BUSSE-HEIKES MODEL

THEORETICAL MODEL

$$\partial_t A_1 = \mathcal{L}_1 A_1 + A_1[1 - |A_1|^2 - g_+ |A_2|^2 - g_- |A_3|^2]$$

$$\partial_t A_2 = \mathcal{L}_2 A_2 + A_2[1 - |A_2|^2 - g_+ |A_3|^2 - g_- |A_1|^2]$$

$$\partial_t A_3 = \mathcal{L}_3 A_3 + A_3[1 - |A_3|^2 - g_+ |A_1|^2 - g_- |A_2|^2]$$

$$g_\pm = 1 + \mu \pm \delta$$

$$\mathcal{L}_j = \nabla^2, \ \partial^2_{x_j}$$

$$\partial_t \vec{A} = -\frac{\delta F_{\mathrm{BH}}}{\delta \vec{A}} + \delta \vec{f}(\vec{A})$$

$$\vec{A} = (A_1, A_2, A_3)$$

$\Longrightarrow$

- $\delta = 0 \rightarrow$ **potential dynamics**
- $\delta \neq 0$
  - $\delta \int d\vec{r} \, \frac{\delta F_{\mathrm{BH}}}{\delta \vec{A}^*} \cdot \vec{f}(\vec{A})^* + \mathrm{c.c.} = 0 \rightarrow$ **nonrelaxational potential flow**
  - **otherwise** $\rightarrow$ **nonpotential**

BUSSE-HEIKES MODEL

ZERO-DIMENSIONAL SYSTEMS

$$\mathfrak{L}_j = 0, \quad A_j = \sqrt{R_j(t)}\, e^{i\theta_j(t)}, \; j = 1,2,3$$

$$\Downarrow$$

$$\left.\begin{aligned}
\dot{R}_1(t) &= 2R_1[1 - R_1 - (1+\mu+\delta)R_2 - (1+\mu-\delta)R_3] \\
\dot{R}_2(t) &= 2R_2[1 - R_2 - (1+\mu+\delta)R_3 - (1+\mu-\delta)R_1] \\
\dot{R}_3(t) &= 2R_3[1 - R_3 - (1+\mu+\delta)R_1 - (1+\mu-\delta)R_2]
\end{aligned}\right\}, \quad
\left.\begin{aligned}
\dot{\theta}_1 &= 0 \\
\dot{\theta}_2 &= 0 \\
\dot{\theta}_3 &= 0
\end{aligned}\right\}$$

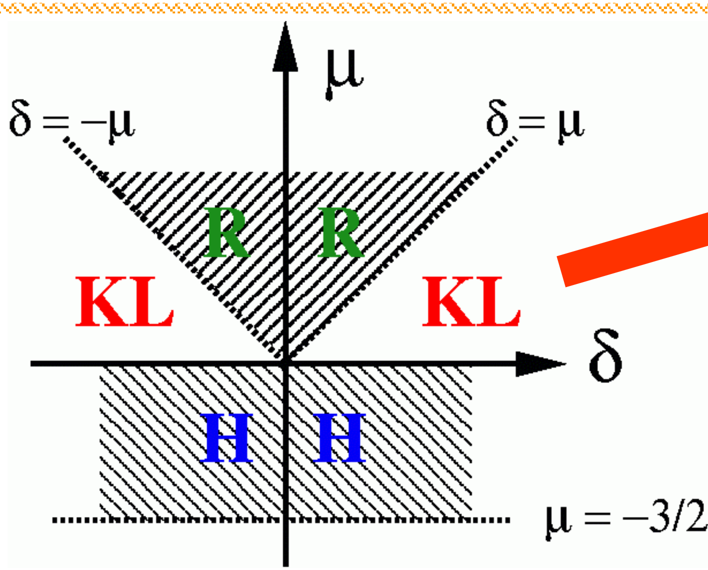
- **Equations for the real variables** $\{R_1, R_2, R_3\}$ **instead of complex variables** $\{A_1, A_2, A_3\}$
- **Similar sets of equations proposed to study population competition dynamics**

