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Self-similarity of complex networks & hidden metric spaces

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Self-similarity of complex networks?

scale-free degree distributions $P(k) \sim k^{-\gamma}$

scale invariant property $f(\lambda v) = \lambda^{\tau} f(v)$

box-covering renormalization procedures

some systems (WWW, biological networks) have degree distributions that remain invariant and have finite “fractal” dimension

C. Song, S. Havlin, H. A. Makse,
Nature 433, 392-395 (2005);
Nature Physics 2, 275-281 (2006)

Where is geometry?

Self-similarity and scale invariance of complex networks are still not well defined in a proper geometrical sense

Lack of a metric structure except lengths of shortest paths

small world property

geometric length scale transformations?

Networks embedded in metric spaces
maybe "hidden"

(as variations of hidden variables)

M. Boguñá and R. Pastor-Satorras, Phys. Rev. E **68**, 036112 (2003).
G. Caldarelli, A. Capocci, P. D. L. Rios, and M. A. Muñoz, Phys. Rev. Lett. **89**, 258702 (2002).
B. Söderberg, Phys. Rev. E **66**, 066121 (2002).

- Geography as an obvious geometrical embedding:
  airport networks, urban networks…

- Hidden metric spaces: WWW (similarity between pages induced by content), social networks (closeness in social space)…

…how to identify these hidden metric spaces…

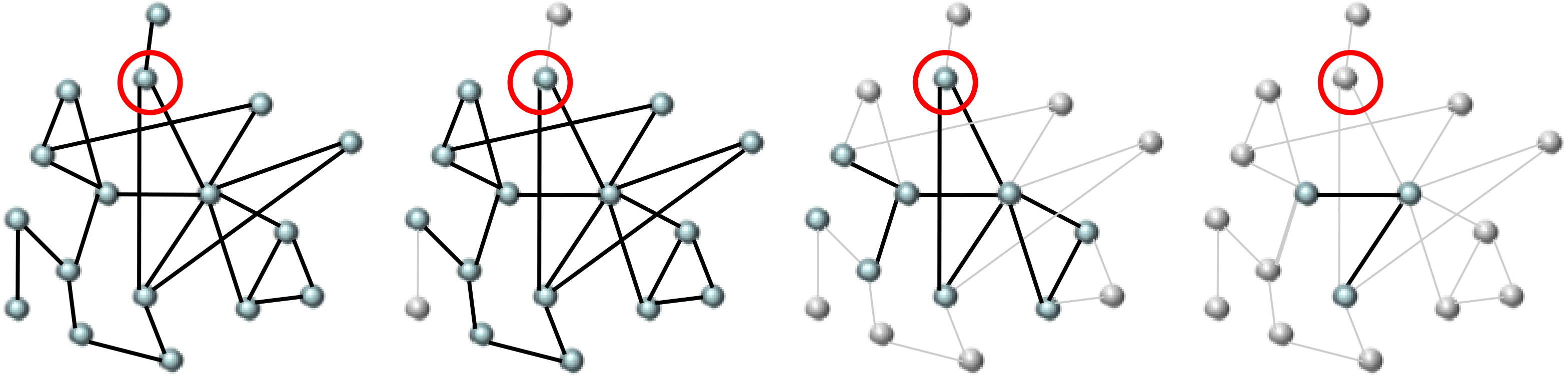
- Some real scale-free networks are self-similar (degree distribution, degree-degree correlations, and clustering) with respect to a simple degree-thresholding renormalization procedure (purely topological)

- A class of hidden variable models with underlying metric spaces are able to accurately reproduce the observed self-similarity properties

We conjecture that hidden geometries underlying some real networks are a plausible explanation for their observed self-similar topologies

degree-thresholding renormalization procedure

$$k > k_T$$



$k_T = 0$ $k_T = 1$ $k_T = 2$ $k_T = 3$

$$k_i / < k_i(k_T) >$$

Hierarchy of nested subgraphs

Topological properties of the subgraphs as a function of
The inner degree
Rescaled
by the average inner degree

