Locality of temperature
structural properties of thermal states

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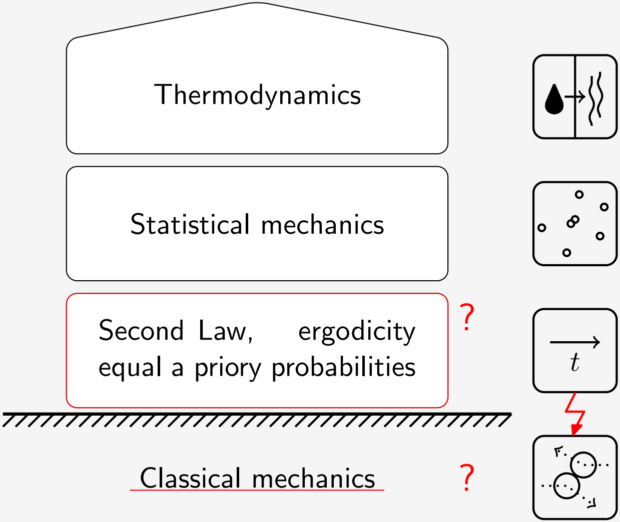
ICFO - The Institute of Photonic Sciences

MPQ - Max Planck Institute of Quantum Optics

2^{nd} Quantum Thermodynamics Conference
2015-04-20

New foundation for statistical mechanics



New foundation for statistical mechanics

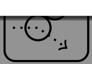
"There is no line of argument proceding from the laws of microscopic mechanics to macroscopic phenomena that is generally regarded by physicists as convincing in all respects."

— E. T. Jaynes [1] (1957)

"Statistical physics [...] has not yet developed a set of generally accepted formal axioms [...]"

— Jos Uffink [2] (2006)

Classical mechanics ?



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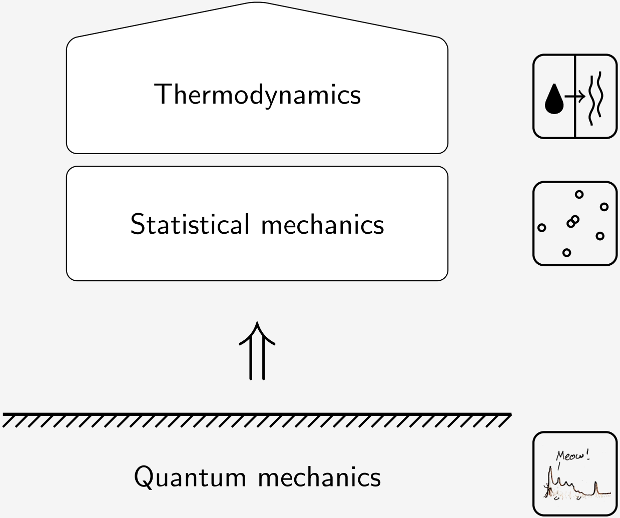


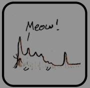
New foundation for statistical mechanics

Thermodynamics

!!! Reviews !!! Reviews !!! Reviews !!! Reviews !!! Reviews !!!

- Shallow but broad but overview:
  J Eisert, M Friesdorf, and C Gogolin, Nature Physics, 11 (2014), 124–130
- In depth review:
  C. Gogolin and J. Eisert (2015), arXiv: 1503.07538v1

Quantum mechanics



Setup

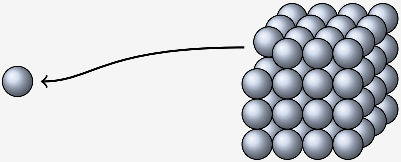
Setting

Subsystem, H_S

d_S

Bath, H_B

$d_B \gg d_S$



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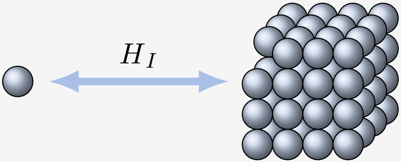
Subsystem, H_S Bath, H_B

d_S $d_B \gg d_S$



Setting

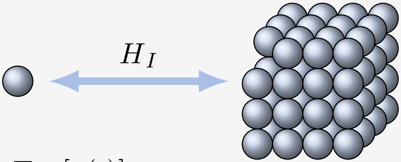
$$H = H_S + H_B + H_I \qquad \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