BUSSE-HEIKES MODEL

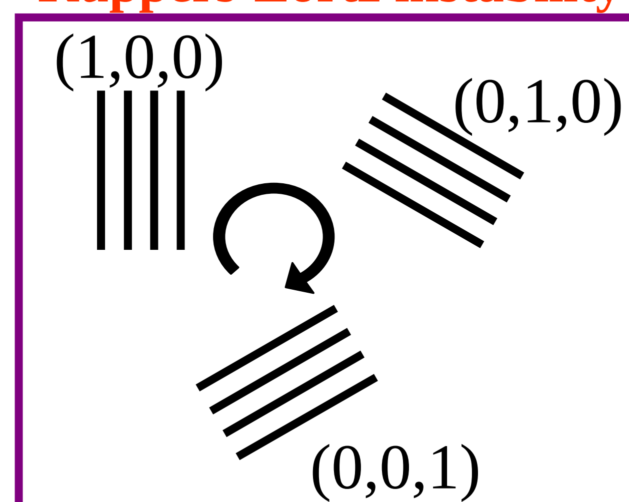
Stationary Solutions

- **Null:** $(R_1, R_2, R_3) = (0,0,0)$
- **Rhombus :** $(R_1, R_2, R_3) = \left(\left\{\frac{\mu+\delta}{\mu(\mu+2)-\delta^2}, \frac{\mu-\delta}{\mu(\mu+2)-\delta^2}, 0\right\}, \ldots\right)$
- **Rolls R:** $(R_1, R_2, R_3) = \{(1,0,0), \ldots\}$
- **Hexagon H:** $(R_1, R_2, R_3) = 1/(3+2\mu)$

Stable somewhere in (μ,δ) parameter space



Küppers-Lortz instability



BUSSE-HEIKES MODEL

The Case $\mu = 0$

- **Orthogonality condition holds** ($\delta\,\mu = 0$) $\Rightarrow$ **nonrelaxational potential flow**

$$\dot{R}_j = \boxed{-\frac{\partial V}{\partial R_j}} + \boxed{\delta f_j}, \quad j = 1,2,3$$

Relaxational part (the $-\frac{\partial V}{\partial R_j}$ term)

Residual dynamics (the δf_j term)

$$V = -(R_1 + R_2 + R_3) + \frac{1}{2}(R_1^2 + R_2^2 + R_3^2) + R_1 R_2 + R_2 R_3 + R_1 R_3$$

- **Equations of motion**

$$x(t) = R_1(t) + R_2(t) + R_3(t) = \frac{1}{\left(\frac{1}{x_0} - 1\right)e^{-2t} + 1}$$

After $t = O(1) \Rightarrow x(t) \approx 1$ **(asymptotic dynamics)**

BUSSE-HEIKES MODEL

- **Eliminating for instance R_1:**

$$\left.\begin{aligned}\dot{R}_2 &= 2\delta\frac{\delta H}{\delta R_3}\\ \dot{R}_3 &= -2\delta\frac{\delta H}{\delta R_2}\end{aligned}\right\} H(R_2, R_3) = R_2 R_3 (1 - R_2 - R_3) \equiv E \quad \text{(‘energy’)}$$

