ETH-Zürich-September 19th 2007

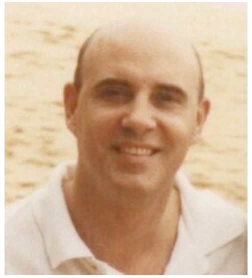
Constructive effects induced by heterogeneity: an application to a model for opinion formation

Raúl Toral

IFISC

CSIC UIB

Raúl Toral (IFISC, Palma de Mallorca)



Claudio Mirasso (IFISC, Palma de Mallorca)

Claudio Tessone (ETH, Zurich)

James D. Gunton (Lehigh U. Pennsylvania, USA)

Summary:

-Disordering role of heterogeneity and noise

-Ordering role of heterogeneity and noise

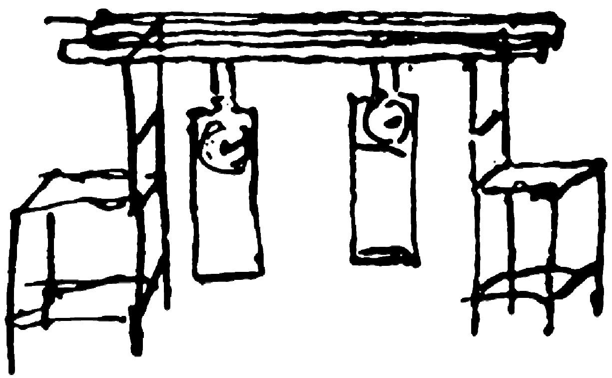
-Stochastic resonance induced by disorder

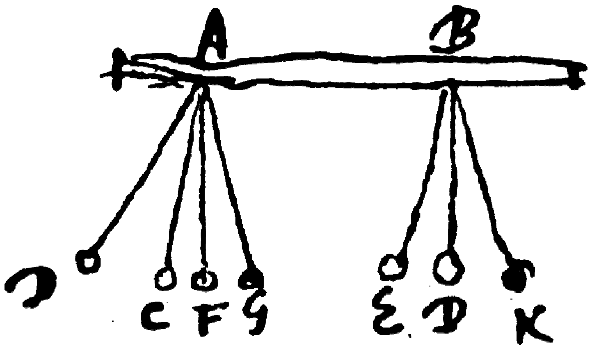
- -Bistable systems
- -Model for opinion formation

Is it possible to have perfect synchrony between two different systems?
Two independent clocks that initially display the same time will, eventually, show different times.
That's why we "Synchronize our clocks before an action"

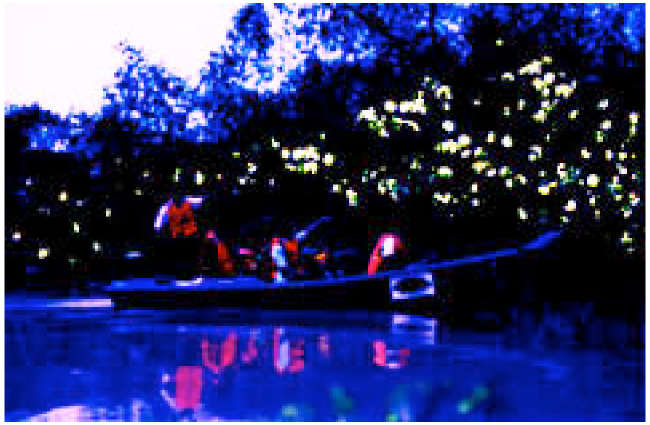
Can we keep two clocks permanently synchronized?

Christiaan Huygens showed that this is possible by coupling them in the appropriate manner
Journal des Sçavans, 16 march 1665





Fireflies in Malaysia, Thailand and New Guinea

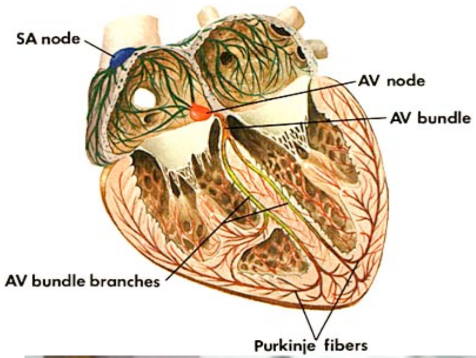


J. Buck & E. Buck, Scientific American May 1976.

S.H. Strogatz, I. Steward, Scientific American, Nov. 1993

The dark side of the moon

Sinoatrial node in heart: pacemaker of
~ 10000 synchronized cells.



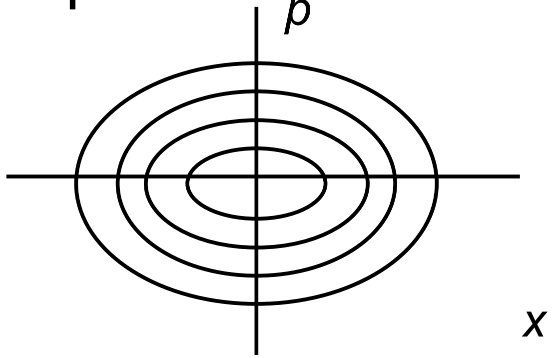
Applause rate in a concert.
Z. Neda et. al. Nature **403**, 849 (2000).

Basic ingredients [Winfree (1967)] : Coupled oscillators

Harmonic oscillator:

$$\dot{x} = \frac{p}{m}$$

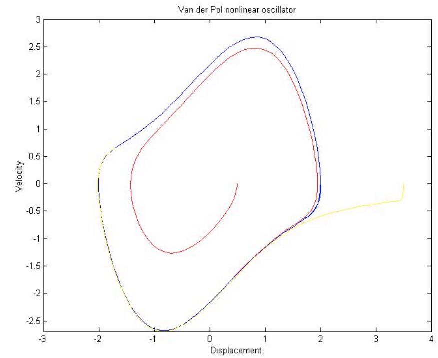
$$\dot{p} = -kx$$



Limit cycle attractor

(van der Pol):

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$



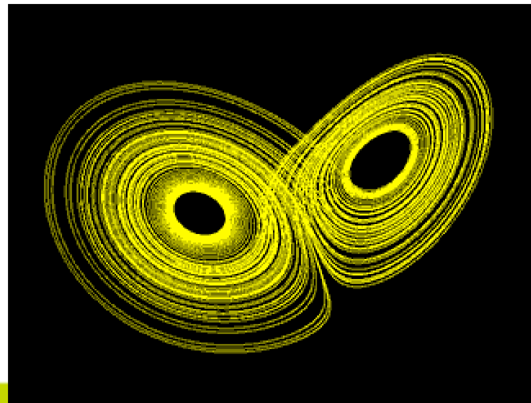
Chaotic attractor

(Lorenz):

$$\dot{x} = p(y - x)$$

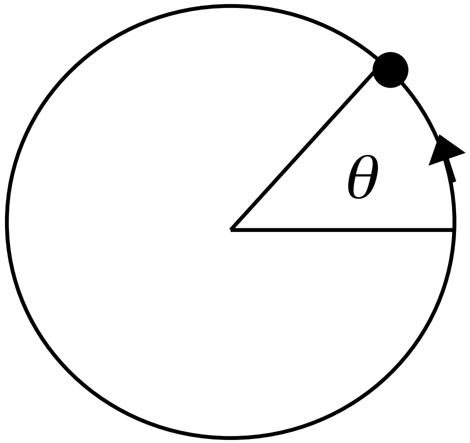
$$\dot{y} = -xz + rx - y$$

$$\dot{z} = xy - bz$$

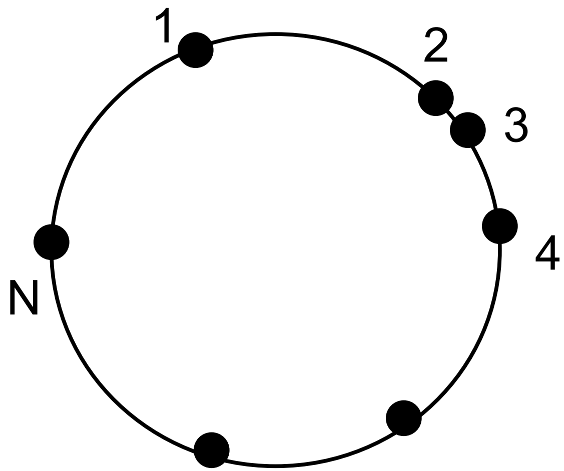


Kuramoto model [1975]

Simplest oscillator: $\dot{\theta} = \omega$



Many of them



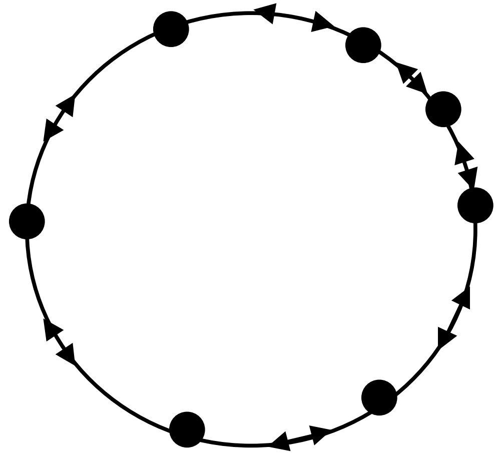
$$\dot{\theta}_j = \omega + \eta_j \quad j = 1,\ldots,N$$

Rotate independently of each other unless.....

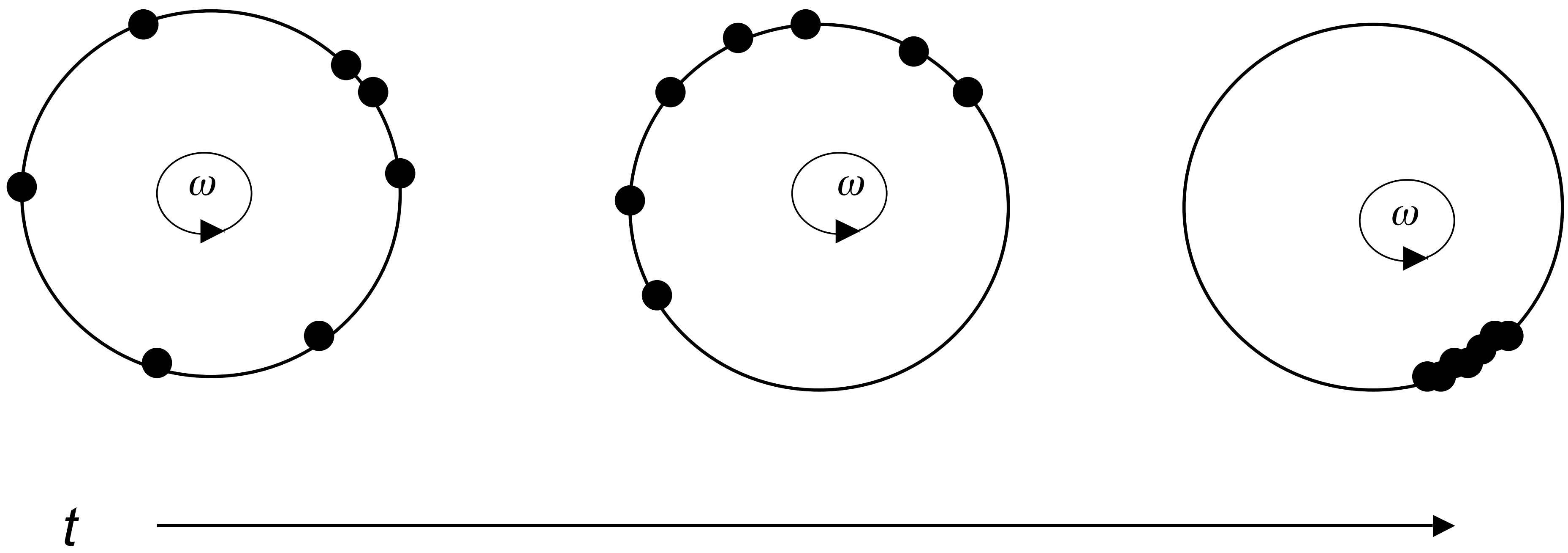
Coupling!!!