BGP map of the Internet at the AS level

SF with exponent 2.1

N=17446

<k>=4.68

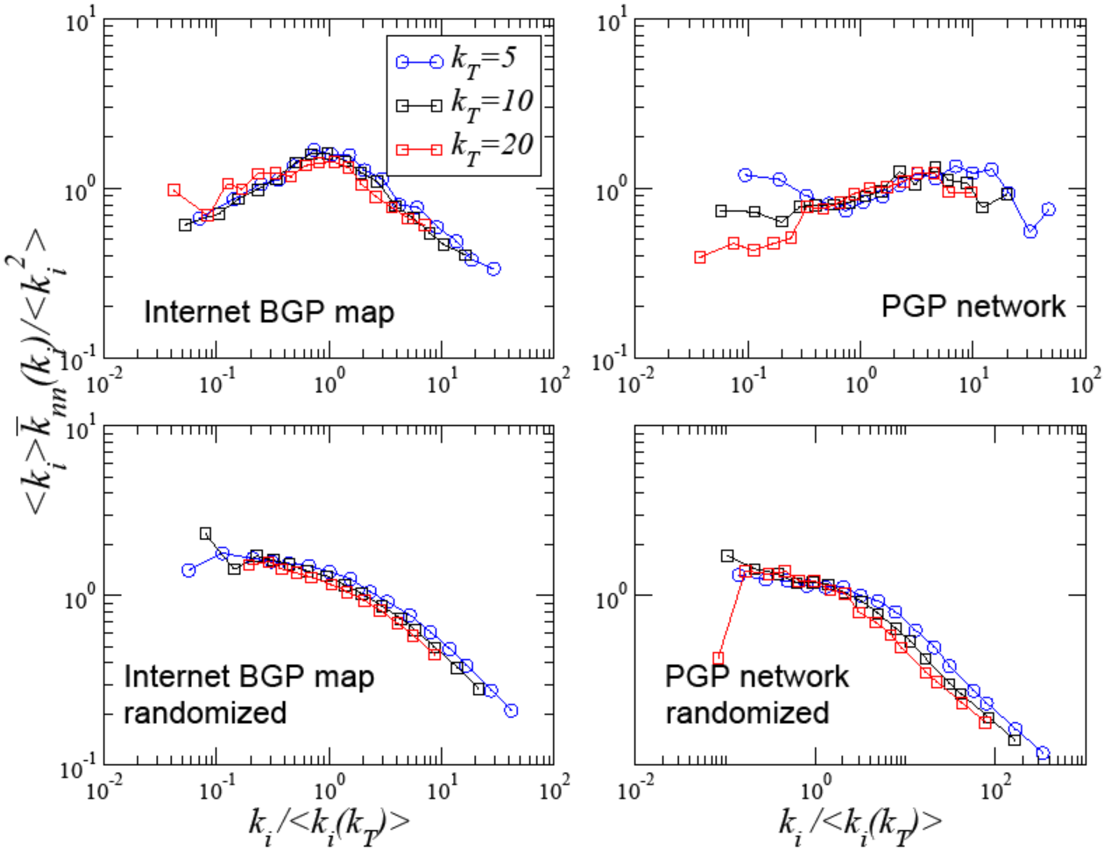
PGP social web of trust

SF with exponent 2.5

N=57243

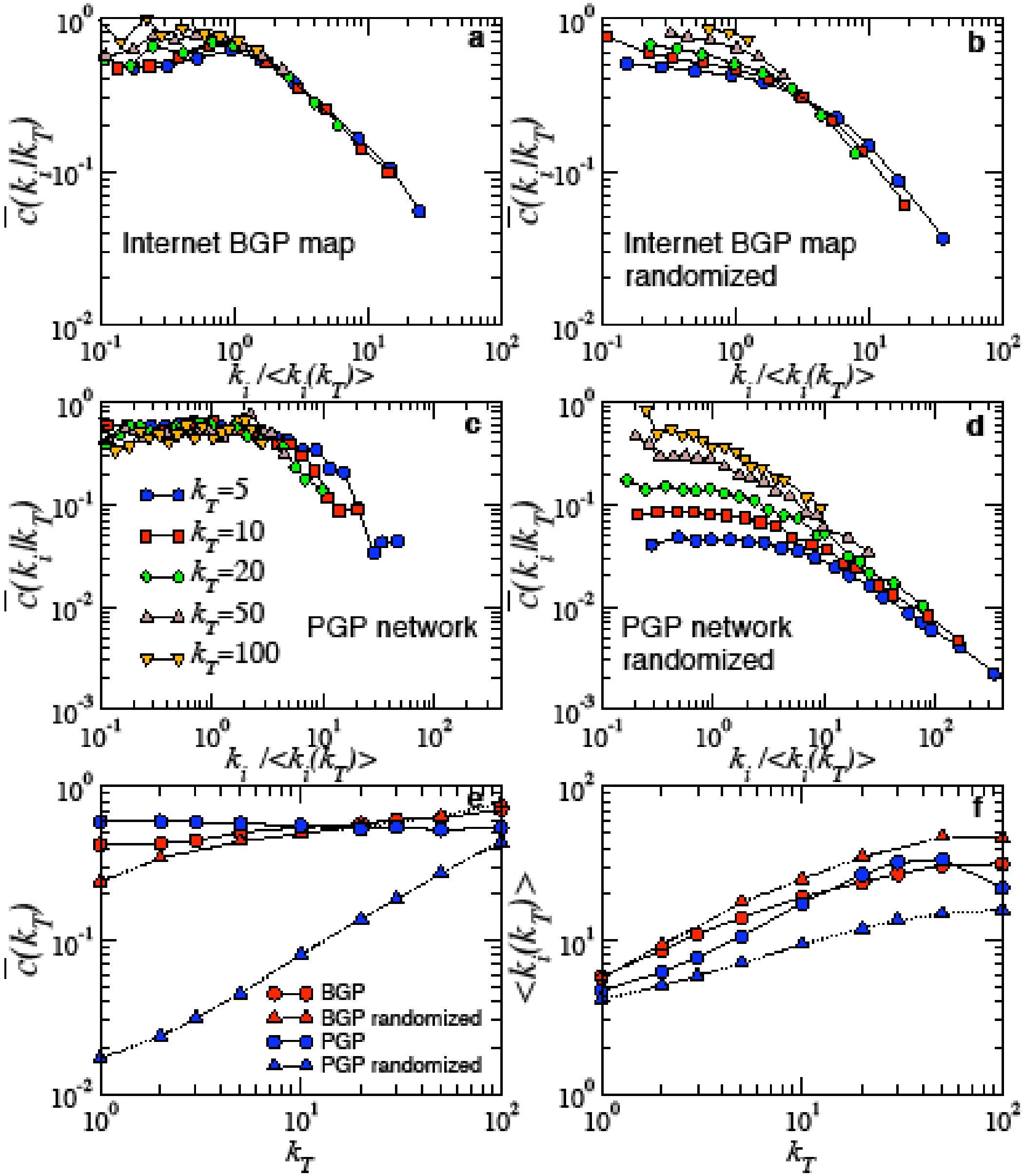
<k>=2.16

(also U.S. airports network)



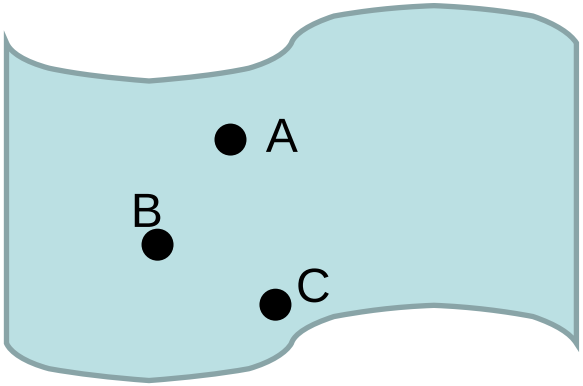
Average nearest neighbors degree

Degree dependent clustering coefficient



A random model like the CM will produce self-similar networks regarding the degree distribution and degree-degree correlations, if the degree distribution of the complete graph is SF….

the key point is to reproduce self-similar clustering

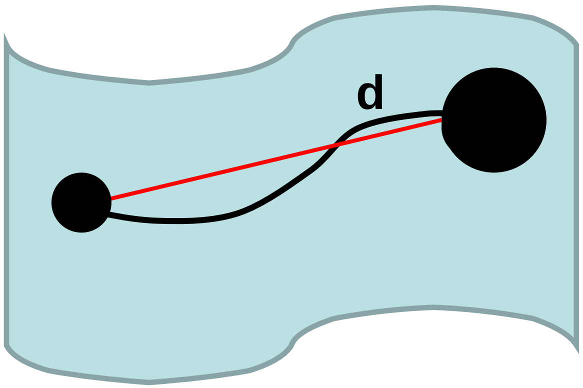


metric space **TRIANGLE INEQUALITY**

Therefore, we focus on clustering as a potential connection between the observed topologies and hidden geometries.