Hamiltonian dynamics

$$E = \frac{R_1(0) R_2(0) R_3(0)}{[R_1(0) + R_2(0) + R_3(0)]^3}$$

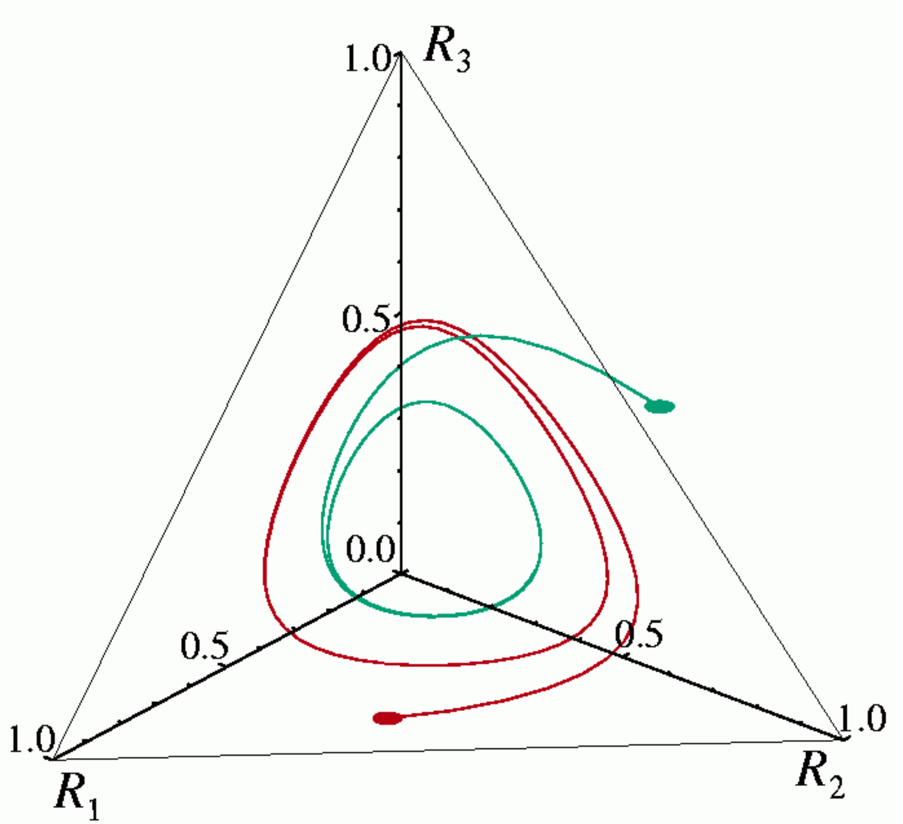
(only depends on initial conditions)

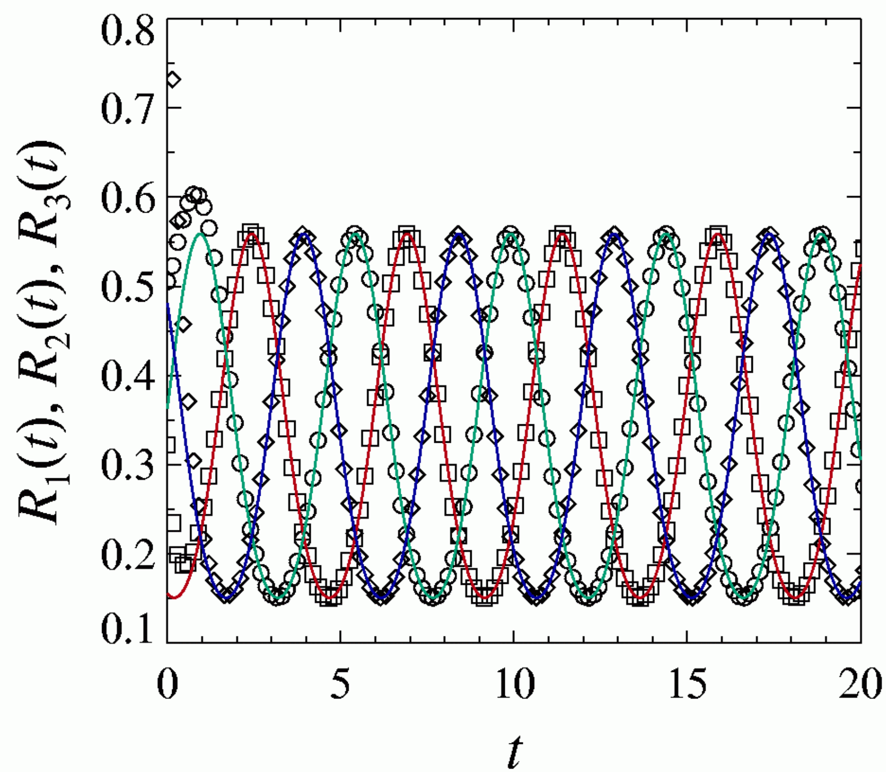
- **Explicit solutions for the period of the orbits and the amplitudes**

$$T(E) = \frac{2}{\delta\sqrt{b(a-c)}} K(q) = \boxed{-\frac{3}{2\delta}\log E} \times (1 + O(E))$$