Kuramoto considers an **ensemble** of globally coupled oscillators:

$$\dot{\theta}_j = \omega + \eta_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j) \quad j = 1,\ldots,N$$



In the homogeneous case: $\eta_j = 0 \qquad j = 1, \cdots, N$



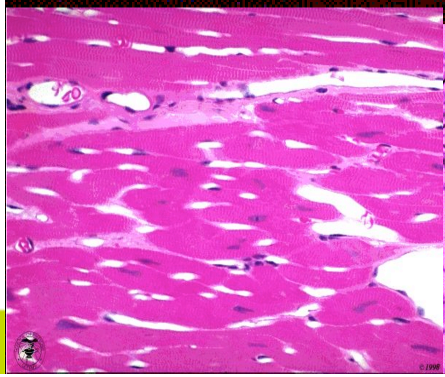
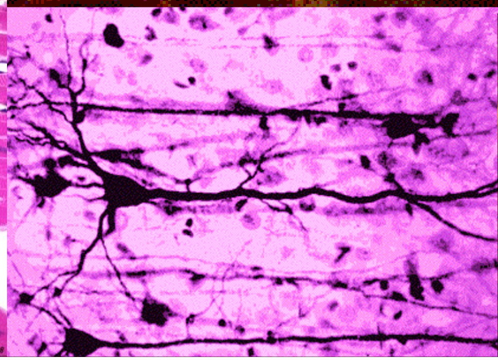
Stable synchronization (coherent behavior) occurs for any $K > 0$

Diversity exists

- There are examples of systems with many different units strongly interacting.
- Despite not all units being identical, the system exhibits, in some situations a macroscopic coherent behavior





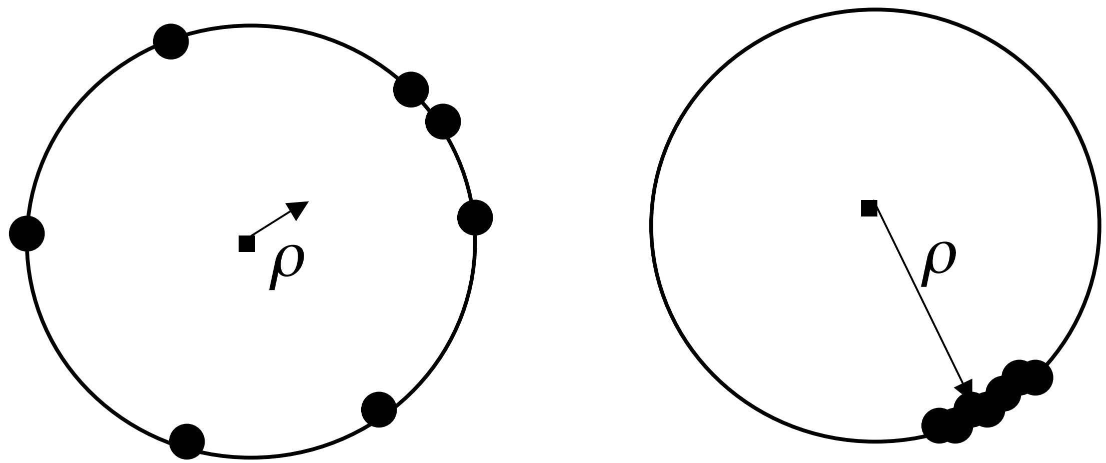

Kuramoto model:

$$\dot{\theta}_j = \omega + \eta_i + \frac{C}{N_j} \sum_j \sin(\theta_j - \theta_i)$$

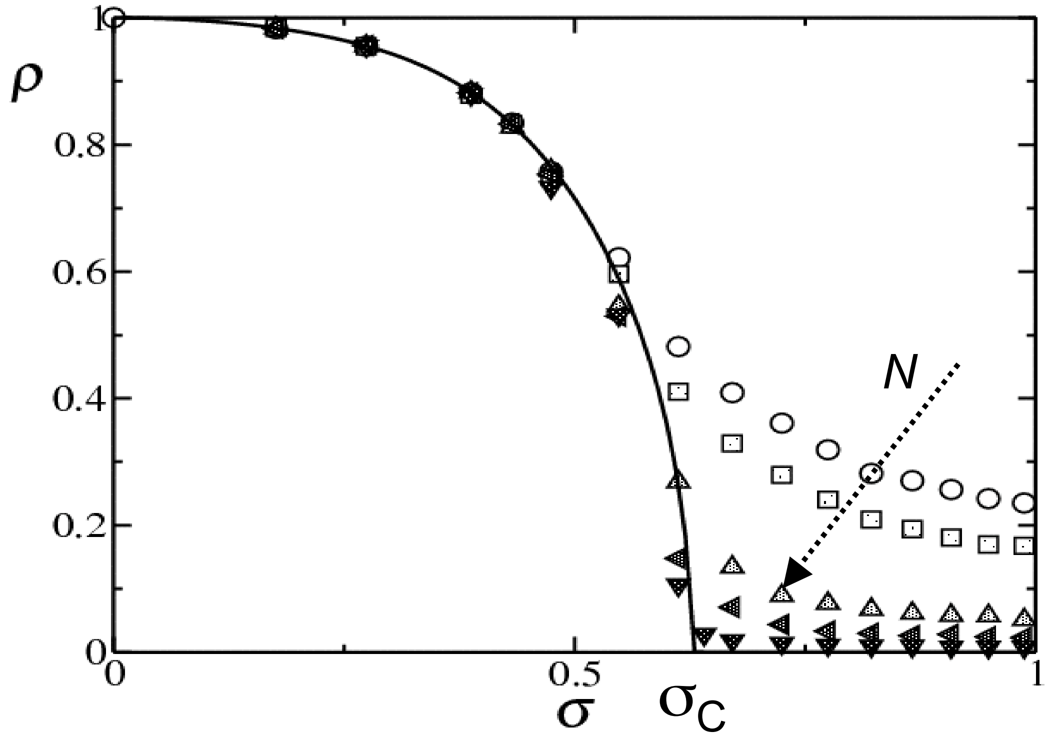
$$\langle \eta_i \rangle = 0, \langle \eta_i^2 \rangle = \sigma^2$$

$$\rho = \left| \frac{1}{N} \sum_i e^{i\theta_i} \right|$$

Radius of the center of mass of the oscillators



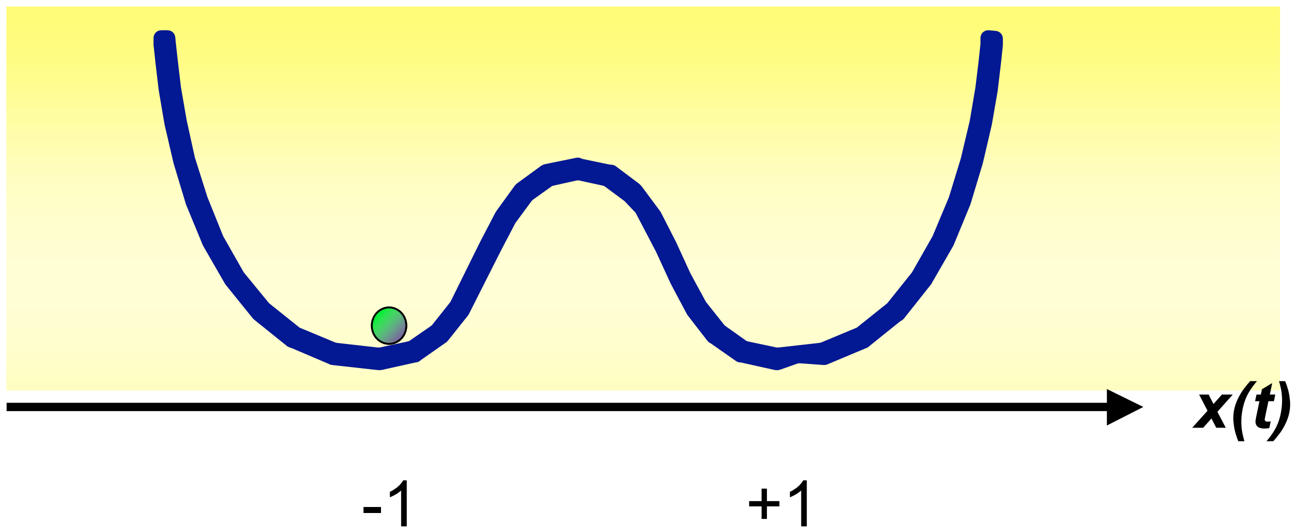
Diversity destroys coherent motion!!



This is a general result

Basic model for binary choice (Landau model):

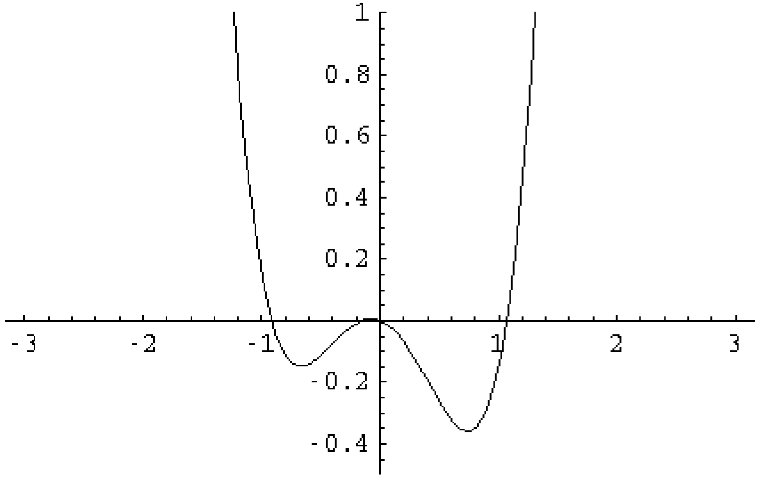
$$\dot{x}(t) = x(t) - x(t)^3$$

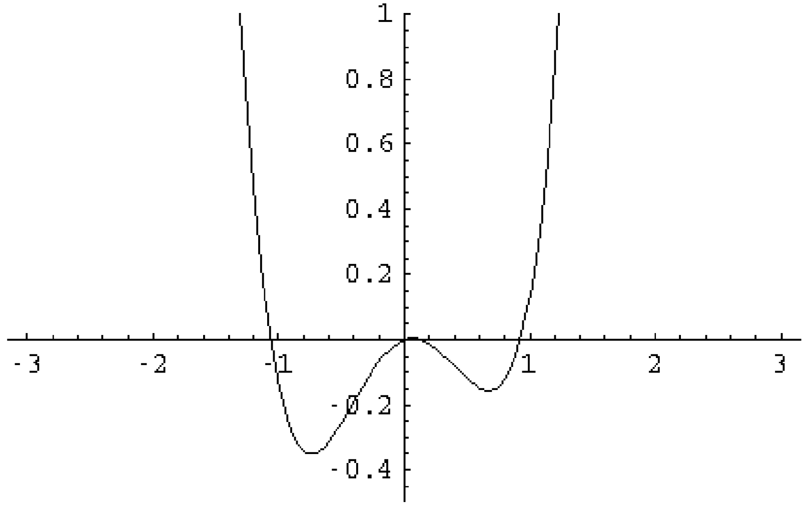


Relaxation in a symmetric potential

Diversity is introduced in the asymmetry of the potential:

$$\dot{x}(t) = x(t) - x(t)^3 + \eta$$





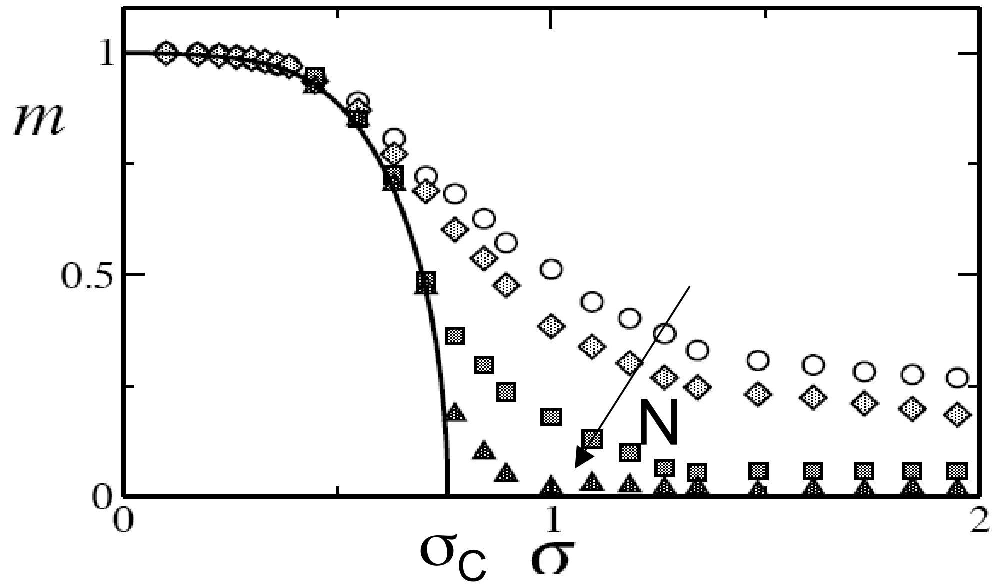
$\eta>0$ favors the well on the right $\eta<0$ favors the well on the left