- A class of hidden variable models with underlying metric spaces are able to accurately reproduce the observed self-similarity properties

$\rho(\kappa)$



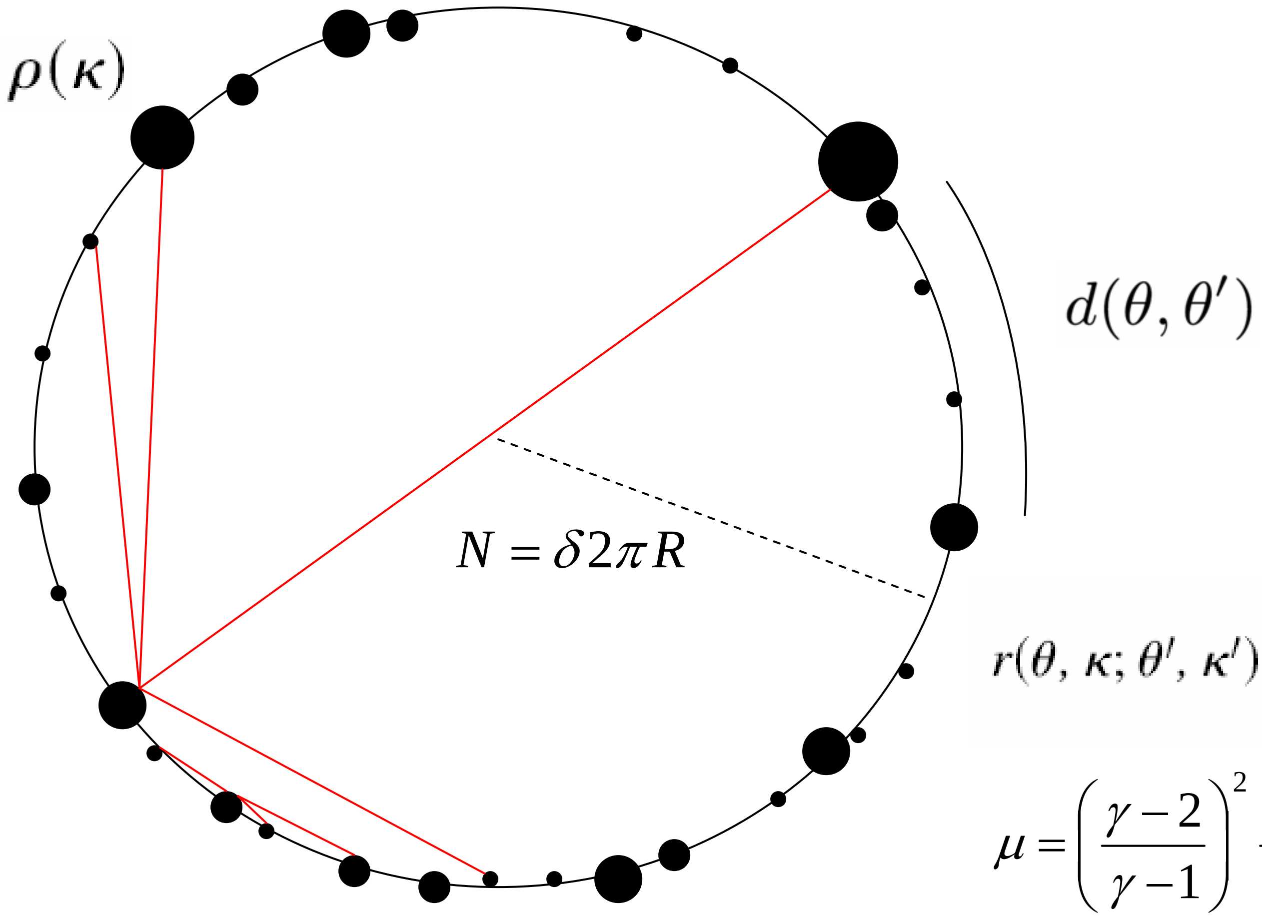
$$r\left(\frac{d}{d_c}\right)$$

Nodes that are close to ech other are more likely to be connected

To control the degree distribution, the characteristic distance depends on the expected degree, and so $\rho(\kappa) \approx P(k)$