Subsystem, H_S

d_S

Bath, H_B

$d_B \gg d_S$



$$\rho^S(t) = \mathrm{Tr}_B[\rho(t)]$$

Equilibration & thermalization

Equilibration

Theorem (Equilibration on average [7])

If H has non-degenerate energy gaps, then for every $\rho(0) = |\psi_0\rangle\langle\psi_0|$ there exists a ω^S such that:

$$\overline{\mathcal{D}\left(\rho^S(t), \omega^S\right)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\mathrm{eff}}}}$$

[5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Physical Review Letters, 100.3 (2008), 30602
[6] P. Reimann, Physical Review Letters, 101.19 (2008), 190403
[7] N. Linden, S. Popescu, A. Short, and A. Winter, Physical Review E, 79.6 (2009), 61103
[8] A. J. Short and T. C. Farrelly, New Journal of Physics, 14.1 (2012), 013063
[9] P. Reimann and M. Kastner, New Journal of Physics, 14.4 (2012), 043020

Equilibration

Theorem

If H has ... $\rangle\langle\psi_0|$

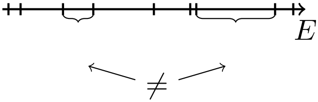
there exis...

Non-degenerate energy gaps

H has non–degenerate energy gaps iff:

$$E_k - E_l = E_m - E_n$$

$$\Longrightarrow k = l \wedge m = n \quad \vee \quad k = m \wedge l = n$$



Intuition: Sufficient for H to be fully interactive

$$H \neq H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2$$

[5] M. Cramer, C.
[6] P. Reimann, P
[7] N. Linden, S. Popescu, A. Short, and A. Winter, Physical Review E, 79.6 (2009), 01103
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Effective dimension

$$d^{\text{eff}} = \frac{1}{\sum_k |\langle E_k|\psi_0\rangle|^4}$$

Intuition: Dimension of supporting energy subspace

[5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Physical Review Letters, 100.3 (2008), 30602
[6] P. Reimann, Physical Review Letters, 101.19 (2008), 190403
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$$\overline{\mathcal{D}\left(\rho^S(t), \omega^S\right)} \leq \frac{1}{2}\sqrt{\frac{d_S^2}{d^{\text{eff}}}}$$

$\Longrightarrow$ If $d^{\text{eff}} \gg d_S^2$ then $\rho^S(t)$ equilibrates on average.

[5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Physical Review Letters, 100.3 (2008), 30602
[6] P. Reimann, Physical Review Letters, 101.19 (2008), 190403
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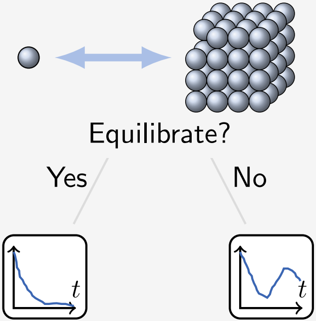
Thermalization



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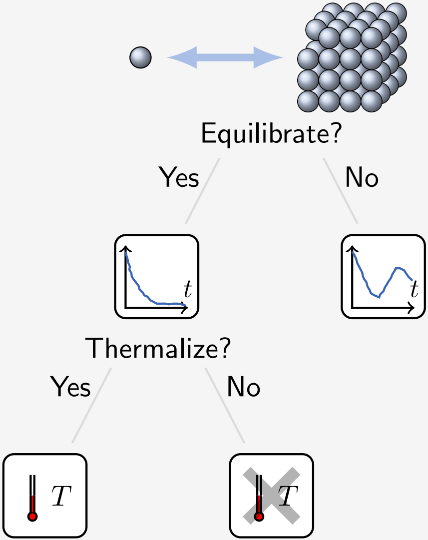

Thermalization is a complicated process

Thermalization implies:

1. Equilibration [5–10]
2. Subsystem initial state independence [11, 12]
3. Weak bath state dependence [13]
4. Diagonal form of the subsystem equilibrium state [14]
5. Thermal state $\omega^S = \mathrm{Tr}_B[\omega] \approx g^S_{H_S}(\beta) \propto \mathrm{e}^{-\beta\, H_S}$ [13, 15, 16]