Landau model:

$$\frac{dx_i}{dt} = x_i - x_i^3 + \eta_i + \frac{C}{N_i}\sum_{j=1}(x_j - x_i)$$

Order parameter: $$m = \left|\frac{1}{N}\sum_{i=1}^{N} x_i\right|$$

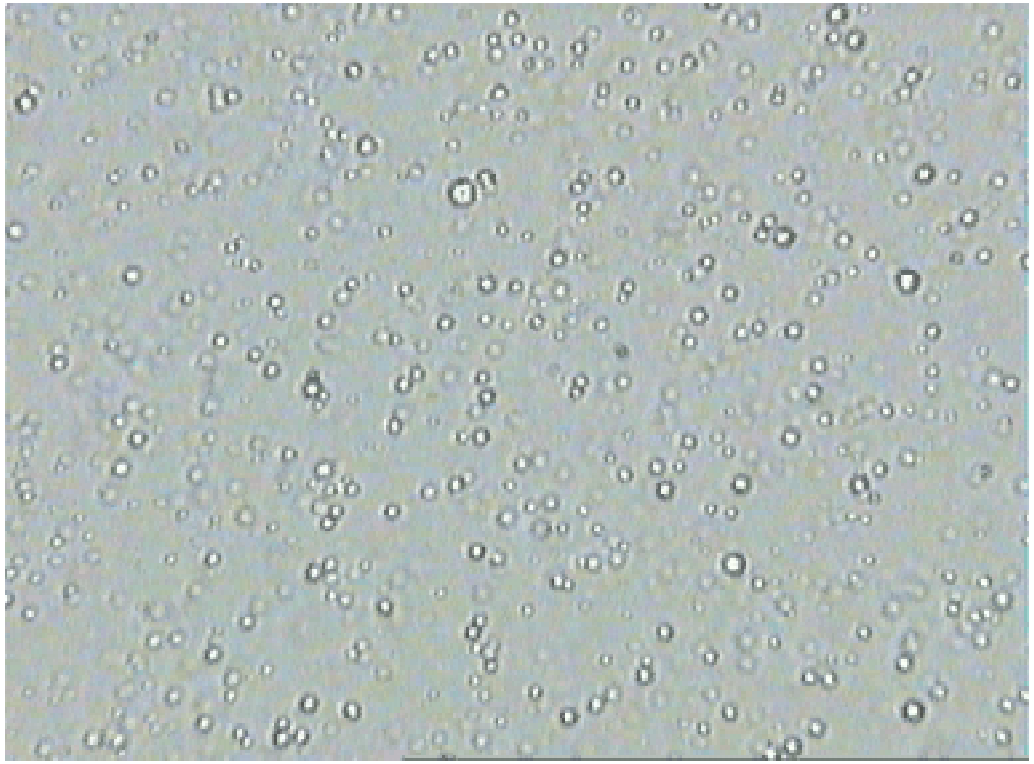
$$\langle \eta_i \rangle = 0$$
$$\langle \eta_i^2 \rangle = \sigma^2$$



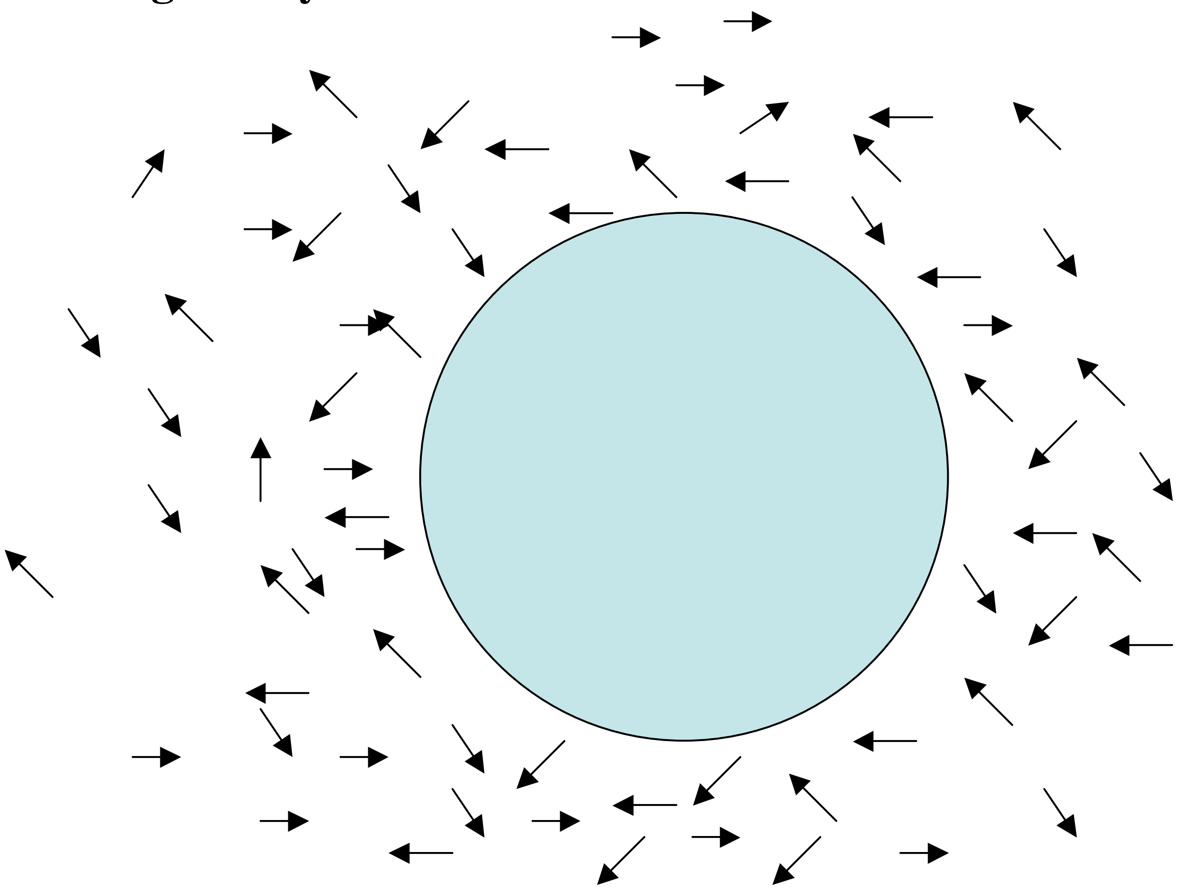
Diversity is not the only source of disorder
“Noise” (fluctuations of microscopic origin) is another

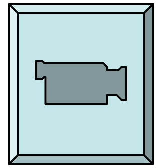
It all started in 1828 with a Scottish botanist

Robert Brown (1828) was looking at pollen in water through a microscope.



Explanation was given by Einstein in 1905

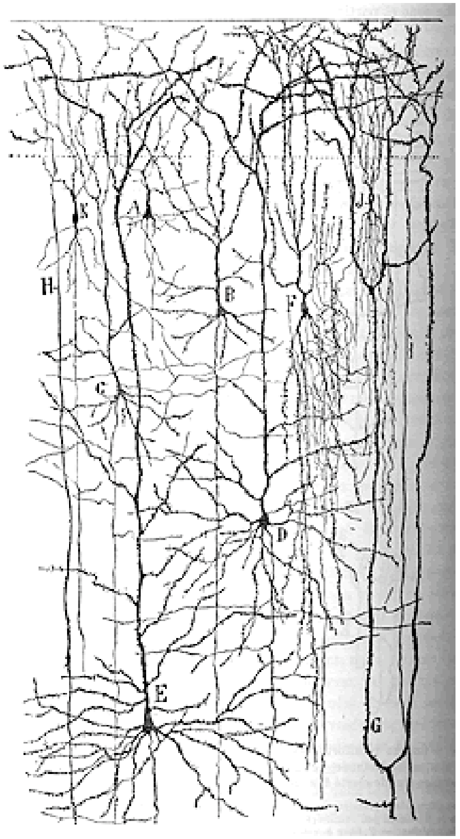




Noise

$$\vec{F} = \vec{F}_{ext} - \gamma \vec{v} + \vec{\xi}(t)$$

Noise everywhere!!!!



Electrical currents with many connections and noise

$$\tau_v \dot{v} = v(v - \frac{1}{2})(1 - v) - w,$$
$$\tau_w \dot{w} = v - w - \beta + \epsilon \cos(\omega t) + \xi(t)$$

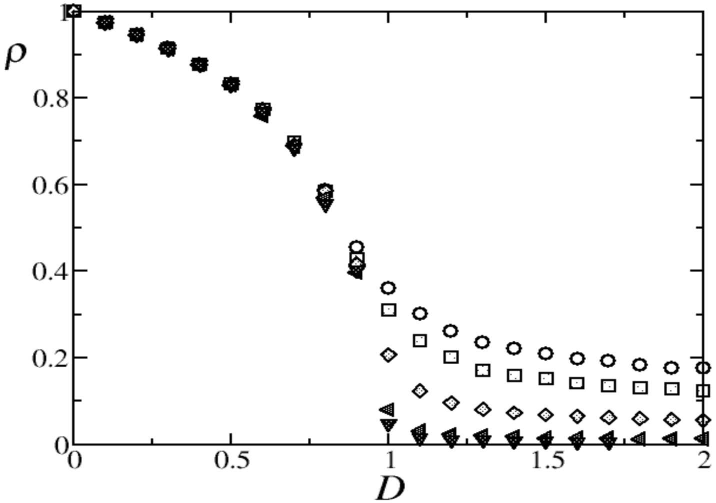
Reduces noise to near thermal.

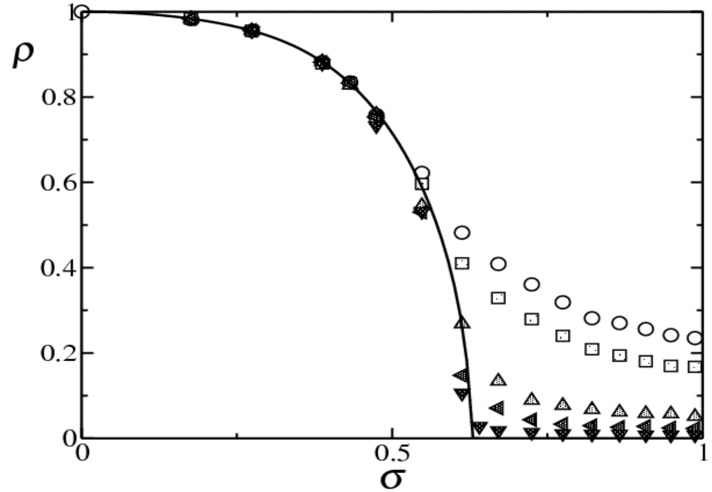
Internal noise: proportional to temperature T

Kuramoto model with noise:

$$\frac{d\theta_i}{dt} = \omega + \xi_i(t) + \frac{C}{N_j}\sum_j \sin(\theta_j - \theta_i)$$

$$\langle \xi_i(t) \rangle = 0, \langle \xi_i^2(t) \rangle = D \qquad \langle \eta_i \rangle = 0, \langle \eta_i^2 \rangle = \sigma^2$$





$$\rho = \left| \frac{1}{N} \sum_i e^{i\theta_i} \right|$$

Landau model with noise:

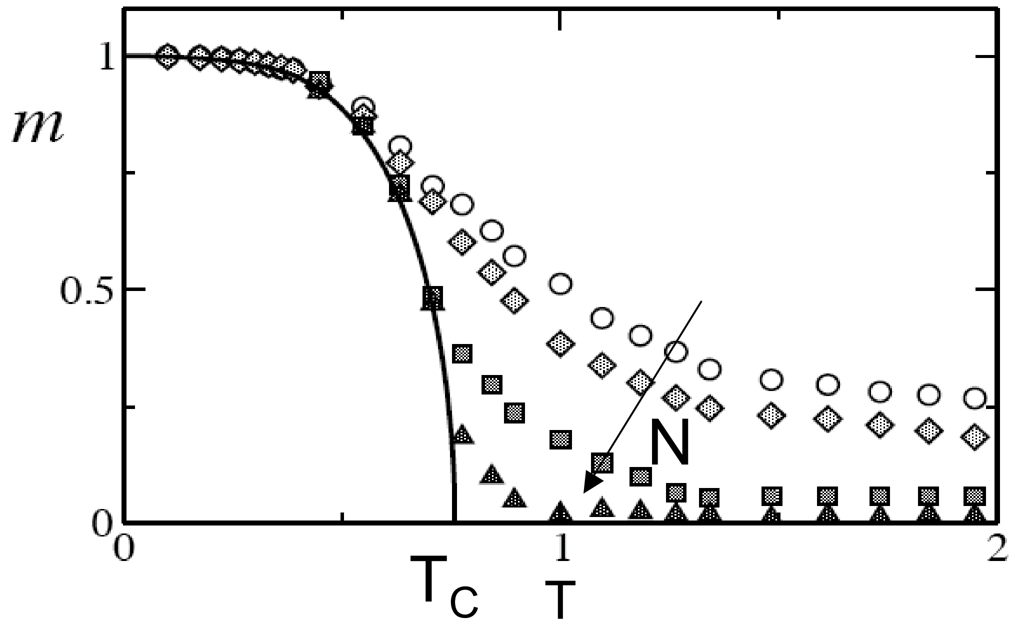
$$\frac{dx_i}{dt} = x_i - x_i^3 + \frac{C}{N_i}\sum_{j=1}(x_j - x_i) + \xi_i(t)$$

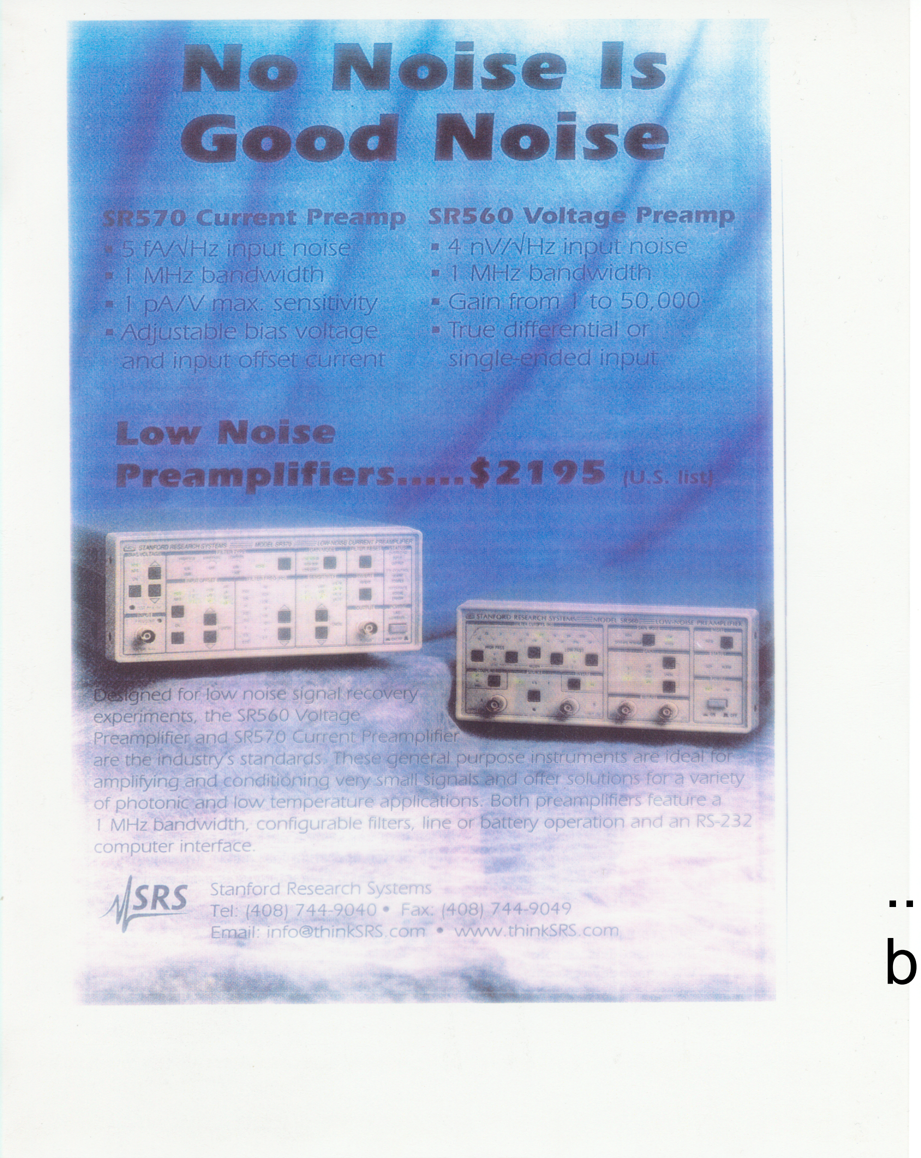
$\xi_i(t)$ is thermal noise

$$\langle \xi_i(t) \rangle = 0$$

$$\langle \xi_i^2(t) \rangle \propto T$$

$$m = \frac{1}{N}\sum_{i=1}^{N} x_i$$



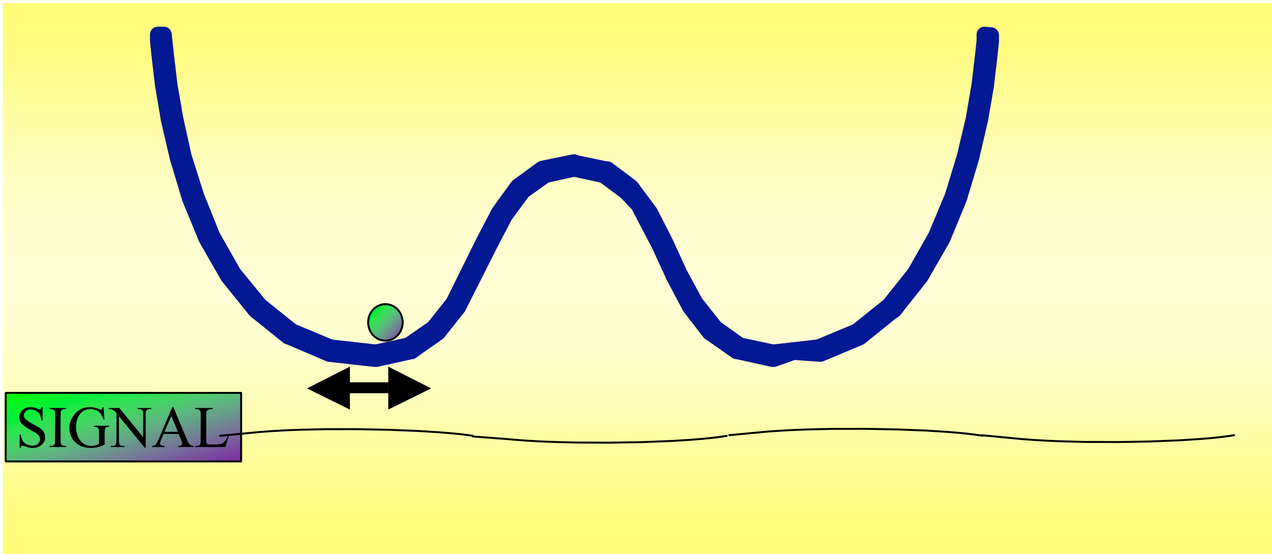


Noise disorders

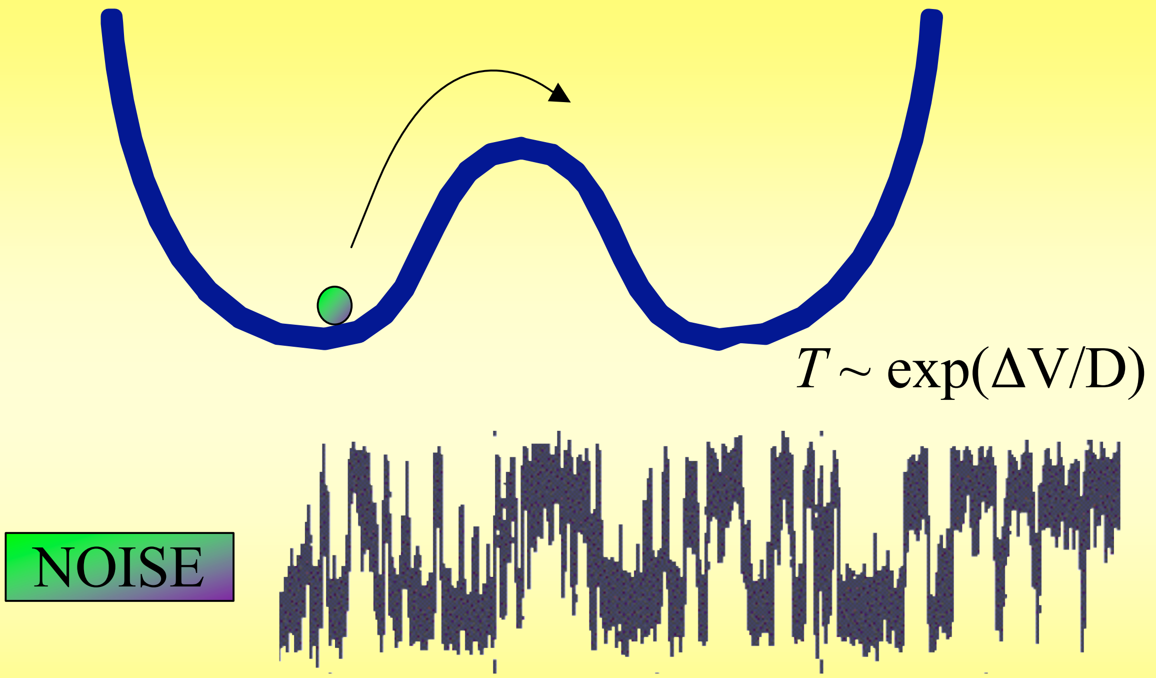
....but some noises are better than others.