$$d_c(\kappa, \kappa') \propto (\kappa\kappa')^{1/D}$$

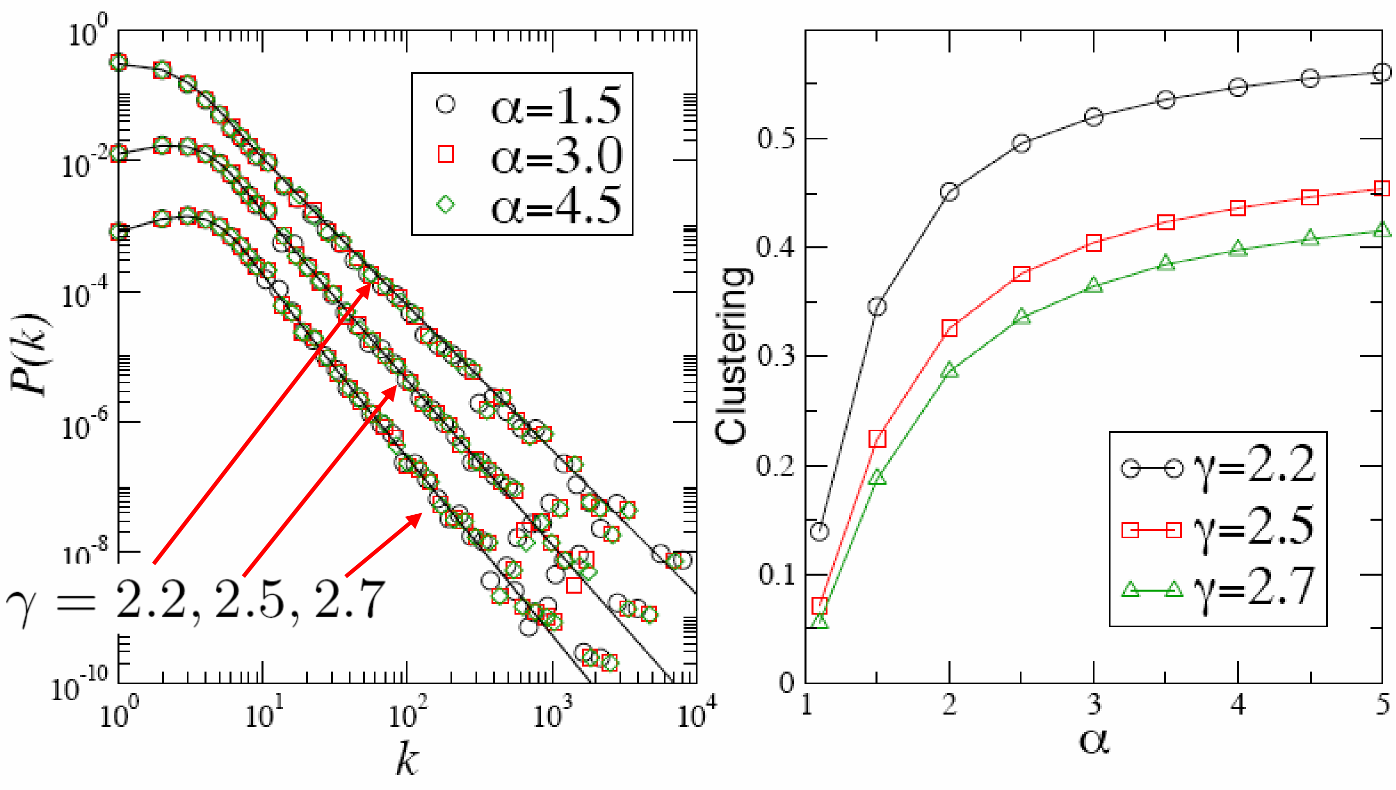
The **S**^{1} model



$$r(\theta, \kappa; \theta', \kappa') = \left(1 + \frac{d(\theta, \theta')}{\mu \kappa \kappa'}\right)^{-\alpha},$$

$$\mu = \left(\frac{\gamma - 2}{\gamma - 1}\right)^2 \frac{(\alpha - 1) < k >}{2\delta \kappa_0^2} \qquad \alpha > 1,$$

The $\mathbf{S^1}$ model



$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma},$$

$$\kappa > \kappa_0 \equiv (\gamma - 2)\langle k\rangle/(\gamma - 1), \quad \gamma > 2$$

$$P(k) = (\gamma - 1)\kappa_0^{\gamma-1}\frac{\Gamma(k + 1 - \gamma, \kappa_0)}{k!}, \quad P(k) \sim k^{-\gamma}$$

The S^1 model

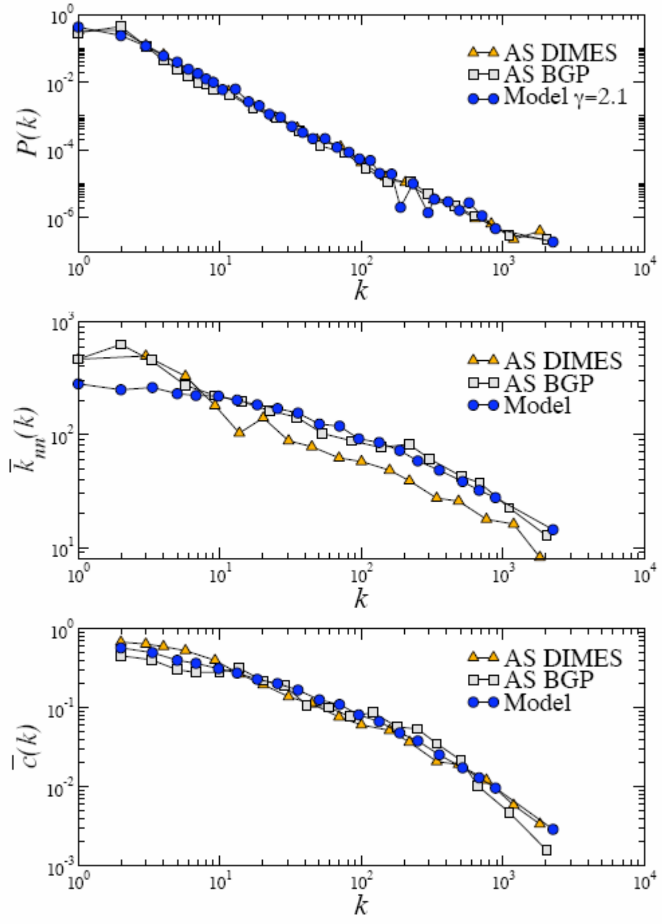


FIG. 1: Degree distribution $P(k)$, average nearest neighbours' degree $\bar{k}_{nn}(k)$, and degree-dependent clustering coefficient $\bar{c}(k)$ generated by our model with $\gamma = 2.1$ and $\alpha = 2$ compared to the same metrics for the real Internet map as seen by BGP data and the DIMES project.

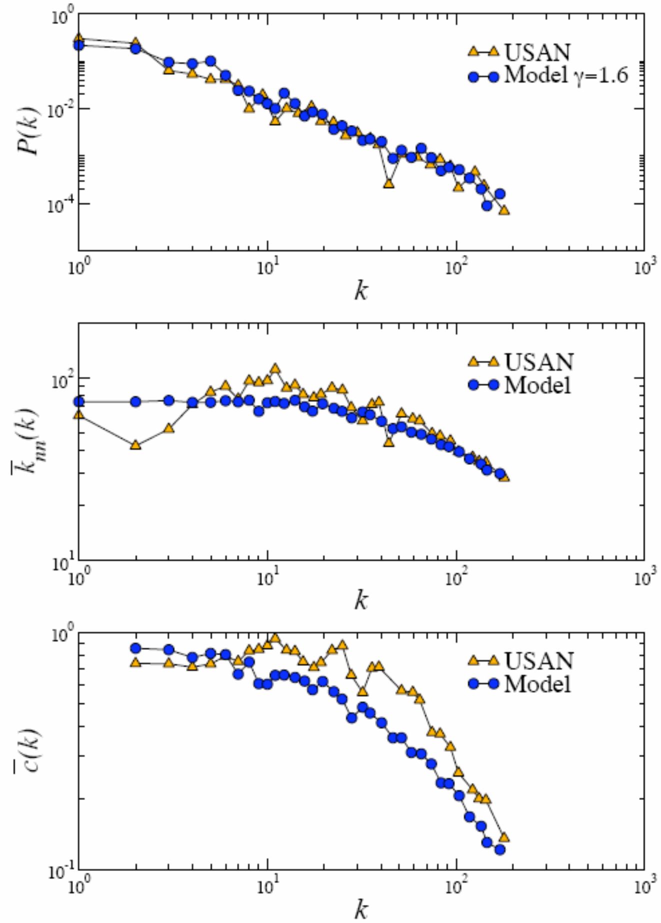


FIG. 2: Degree distribution $P(k)$, average nearest neighbours' degree $\bar{k}_{nn}(k)$, and degree-dependent clustering coefficient $\bar{c}(k)$ generated by our model with $\gamma = 1.6$, $\alpha = 5$ and a cut-off at $k_c = 180$ compared to the same metrics for the real US airport network.