[5] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Physical Review Letters, 100.3 (2008), 30602
[6] P. Reimann, Physical Review Letters, 101.19 (2008), 190403
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[10] J. Gemmer, M. Michel, and G. Mahler, vol. 784, Lecture Notes in Physics, Berlin, Heidelberg: Springer, 2009
[12] A. Hutter and S. Wehner, Physical Review A, 87.1 (2013), 012121
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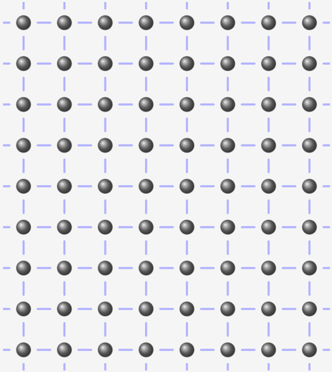
[15] M. P. Mueller, E. Adlam, L. Masanes, and N. Wiebe (2013), arXiv: 1312.7420
[16] F. G. S. L. Brandão and M. Cramer (2015), arXiv: 1502.03263v1
...

Locality of temperature

The setting

- Local Hamiltonian (spins or fermions)

$$H := \sum_{\lambda \in \mathcal{E}} h_\lambda$$



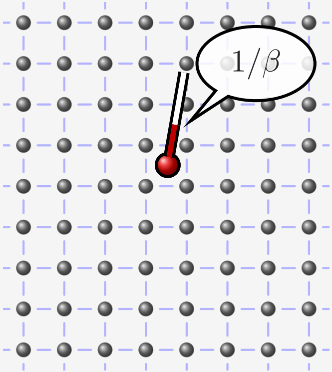
The setting

- Local Hamiltonian (spins or fermions)

$$H := \sum_{\lambda \in \mathcal{E}} h_\lambda$$

- Thermal state

$$g(\beta) := \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\right]}$$



The setting

- Local Hamiltonian truncated to $S \subset V$

$$H_S := \sum_{\lambda \in \mathcal{E}(S)} h_\lambda$$

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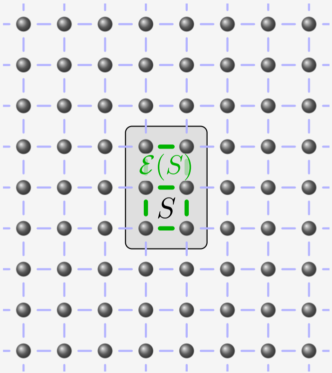
$$g(\beta) := \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\right]}$$



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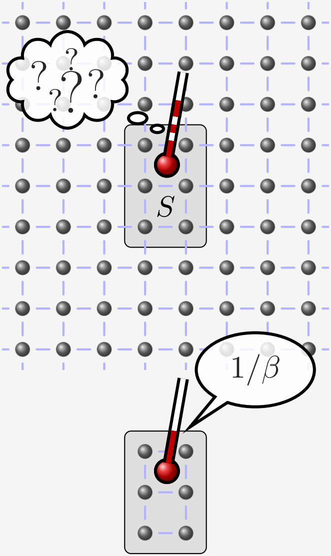


The setting

- Local Hamiltonian truncated to $S \subset V$

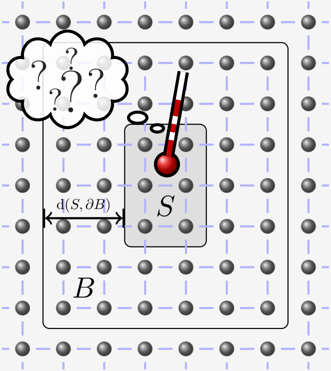
$$H_S := \sum_{\lambda \in \mathcal{E}(S)} h_\lambda$$

- Thermal state

$$g_B(\beta) := \frac{\mathrm{e}^{-\beta\, H_B}}{\mathrm{Tr}\left[\mathrm{e}^{-\beta\, H_B}\right]}$$

- Introduce buffer region

$$\mathrm{Tr}_{S^c}[g_B(\beta)] \approx \mathrm{Tr}_{S^c}[g(\beta)] \;?$$



This can be made rigorous:

Generalized covariance

$$\mathrm{cov}_\rho^\tau(A, A') := \mathrm{Tr}[\rho^\tau A\, \rho^{1-\tau} A'] - \mathrm{Tr}[\rho\, A]\, \mathrm{Tr}[\rho\, A']$$