$E \to 0$

BUSSE-HEIKES MODEL





$\mu = 0, \delta = 1.3$

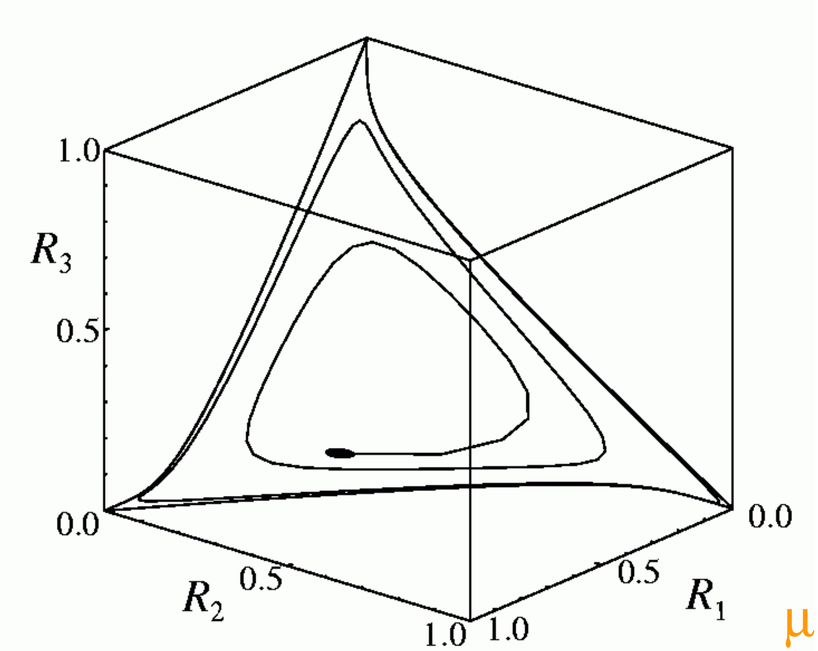
BUSSE-HEIKES MODEL

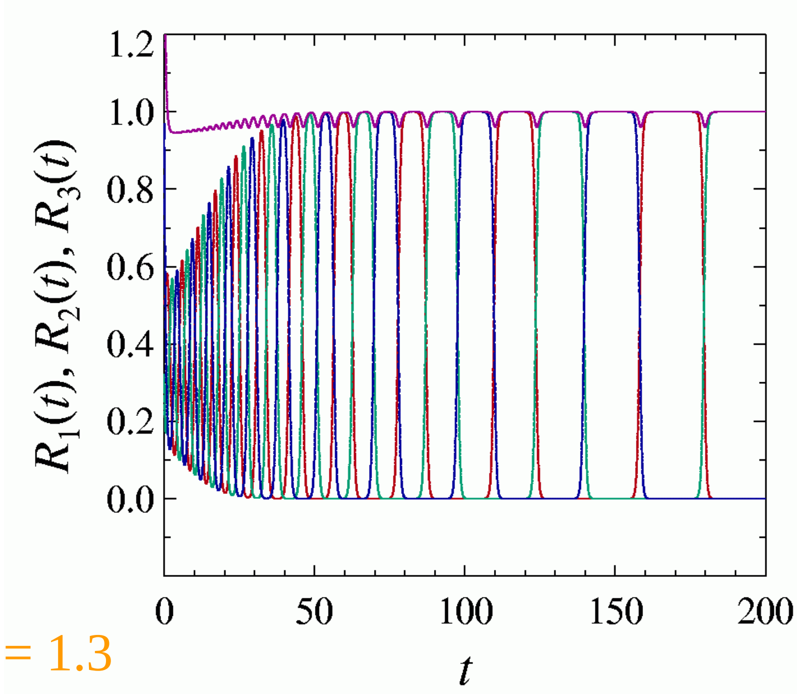
The case $\mu \gtrsim 0$

- **Time-dependent energy**

$$E(t) = \frac{R_1 R_2 R_3}{(R_1 + R_2 + R_3)^3} \rightarrow \frac{dE}{dt} = -4\mu f(t) E, \quad f(t) > 0$$

- **Motion occurs near the plane $R_1+R_2+R_3=1$ [after a transient time of order 1]**