The $\mathbf{S}^1$ model

- $\rho(\kappa)$ controls the degree distribution, SF
- independently, α controls the level of clustering , strong clustering
- given α, the parameter $\mu = \left(\frac{\gamma-2}{\gamma-1}\right)^2 \frac{(\alpha-1)<k>}{2\delta\kappa_0^2}$ controls the average degree
- if $1<\alpha<2,\ 2<\gamma<3$, small-world!!! but underlying metric space!

$$p(d, \kappa|\kappa') = \frac{2}{\kappa'}\rho(\kappa)\left(1+\frac{d}{\mu\kappa\kappa'}\right)^{-\alpha}.$$

D=1

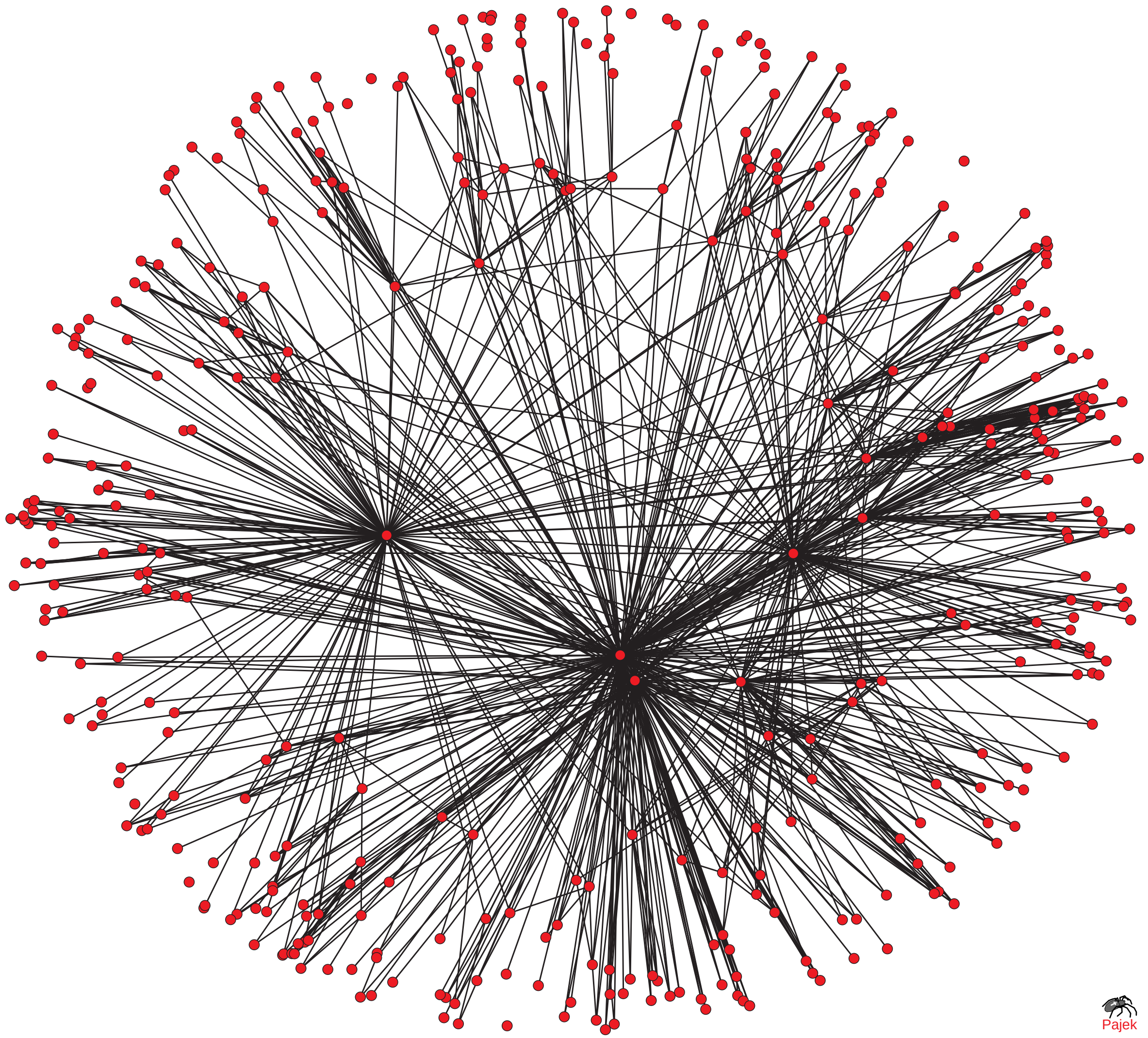
$$\bar{d}(\kappa') = \int x p(x|\kappa')dx \xrightarrow{N,R\to\infty} \infty$$