Stochastic Resonance (1981)

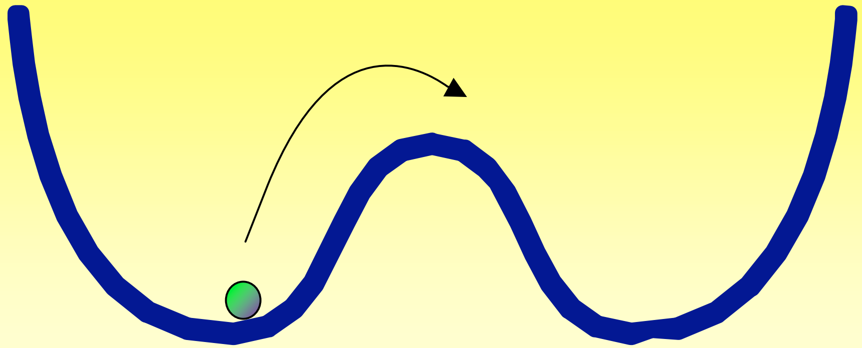
-Amplification of sub-threshold signals helped by noise



$$\dot{x} = x - x^3 + A\cos(\Omega t)$$

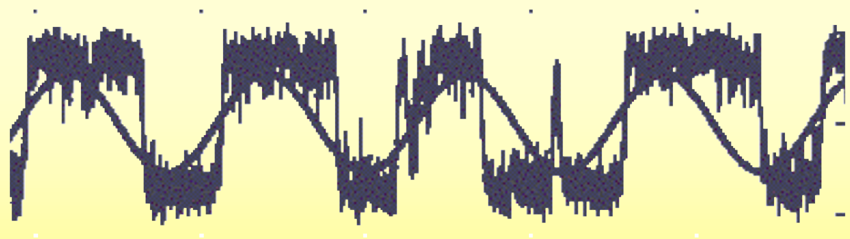


$$\dot{x} = x - x^3 + D\,\xi(t)$$



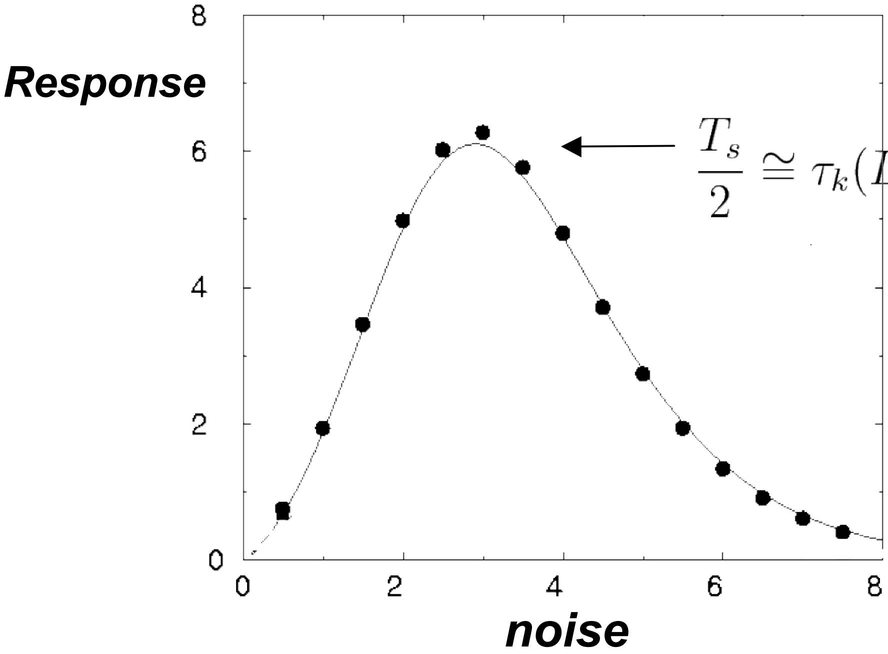
Matching between T and Ω

NOISE

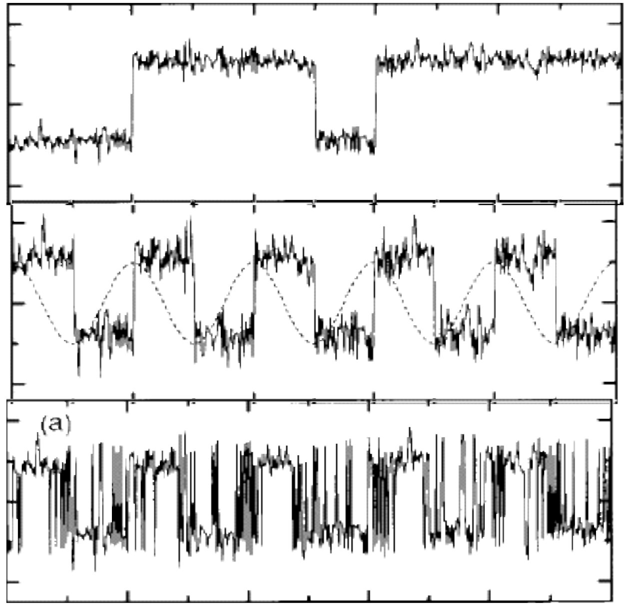


$$\dot{x} = x - x^3 + A\cos(\Omega t) + D\,\xi(t)$$

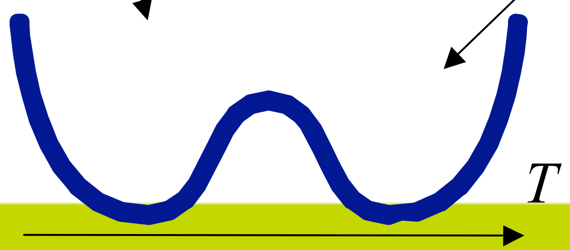
$$\frac{d}{dt}x(t) = -x(t)^3 + b\,x(t) + A\,\sin\left(\frac{2\pi}{T_s}t\right) + \xi(t)$$



$$\frac{T_s}{2} \cong \tau_k(D)$$



Earth climate



Weak signal: earth eccentricity (~ 3 . 10^4 y)
Noise: Changes in Sun luminosity due to
solar stains, clouds, CO_2, etc...

Global temperature changes
Bi-stable climate potential

New results begin here

The role of diversity in a forced system

What is the role of diversity when a weak external periodic forcing acts upon a system composed by many, diverse, bistable units ?

Diversity can make the response easier

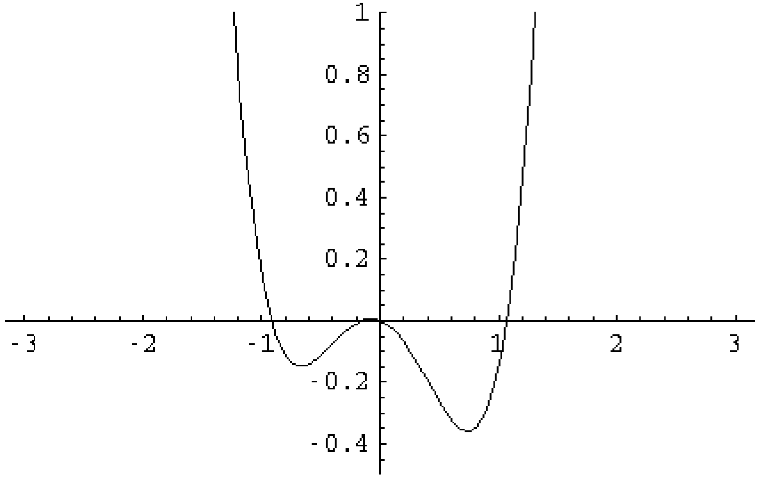
If a population makes only umbrellas, they are O.K. only in rainy years.
If they make only swim suits, they are O.K. in sunny years.
But if a fraction of the population makes umbrellas and another fraction swim suits, they can be O.K. always ….

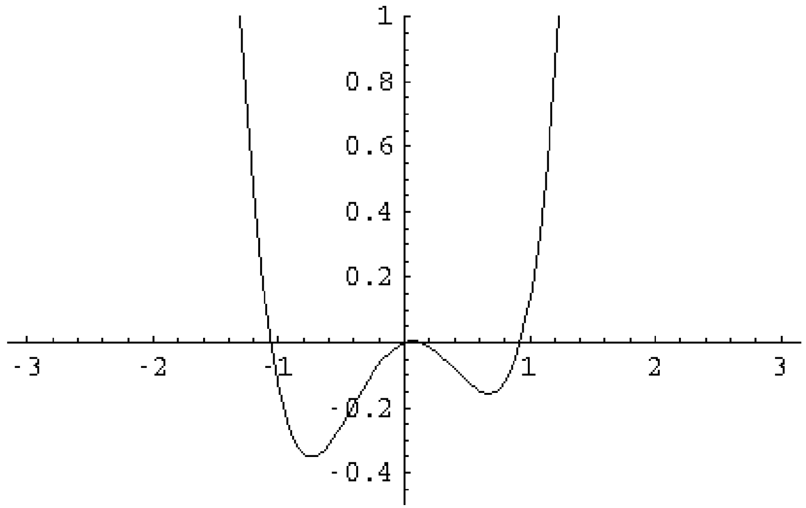
…provided they is some coupling between people.

Never put all the eggs in the same basket!!

Diversity is introduced in the asymmetry of the potential:

$$\dot{x} = x - x^3 + a$$





a>0 favors the well on the right (positive forces)

a<0 favors the well on the left (negative forces)

An ensemble of coupled bistable units

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + \underbrace{a_i}_{\text{diversity parameter}} + \underbrace{\frac{C}{N}\sum_j (x_j(t) - x_i(t))}_{\text{global coupling}} + \underbrace{A \sin(\omega t)}_{\text{periodic signal}}$$

diversity parameter

global coupling

periodic signal
...weak ...

- a_i is a parameter distributed according to a Gaussian distribution function $g(a)$, of mean 0 and variance σ^2

... **no** noise ...

applet

Some theory:

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + a_i + \frac{C}{N}\sum_j (x_j(t) - x_i(t)) + A\sin(\omega t)$$

The macroscopic variable $$X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$$

$$\dot{X} = X(1 - 3M) - X^3 + A\sin(\omega t)$$

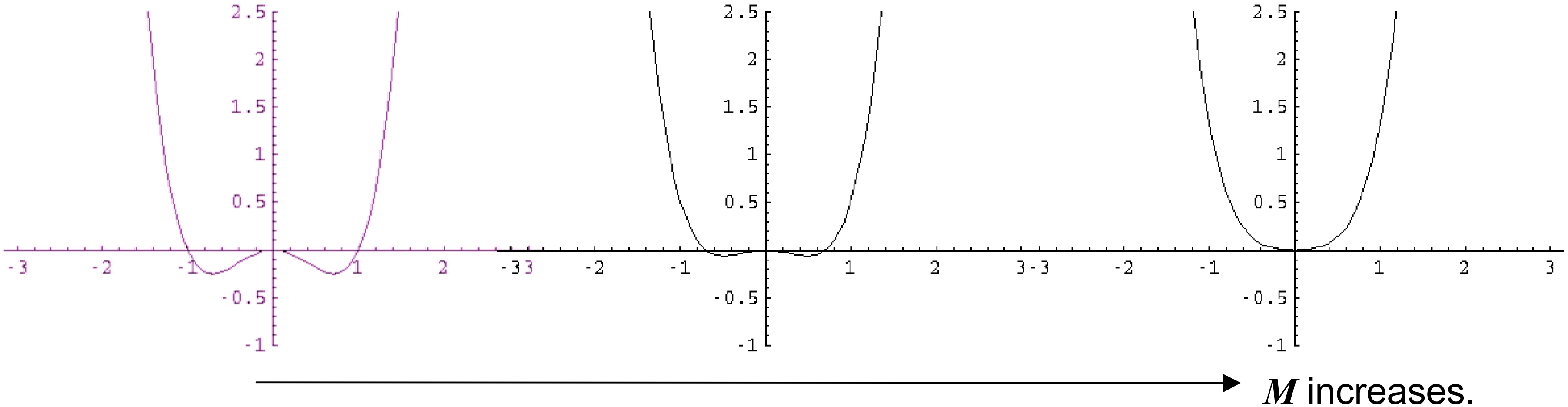
$$\frac{dX}{dt} = -\frac{\partial V(X,M)}{\partial X} + A\sin(\omega t)$$

$$M = \frac{1}{N}\sum_j (X - x_j)^2 = \frac{1}{N}\sum_j \delta_j^{\,2}$$

Effective global potential

$$V(X,M) = \frac{X^4}{4} - \frac{X^2(1-3M)}{2} \qquad \frac{1}{N}\sum_i \delta_i^2 = M$$

- No diversity implies ***M*=0** because of the coupling
- Increasing diversity, ***M*** increases and the barrier between minima lowers. Eventually, the system undergoes a transition from a bistable to a monostable potential



M increases.