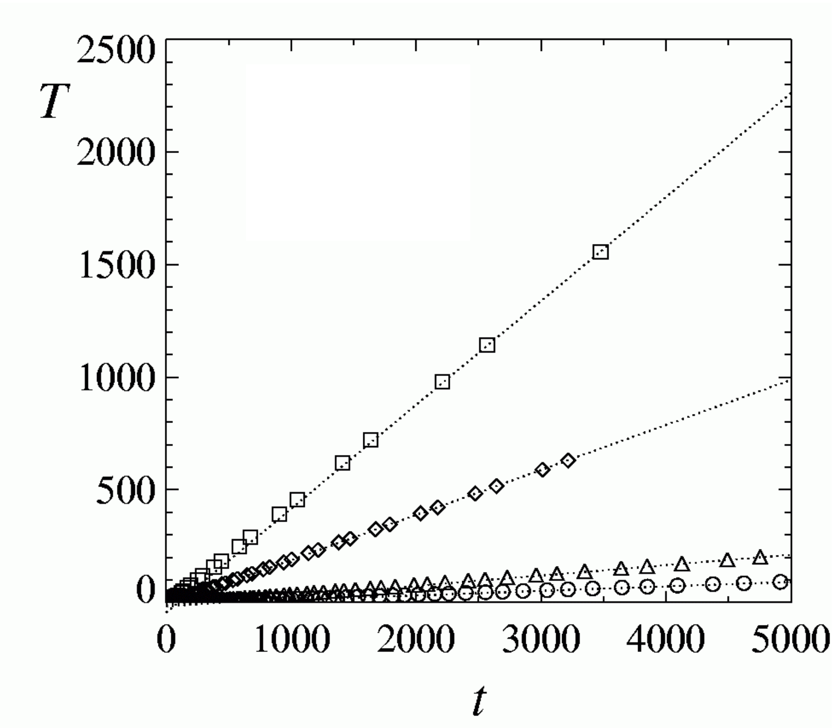
$\mu = 0.1, \delta = 1.3$

BUSSE-HEIKES MODEL

- **Period of the orbits is function of time**

$$E(t) = E(0)\exp(-4\mu\int_{t_0}^{t} f(t')\,dt') \approx E(t_0)\,e^{-4\mu(t-t_0)}$$

$$T = T(E(t)) \xrightarrow[\text{(small energies)}]{\text{long times}} -\frac{3}{2\delta}\log E(t) = T_0 + \frac{6\mu}{\delta}t$$



△ δ=1.3, μ=0.01
○ δ=3, μ=0.01
□ δ=1.3, μ=0.1
◇ δ=3, μ=0.1

lines with slope $6\mu/\delta$

BUSSE-HEIKES MODEL

Busse-Heikes Model with Noise

- **Period increasing with time is unphysical.**
  Solution: addition of fluctuations

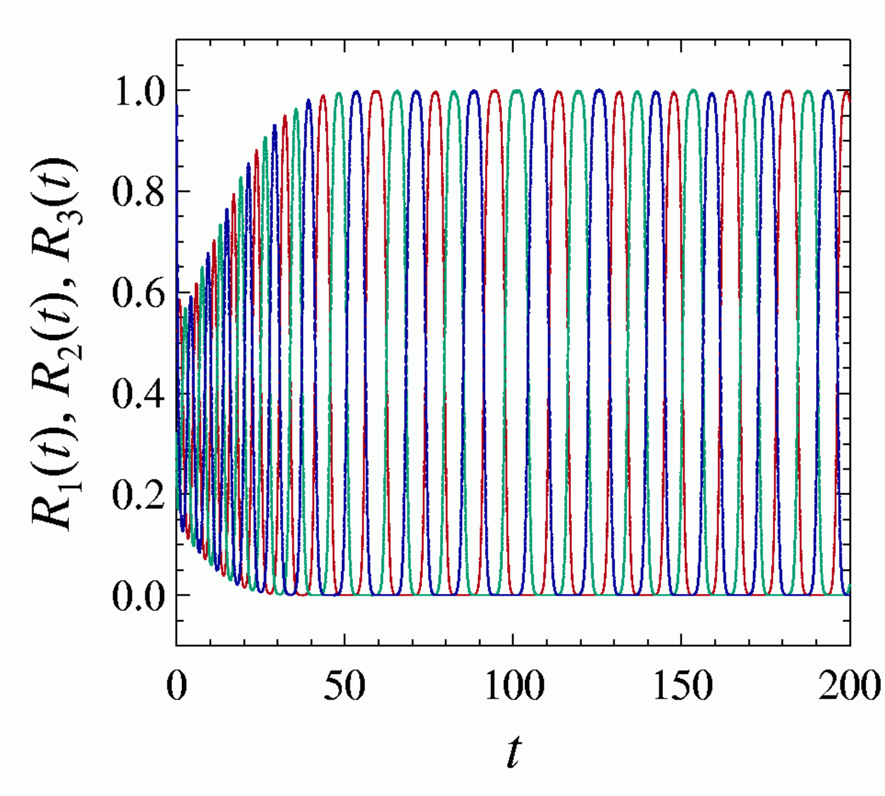
$$\dot{A}_1 = A_1[1-|A_1|^2-(1+\mu+\delta)|A_2|^2-(1+\mu-\delta)|A_3|^2]+\xi_1(t)$$
$$\dot{A}_2 = A_2[1-|A_2|^2-(1+\mu+\delta)|A_3|^2-(1+\mu-\delta)|A_1|^2]+\xi_2(t)$$
$$\dot{A}_3 = A_3[1-|A_3|^2-(1+\mu+\delta)|A_1|^2-(1+\mu-\delta)|A_2|^2]+\xi_3(t)$$