$$p(d|\kappa') \sim d^{-\alpha} \text{ when } \alpha < \gamma - 1$$

$$p(d|\kappa') \sim d^{1-\gamma} \text{ when } \alpha > \gamma - 1$$

Self-similarity of the **S¹** model

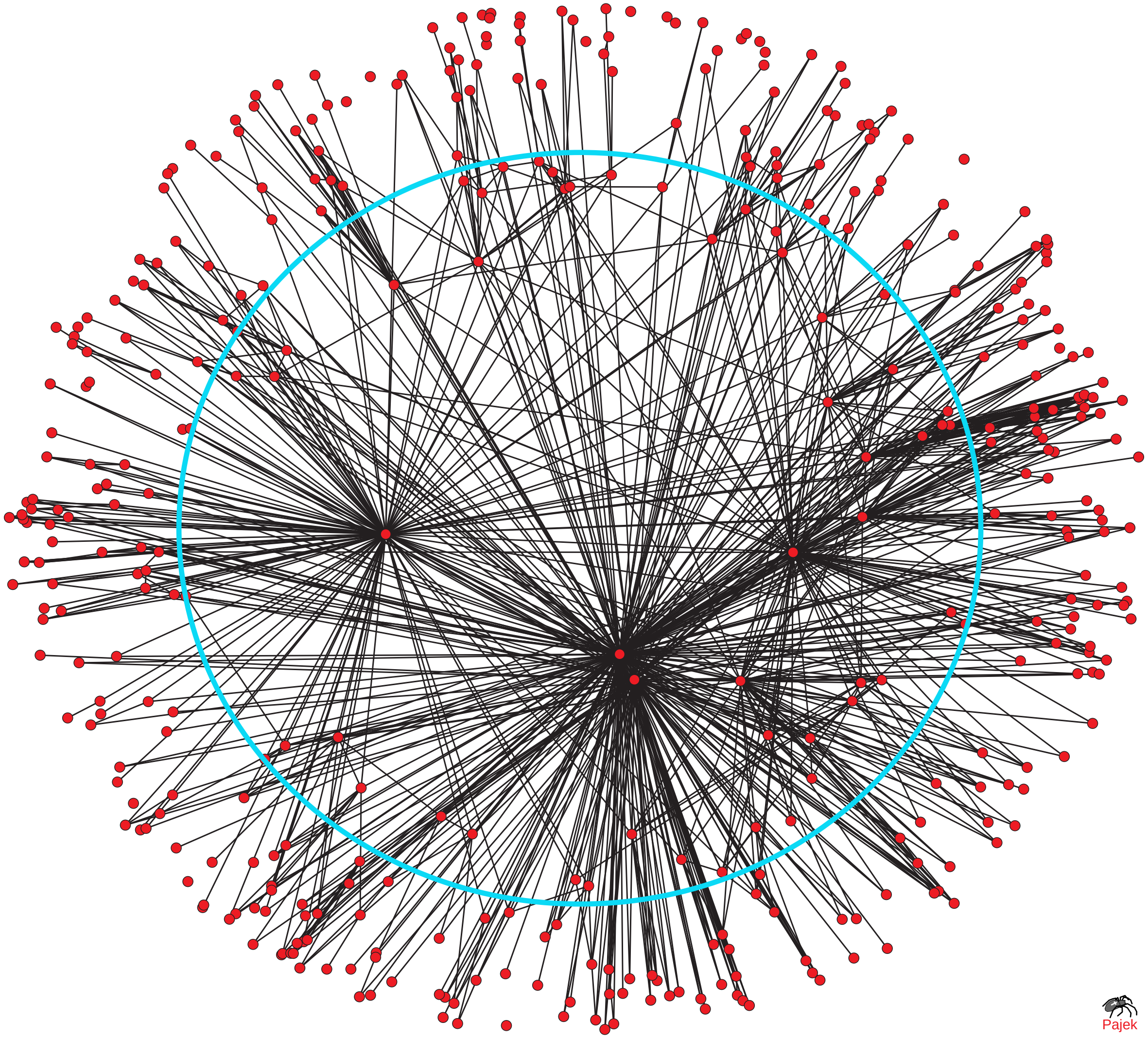
Nodes in the complete graph can be represented such that higher degree nodes are closer to the center of the S¹ ring



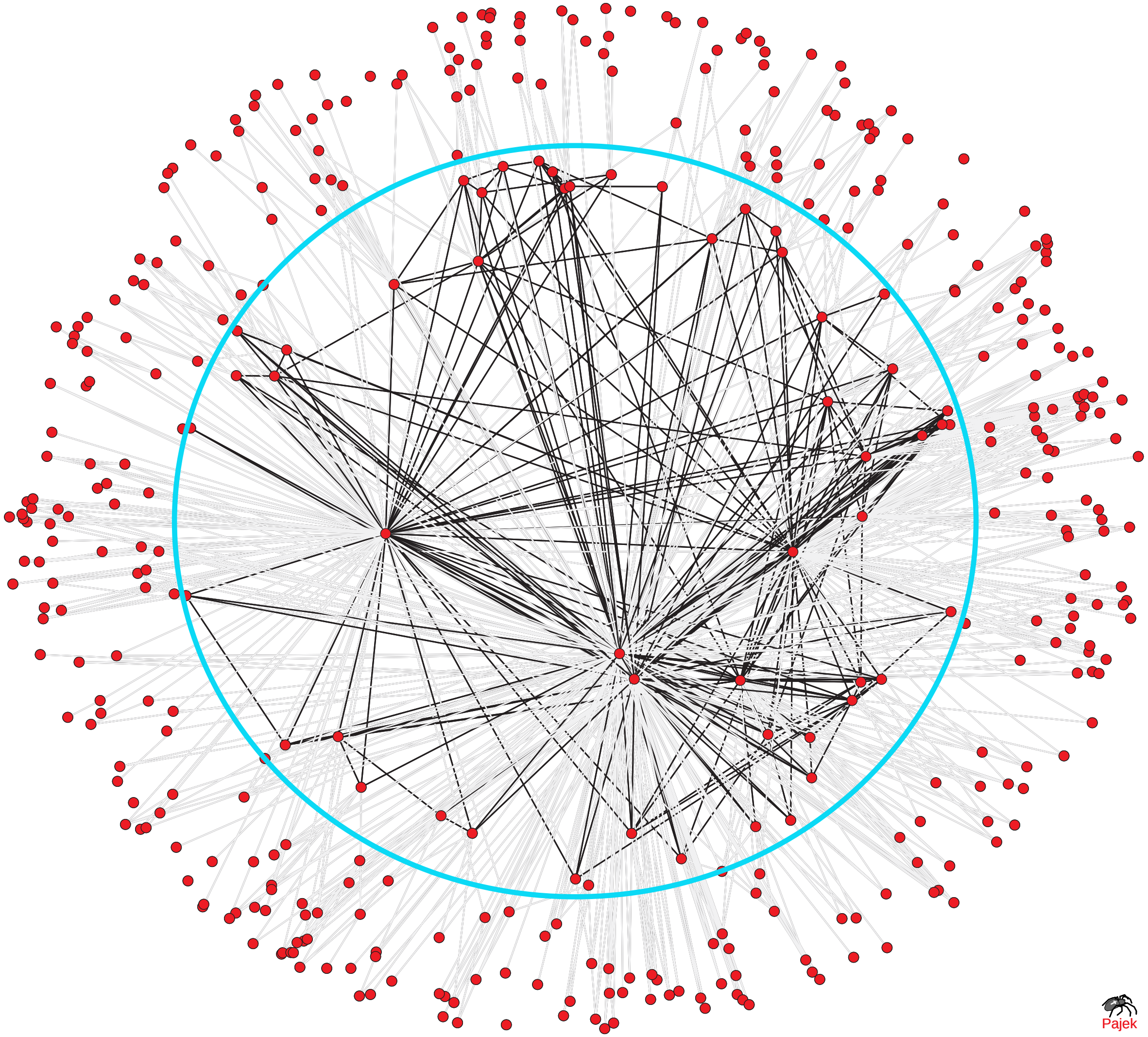
Initial graph

Self-similarity of the $\mathbf{S}^1$ model

A subgraph is formed by all nodes and their connections inside a circle of radius r