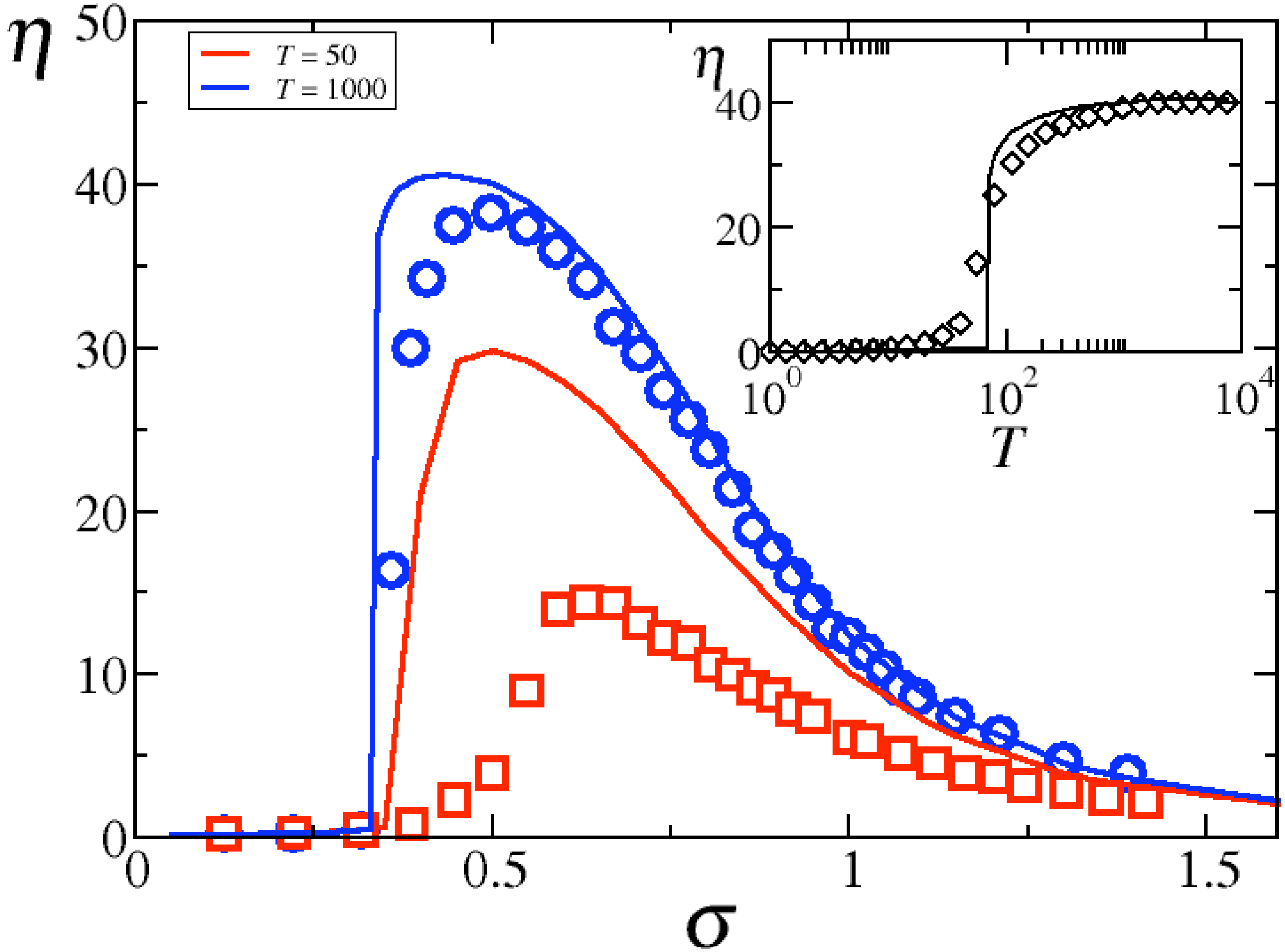
Linear regime ---------------------->non-linear regime-------------------------->linear regime

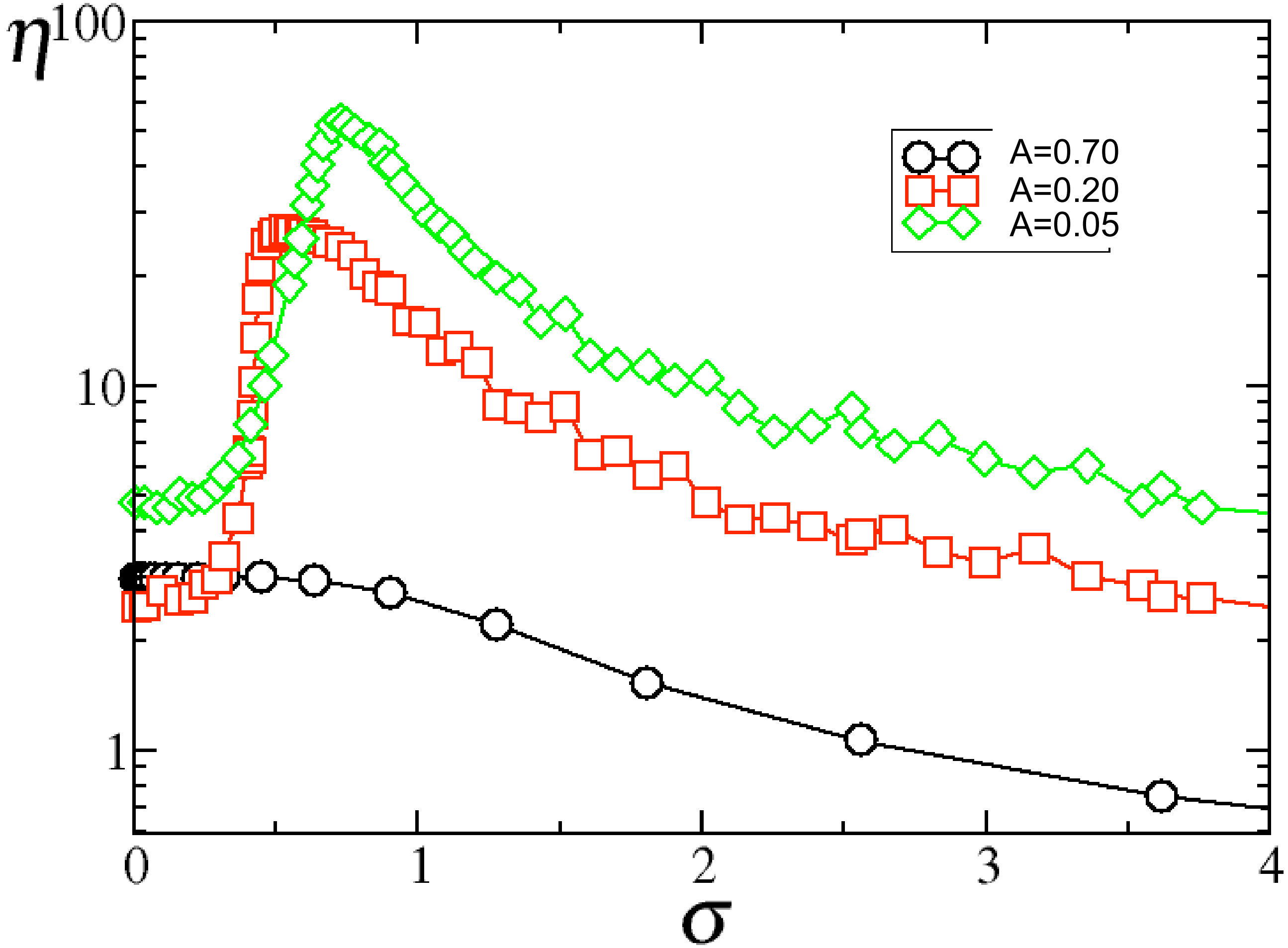
Effective global potential

$$V(X,M) = \frac{X^4}{4} - \frac{X^2(1-3M)}{2} \qquad \frac{1}{N}\sum_i \delta_i^2 = M$$

- No diversity implies $M=0$ because of the coupling
- Increasing diversity, M increases and the barrier between minima lowers. Eventually, the system undergoes a transition from a bistable to a monostable potential

This is independent on the mechanism that originates the increase in M


η
50
40
30
20
10
0
0
0.5
1
1.5
σ
T = 50
T = 1000
η
40
20
0
10^0
10^2
10^4
T



η
100
10
1
0
1
2
3
4
σ
A=0.70
A=0.20
A=0.05

Bistable system: *summary*

We have demonstrated the existence of a mechanism by which a weak (sub-threshold on the average) forcing is optimally followed by a large coupled bistable system in the presence of diversity

How general is this phenomenon?

Excitable system

As a paradigmatic example of excitable system, we studied the FitzHugh-Nagumo model

$$\epsilon \dot{x} = x - \frac{1}{3}x^3 - y$$
$$\dot{y} = x + a + D\xi(t)$$

For a single unit, for $|a|<1$ The system is oscillatory; otherwise, it is excitable (there is a Hopf bifurcation on this parameter).

Let us couple many, diverse, FitzHugh-Nagumo units

$$\epsilon \dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{k}{N}\sum_{j=1}^{N}(x_j - x_i)$$

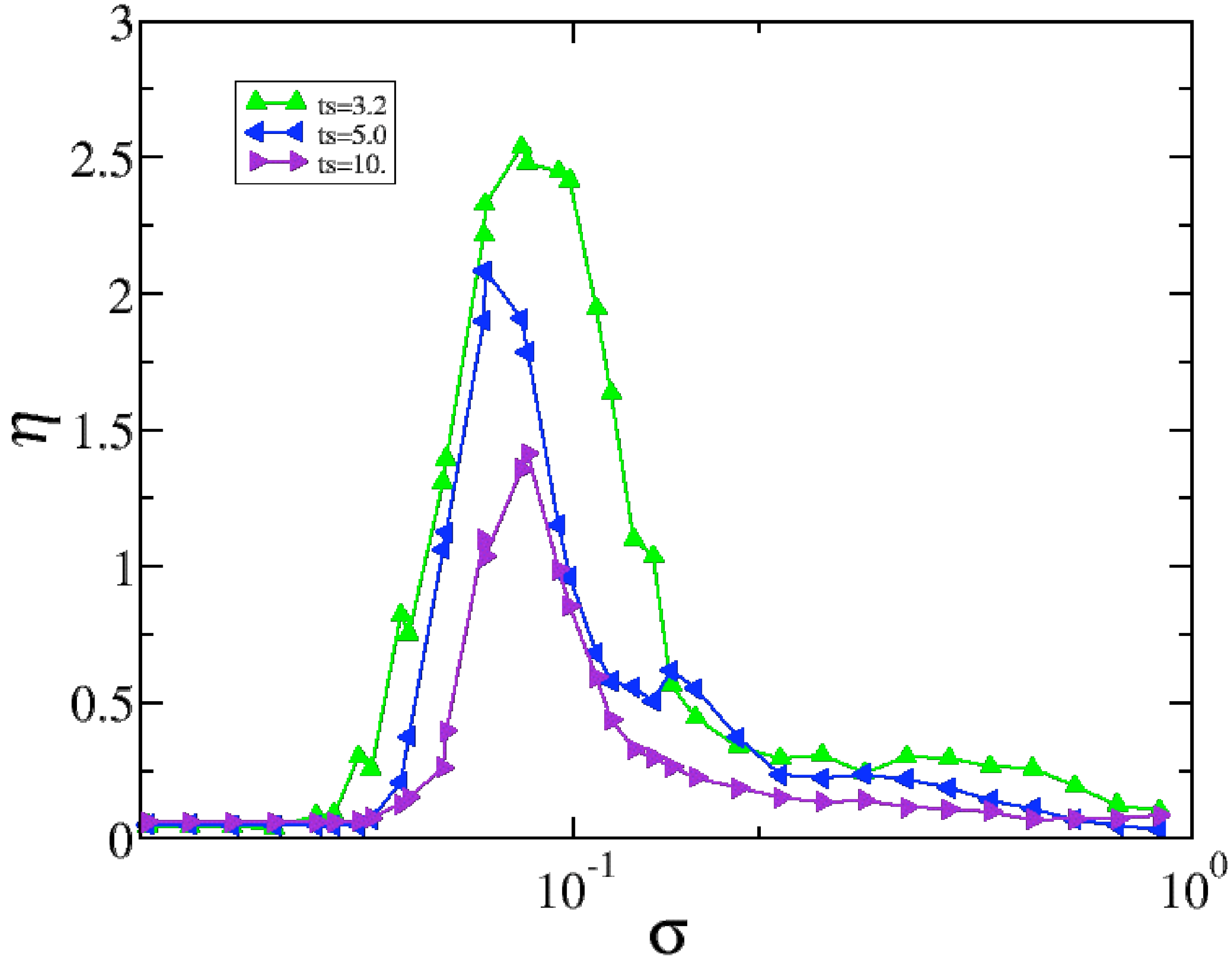
$$\dot{y}_i = x_i + a_i + \cancel{D\xi_i(t)} + A\sin\left(\frac{2\pi}{T_s}t\right)$$

$D=0$

The diversity is put in this parameter, the one that controls the bifurcation, distributed according to a Gaussian distribution of mean *a* and variance σ^2

applet

Excitable system: *numerical results*



Model for opinion formation

[M. Kuperman, D. Zanette, *Eur. Phys. Jour. B*, **26** 387 (2002)]