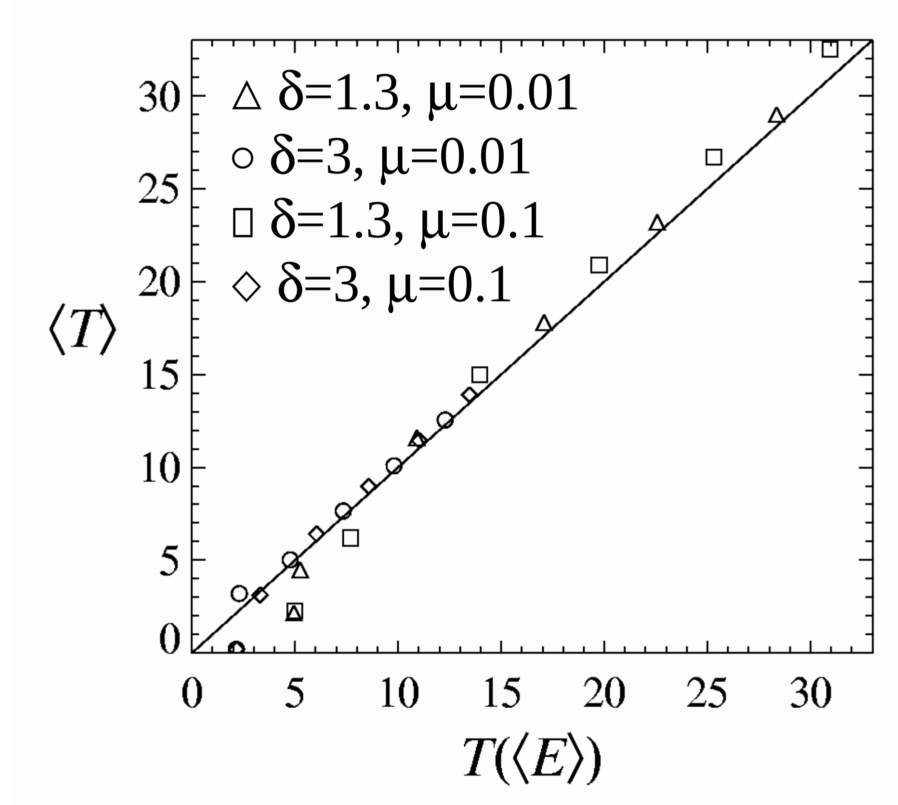
White noise processes: $\langle\xi_i(t)\xi_j^*(t')\rangle=2\varepsilon\delta(t-t')\delta_{ij}$

- **Noise prevents $E(t)$ from decaying to zero.**
- **Fluctuating period is established (but periodic on average)**

$$E(t)\rightarrow\langle E\rangle \;\Rightarrow\; T(t)\rightarrow\langle T\rangle = T(\langle E\rangle)$$