Self-similarity of the **S¹** model



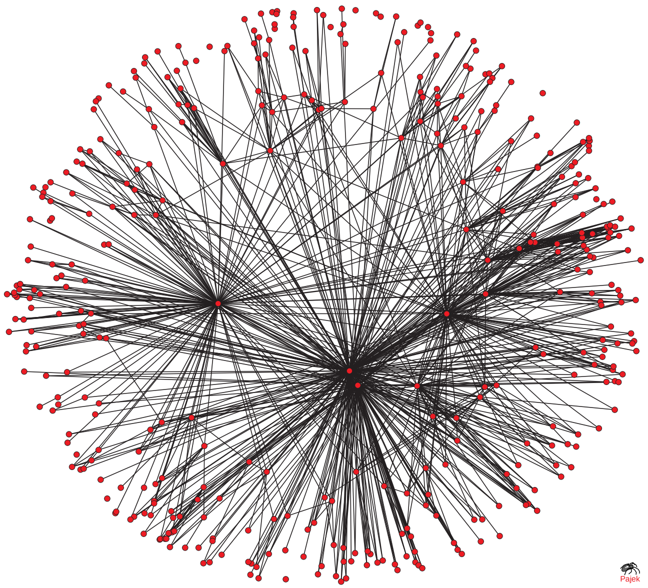
Subgraph

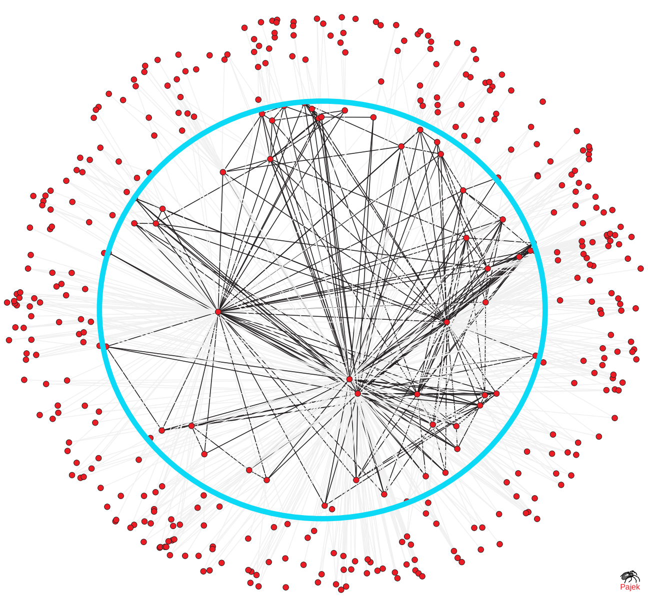
Self-similarity of the **S¹** model

$$\rho(\kappa) = (\gamma - 1)\frac{\kappa_0^{\gamma-1}}{\kappa^{\gamma}} \qquad N = \delta 2\pi R$$

$$r(\theta, \kappa; \theta', \kappa') = \left(1 + \frac{d(\theta, \theta')}{\mu \kappa \kappa'}\right)^{-\alpha},$$

$$\mu = \left(\frac{\gamma - 2}{\gamma - 1}\right)^2 \frac{(\alpha - 1) < k >}{2\delta \kappa_0^2}$$





$$\rho_r(\kappa) = (\gamma - 1)\frac{\kappa_T^{\gamma-1}}{\kappa^{\gamma}} \qquad N_r = \delta_r 2\pi R$$

$$r(\theta, \kappa; \theta', \kappa') = \left(1 + \frac{d(\theta, \theta')}{\mu \kappa \kappa'}\right)^{-\alpha},$$

$$\mu = \left(\frac{\gamma - 2}{\gamma - 1}\right)^2 \frac{(\alpha - 1) < k >_r}{2\delta_r \kappa_T^2}$$

$$\delta_r = \delta \left(\frac{\kappa_0}{\kappa_T}\right)^{\gamma-1}$$

$$\delta_r = \delta \left(\frac{\kappa_0}{\kappa_T}\right)^{\gamma-1}$$

$$< k >_r = < k > \left(\frac{\kappa_T}{\kappa_0}\right)^{3-\gamma}$$

Self-similarity of the **S¹** model

$G(\kappa_T)$ are replicas of G

after

$$<k>_r = <k>\left(\frac{\kappa_T}{\kappa_0}\right)^{3-\gamma}$$

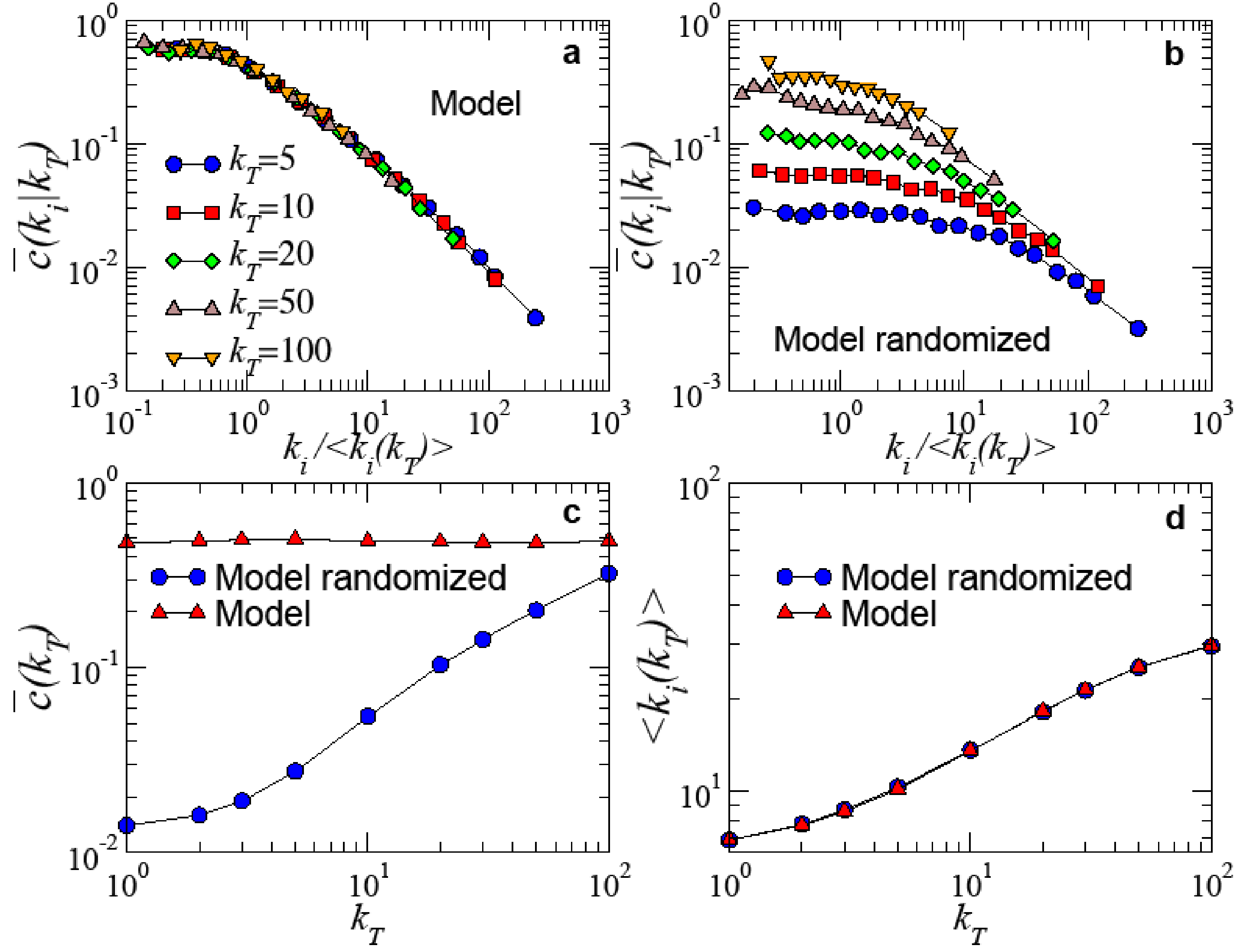
In particular, clustering spectrum and clustering…