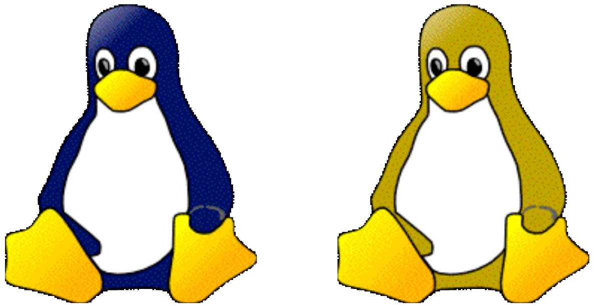
Opinion is a binary variable:

Individuals have an opinion: $\mu_i = +1,-1$

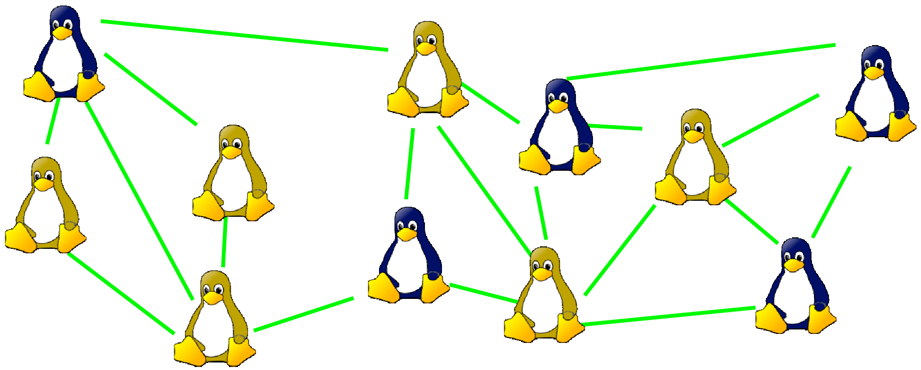
Opinion $\mu_i(t)$ changes by 3 effects:

- **Social pressure**
- **Advertising**
- **Random effects**

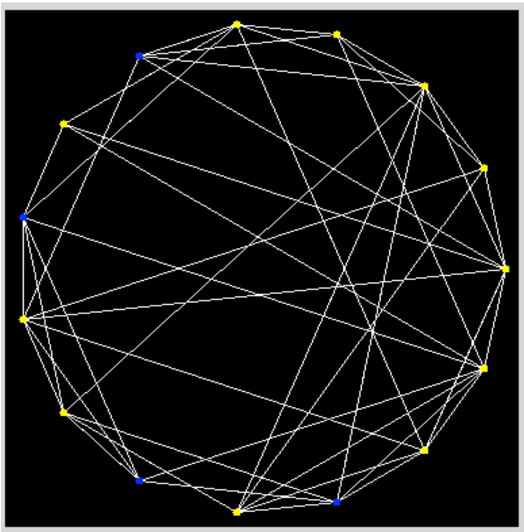
-System formed by *N* individuals which have one of two opinions



-Each individual has a set of neighbors



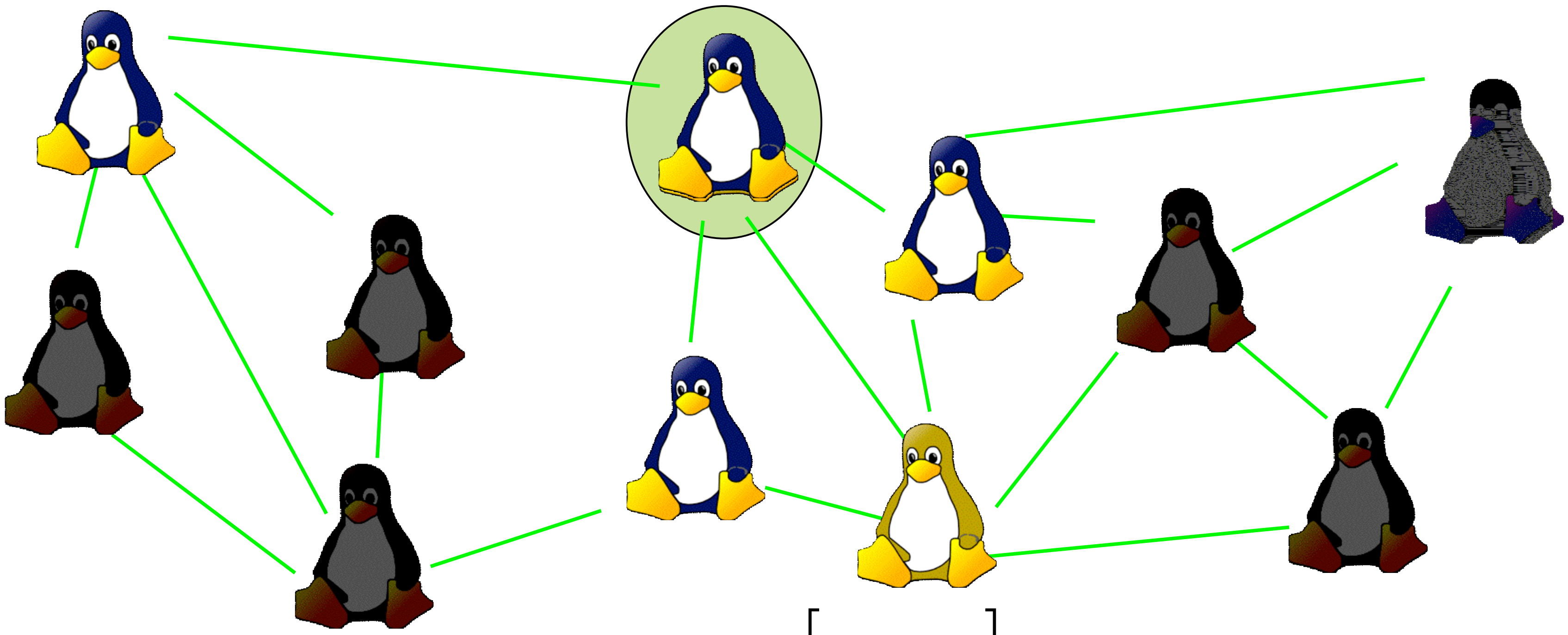
The network of neighbors is a small-world one:



- A regular 1D network built up to *k* neighbors
- with probability *p* each link is rewired, i.e. another destination node is selected

- Opinion Update: 1 *social pressure*

An individual is randomly chosen and takes the majority opinion of his neighbors

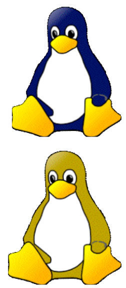


$$\mu_i(t) = sign\left[\sum_{j \in n(i)} \mu_j(t)\right]$$

- Opinion Update: 2 *advertising*

The preference for one of each opinions is assumed to change periodically in the form

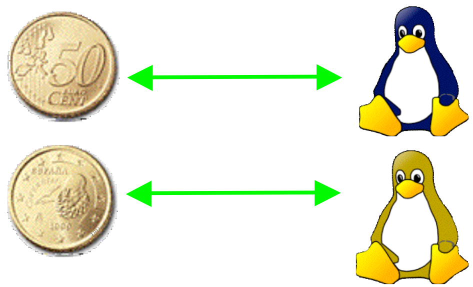
$$\cos(\omega t) \begin{cases} < 0 \longleftrightarrow \\ > 0 \longleftrightarrow \end{cases}$$