BUSSE-HEIKES MODEL





$\mu = 0.1,\ \delta = 1.3,\ \varepsilon = 10^{-6}$

$\varepsilon = 10^{-2}$ — 10^{-6}

BUSSE-HEIKES MODEL

- **Average energy**

$$P_{st}(\mu=0) \propto \exp[-V(\mu=0)/\varepsilon] \rightarrow P_{st}(\mu) \propto \exp[-\Phi(\mu)/\varepsilon] \approx \exp[-V(\mu)/\varepsilon]$$

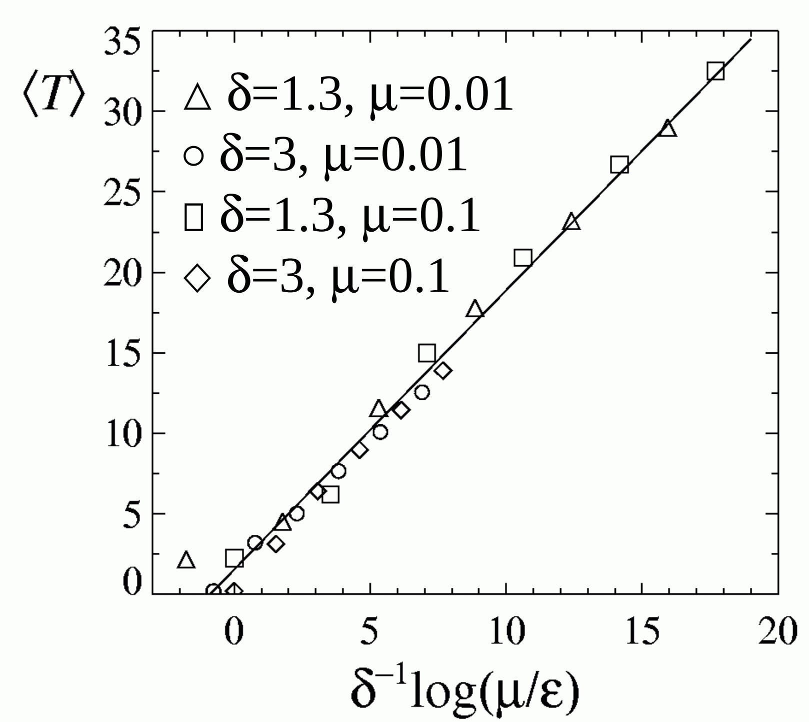
μ small

$$\langle E \rangle \cong \frac{\int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^\infty dR_3\, E \exp(-V/\varepsilon)}{\int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^\infty dR_3 \exp(-V/\varepsilon)}$$

Saddle-point type integration:

$\langle E \rangle \approx (\varepsilon/\mu)^2$, $\varepsilon \rightarrow 0$, μ small

$$\Rightarrow \quad \langle T \rangle = T(\langle E \rangle) \approx \frac{3}{\delta} \log(\mu/\varepsilon)$$



$\varepsilon = 10^{-2}$ — 10^{-6}

BUSSE-HEIKES MODEL

ONE-DIMENSIONAL SYSTEMS

- **Three competing nonconserved real order parameters with short-range interactions**

$$\dot{A}_1 = \partial_{xx} A_1 + A_1[1 - A_1^2 - (1+\mu+\delta)A_2^2 - (1+\mu-\delta)A_3^2]$$
$$\dot{A}_2 = \partial_{xx} A_2 + A_2[1 - A_2^2 - (1+\mu+\delta)A_3^2 - (1+\mu-\delta)A_1^2]$$
$$\dot{A}_3 = \partial_{xx} A_3 + A_3[1 - A_3^2 - (1+\mu+\delta)A_1^2 - (1+\mu-\delta)A_2^2]$$

- **Prothotypical nonpotential problem to study domain growth and dynamical scaling**