$$\bar{c}(\kappa|\kappa_T) = f(\kappa/\kappa_T), \quad \bar{c}(k_i|k_T) \approx f(k_i/k_T^{3-\gamma}) = \tilde{f}[k_i/\langle k_i(k_T)\rangle],$$

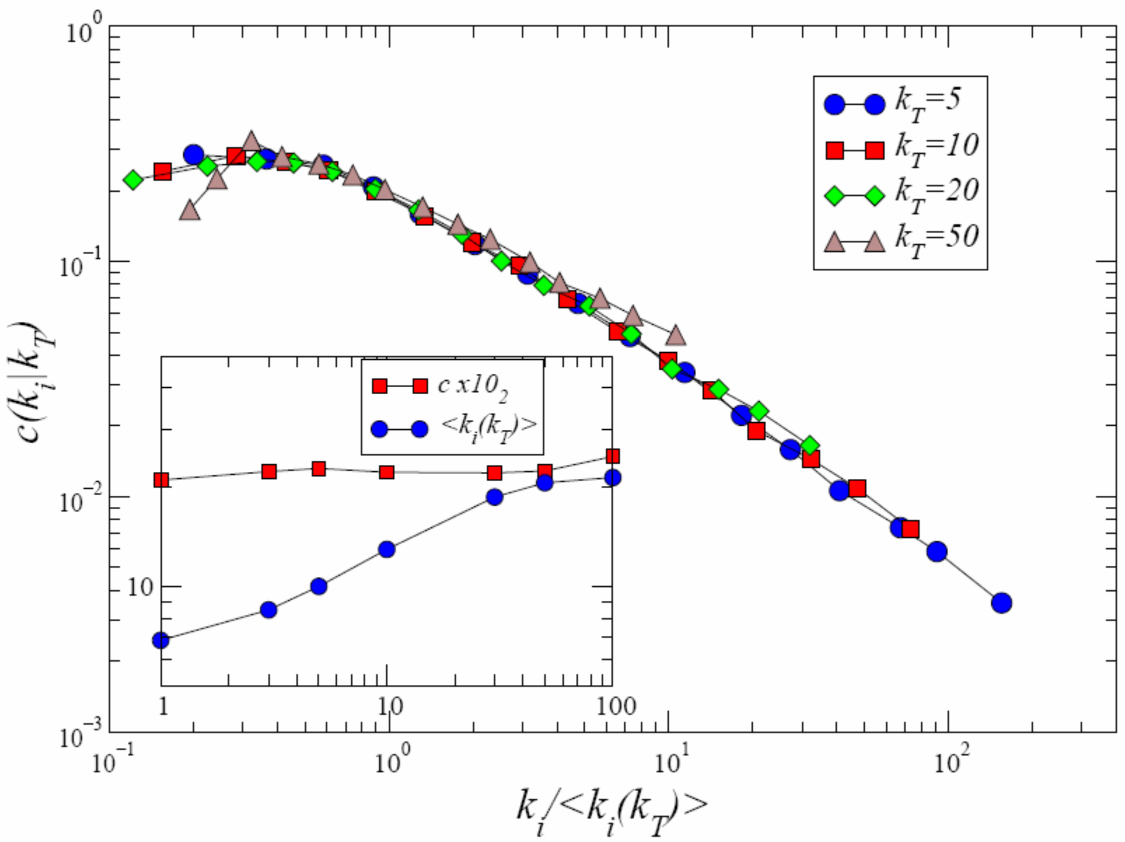
$$\bar{c}(\kappa_T) = \int_{\kappa_T} d\kappa \rho(\kappa|\kappa_T)\bar{c}(\kappa|\kappa_T),$$

Independent of κ_T

The $\mathbf{S^1}$ model



The **S²** model


10^0
10^{-1}
10^{-2}
10^{-3}
$c(k_i|k_T)$
k_T=5
k_T=10
k_T=20
k_T=50
c x10$_2$
<$k_i(k_T)$>
10
1
10
100
10^{-1}
10^0
10^1
10^2
k_i/<$k_i(k_T)$>

In summary

hidden geometries underlying the observed topologies of some complex networks appear to provide a simple a natural explanation of their degree-renormalization self-similarity

real networks could be, after all, embedded in metric spaces

Future work

- **fractality,** C. Song, S. Havlin, H. A. Makse, Nature 433, 392-395 (2005); Nature Physics 2, 275-281 (2006). K.I. Goh, G. Salvi, B. Kahng, and D. Kim, PRL 96, 018701 (2006); J.S. Kim, K.I. Goh, B. Kahng, and D. Kim, New J. Phys 9, 177 (2007);
- **k-core self-similarity**, J. I. Alvarez-Hamelin, L. Dall'Asta, A. Barrat and A. Vespignani , Networks and Heterogeneous Media, 3, 395-411 (2008).

THANK YOU for your attention

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Self-Similarity of Complex Networks and Hidden Metric Spaces

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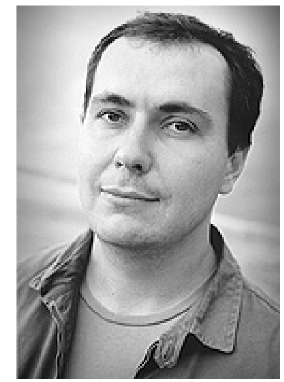

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