lambda_n |\xi_n\rangle\langle\xi_n| \qquad \rho(\beta) = \sum_{n=1}^{d_{\mathcal{O}} d_{\mathcal{K}}} p_n |\xi_n\rangle\langle\xi_n|$$

- Maximal information erasure when $U\,|\xi_n\rangle = |\Psi\rangle \otimes |\phi_j\rangle$ for the $d_{\mathcal{K}}$ largest probabilities p_n

$$\beta \Delta Q = S(\rho' || \rho(\beta)) = \sum_{n=1}^{d_{\mathcal{O}} d_{\mathcal{K}}} q_n^U \log\left(\frac{1}{p_n}\right) - S(\rho')$$

$$q_n^U := \sum_{m=1}^{d_{\mathcal{O}} d_{\mathcal{K}}} p_m |\langle\xi_m|U|\xi_n\rangle|^2$$

- Minimise ΔQ by majorising $\{q_n^U\}_n$
$\Longrightarrow$ as $\{U\,|\xi_n\rangle\}_{n=1}^{d_{\mathcal{K}}}$ are product vectors, then $\{|\xi_n\rangle\}_{n=1}^{d_{\mathcal{K}}}$ must be product vectors also



$$d_{\mathcal{O}} = d_{\mathcal{K}} = 2$$

$$\lambda_1 = 0, \quad \lambda_{n+1} - \lambda_n = 1$$

- γ is the Schmidt coefficient of $\{|\xi_n\rangle\}_n$
- As $\gamma \to 1$, the vectors $\{|\xi_n\rangle\}_n$ become separable
- In this limit, when $\beta \sim 1$, $\Delta Q - \Delta S/\beta$ becomes negative, thereby "violating" Landauer's limit

Conclusions

- Determined the unitary operator that purifies an object with a desired probability, with the minimal consequent heat dissipation

- For a reservoir composed of a harmonic oscillator, minimal heat dissipation of full qubit erasure is the thermal energy of the reservoir, achieved when the frequency becomes vanishingly small.

- For a reservoir composed of a spin chain of length N, can achieve the same probability of qubit erasure as with a harmonic oscillator, but with a smaller heat cost.

- Harmonic oscillator most robust to dephasing when it is “like” a spin chain

- Enumerated two alterations to the set-up of information erasure so as to dissipate less heat than required by Landauer’s principle, but in such a way that we do not make a category error regarding heat and temperature.

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