[17] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert (2013), arXiv: 1309.0816v2

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Theorem (Truncation formula [17])

For any observable $A = A_S \otimes \mathbb{1}$

$$\mathrm{Tr}[A\, g_B(\beta)] - \mathrm{Tr}[A\, g(\beta)] = \beta \int_0^1 \int_0^1 \mathrm{cov}^\tau_{g(s,\beta)}(H_{\partial B}, A)\, \mathrm{d}\tau\, \mathrm{d}s\,,$$

where $g(s, \beta)$ is thermal state of $H(s) := H - (1 - s)\; H_{\partial B}$.

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$$\mathrm{cov}^{\tau}_{\rho}(A, A') := \mathrm{Tr}[\rho^{\tau} A\, \rho^{1-\tau} A'] - \mathrm{Tr}[\rho\, A]\, \mathrm{Tr}[\rho\, A']$$

exactly captures the response of local expectation values.

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Clustering of correlations

Theorem (Clustering of correlations at high temperature [17])

Let $J := \max_\lambda \|h_\lambda\|_\infty$, then for every $\tau \in [0,1]$ and $\beta < \beta^*(J,\alpha)$

$$\left|\mathrm{cov}^{\tau}_{g(\beta)}(A, A')\right| \leq C\ \mathrm{e}^{-\,\mathrm{d}(A,A')\,/\,\xi(\beta\, J,\alpha)}$$

with $\alpha = \alpha(\mathcal{E})$ the lattice animal constant.

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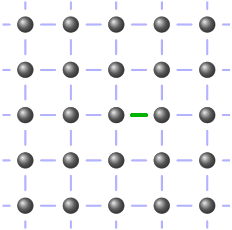

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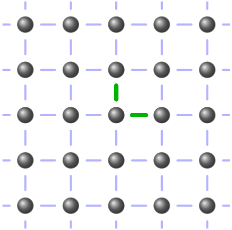

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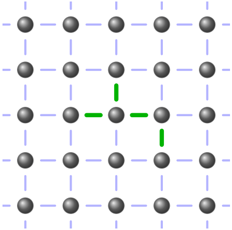

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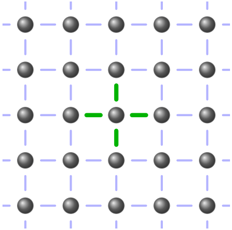

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$$\implies \mathcal{D}\left(g^S(\beta), {g_B}^S(\beta)\right) \leq C'\ \mathrm{e}^{-\mathrm{d}(S,\partial B)/\xi(\beta\,J,\alpha)}$$

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Implications

$$\beta < \beta^*(J,\alpha) \implies \mathcal{D}\left(g^S(\beta), {g_B}^S(\beta)\right) \leq C' \, \mathrm{e}^{-\mathrm{d}(S,\partial B)/\xi(\beta\, J,\alpha)}$$

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Local stability of thermal states

$g^S(\beta)$ only depends exponentially weakly on far away terms of the Hamiltonian.

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$g^S(\beta)$ only depends exponentially weakly on far away terms of the Hamiltonian.

Classical simulability with cost independent of total system size

Local expectation values can be calculates with cost independent of the total system size.

A universal bound on phase transitions

Universal critical temperature

The critical temperature

$$\frac{1}{\beta^* J} = \frac{2}{\ln\left((1+\sqrt{1+4/\alpha})/2\right)}$$

upper bounds the physical critical temperatures of all possible models.

A universal bound on phase transitions

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upper bounds the physical critical temperatures of all possible models.

Example: 2D square lattice ($\alpha \leq 4\,\mathrm{e}$)

- The bound:
  $1/(\beta^*\, J) = 2/\ln((1+\sqrt{1+1/\mathrm{e}})/2) \approx 24.58$
- Ising model (ferromagnetic, isotropic) phase transition at:
  $1/(\beta_c\, J) = 2/\ln(1+\sqrt{2}) \approx 2.27$

References

Thank you for your attention!

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[2] J. Uffink, http://philsci-archive.pitt.edu/id/eprint/2691, 2006.

[3] J Eisert, M Friesdorf, and C Gogolin, Nature Physics, 11 (2014), 124–130, arXiv: 1408.5148.

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