With probability $|\varepsilon \cos(\omega t)|$ the favored opinion is taken

- Opinion Update: 3 *random choice*

With probability **η** a random opinion is taken

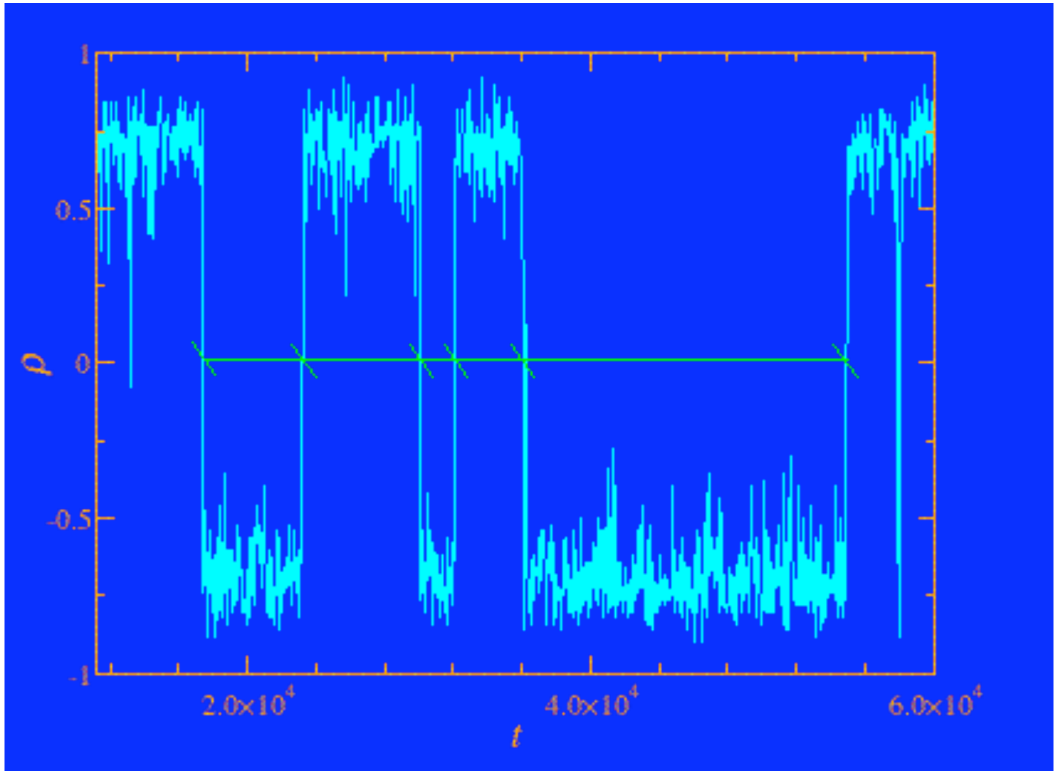


-Steps **1,2,3** are applied ***CONSEQUTIVELY***

-After each repetition, *t* increases by 1/N

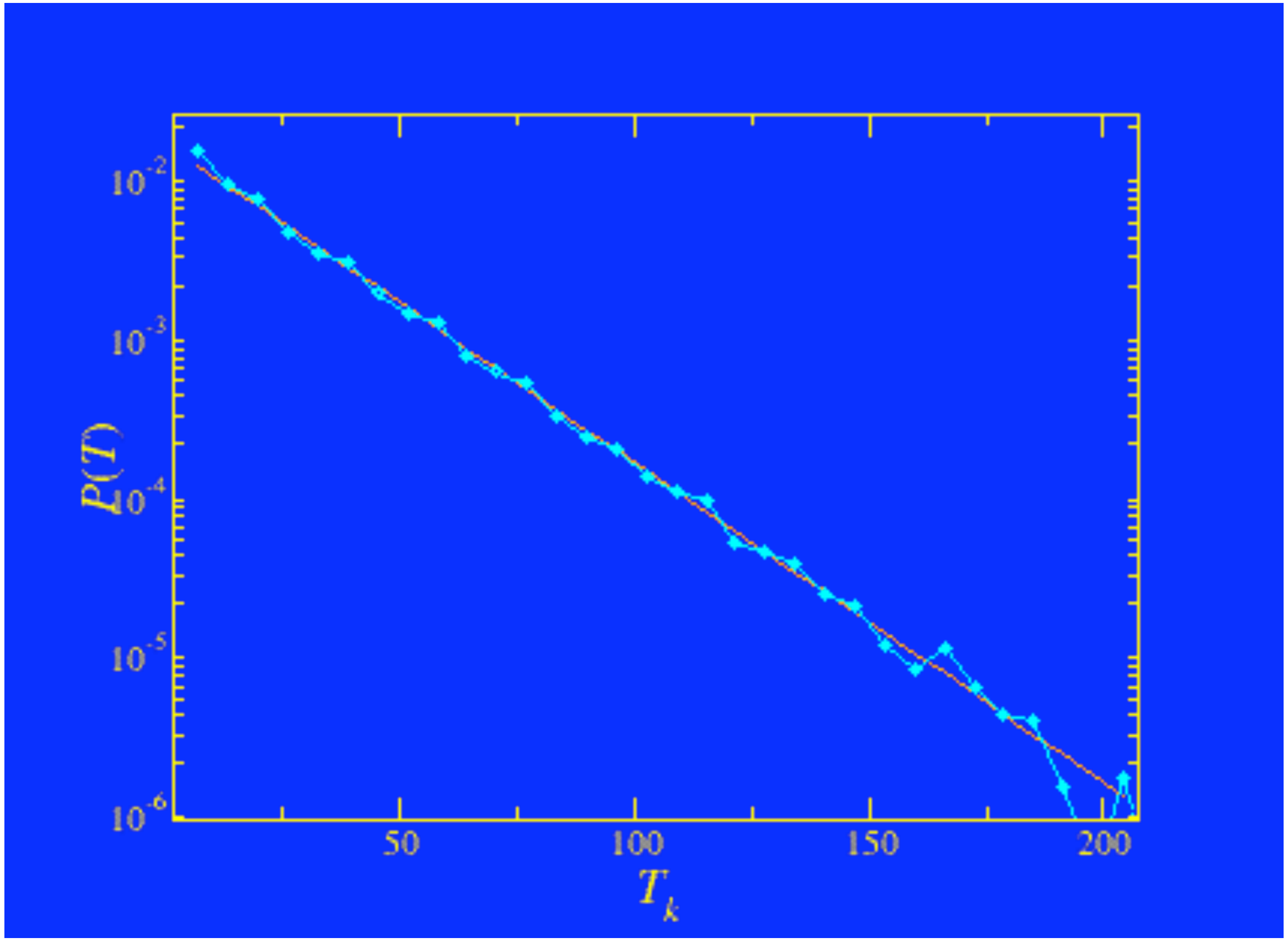
Results: Dynamical Evolution

i) In absence of external forcing, behaves as a bistable system

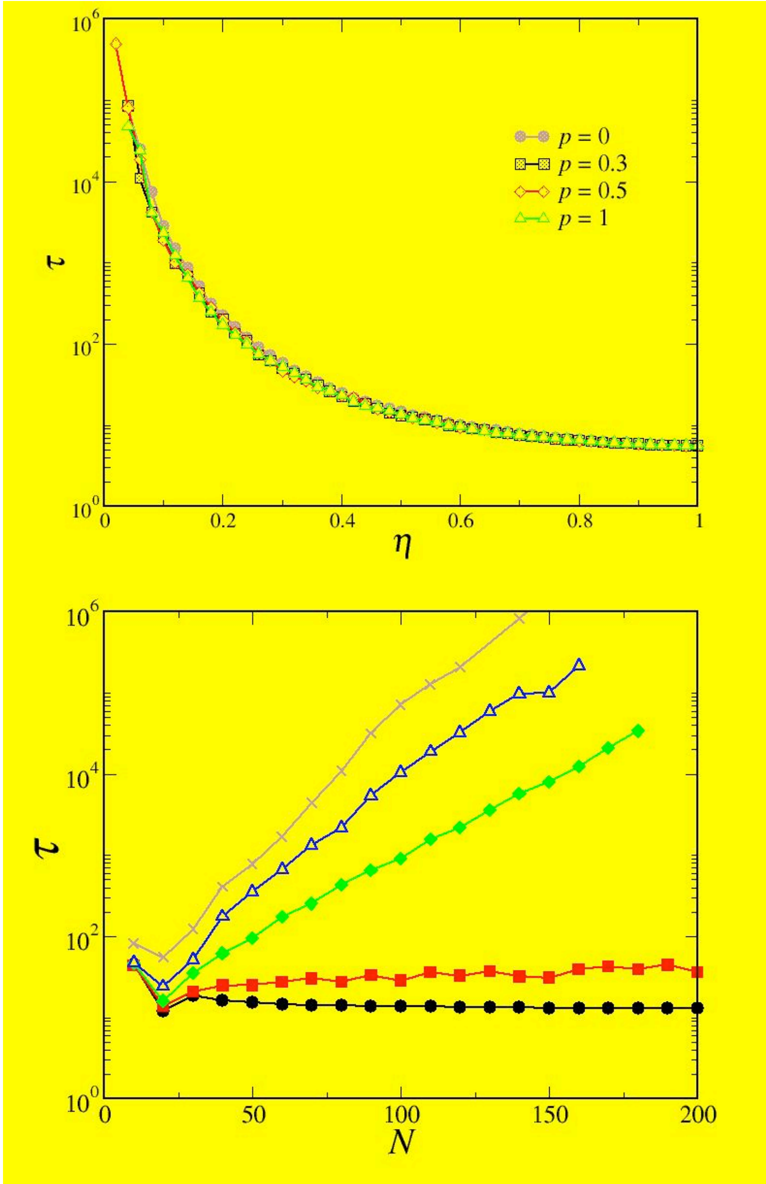


ii) the residence time distribution follows Kramer's law

$$p(T) = \tau \exp(-T/\tau)$$



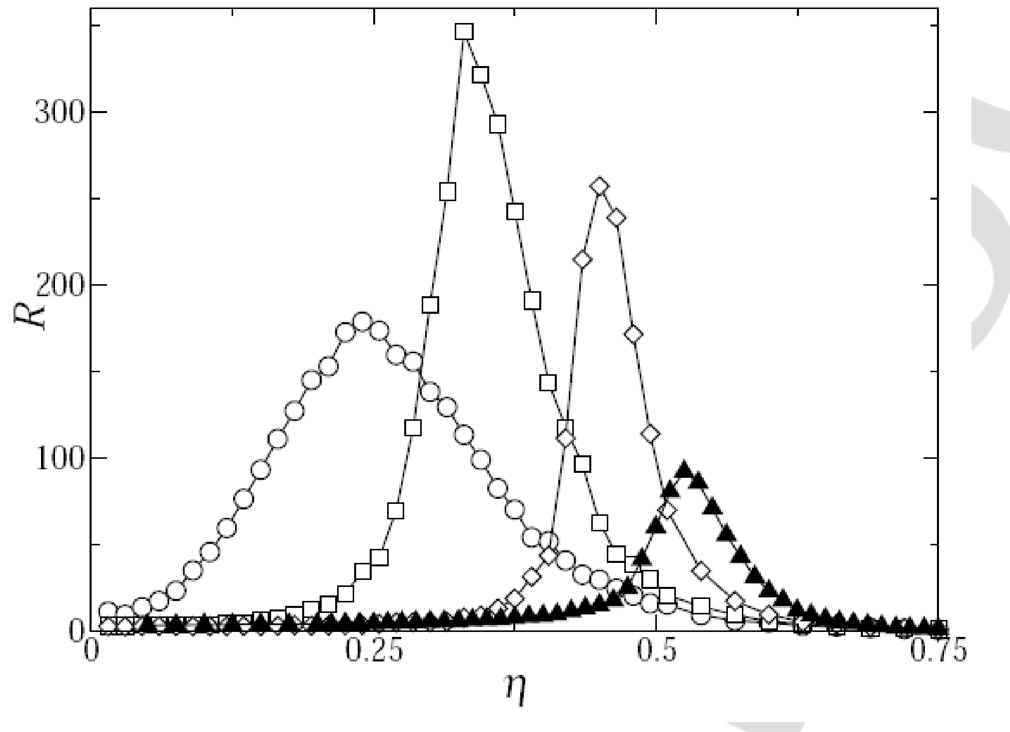
τ decreases with noise rate



τ increases with systems size

-We have all the ingredients for stochastic resonance:

-Bistable system (two opinions)
-Coupling (social pressure)
-Noise (random choice "free will")
-External forcing



Diversity introduced through ***preferences*** θ_i:

$$\langle \theta_i \rangle = 0, \qquad \langle \theta_i^2 \rangle = \sigma^2$$

Step 1: social pressure

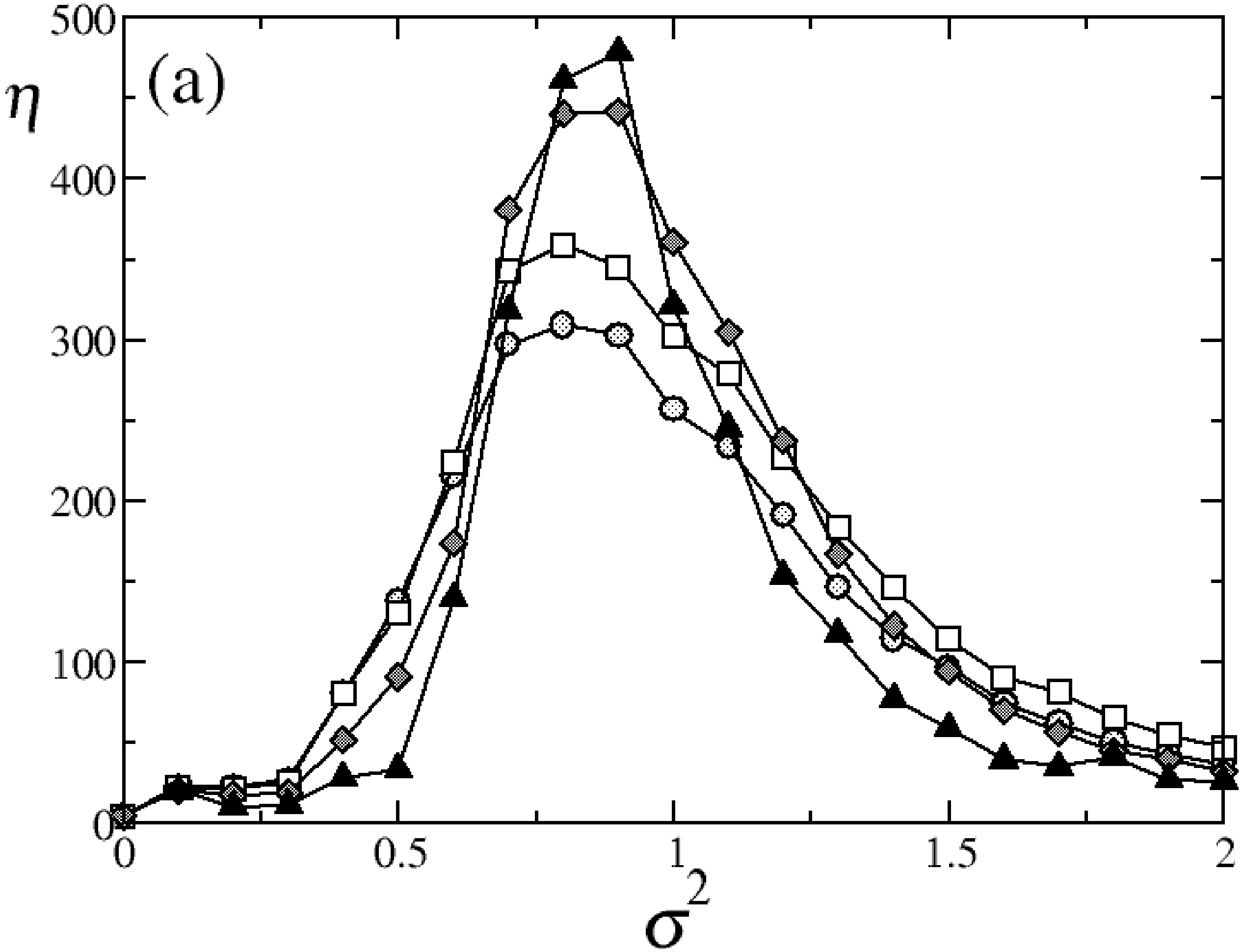
$$\mu_i(t) = sign\left[\sum_{j \in n(i)} \mu_j(t) + \theta_i \right]$$

Step 2: advertising

With probability $\left| \varepsilon \sin(\omega t) + \theta_i \right|$

set $\mu_i(t) = sign\left[\sin(\omega t) + \alpha \theta_i \right]$

Diversity-induced Resonance



Conclusions

- In bistable coupled systems, there is an optimal amount of the dispersion that optimizes the response to an external forcing.
- This is a rather general phenomenom similar to that of Stochastic Resonance. It also happens in excitable systems.
- In a majority opinion formation model, an external influence works optimally under the right amount of diversity in individual preferences.
- Any source of disorder leads to similar results
- Real systems could, or could not, be taking advantage of these mechanisms. Experimental verification in some candidate systems is most welcome to check whether these diversity-induced effects are taking place.

Thank you... for being different.

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