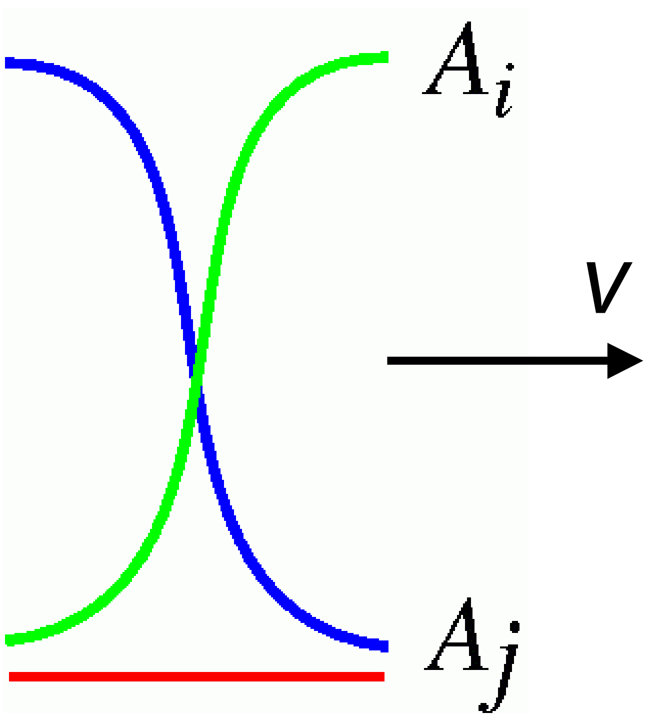
- **We focus on the region below the KL instability: coexistence of three stable homogeneous states under nonpotential dynamics ($\delta \neq 0$)**

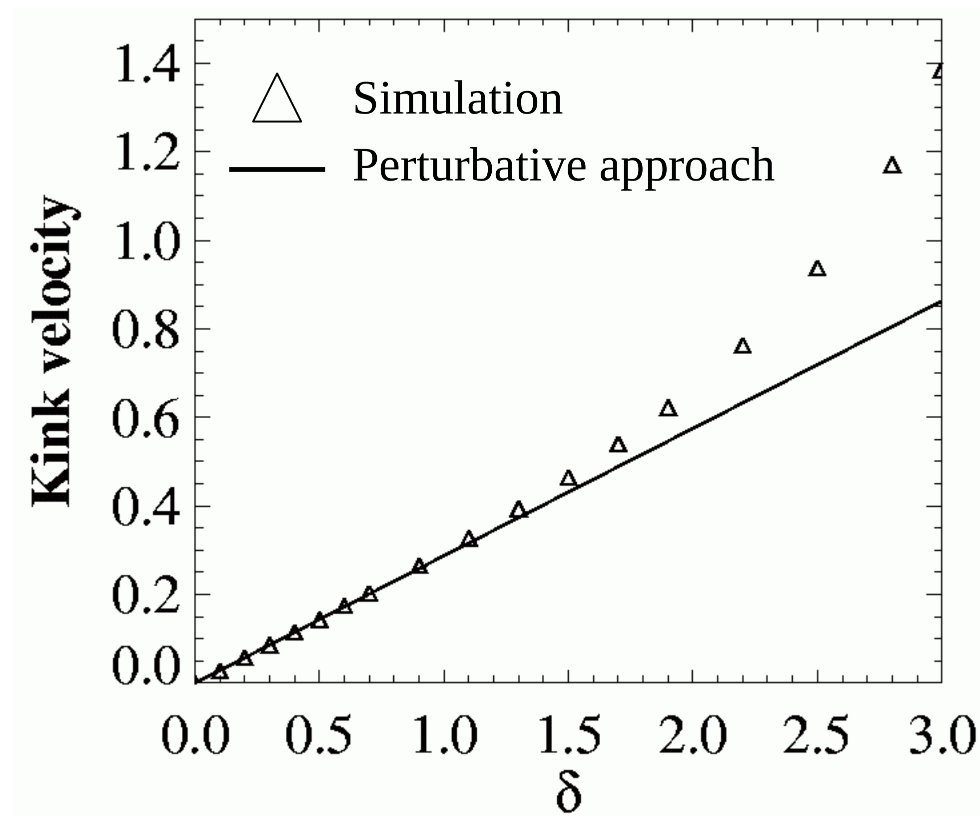
BUSSE-HEIKES MODEL

Dynamics of an Isolated Kink

- **Isolated kink moves due to nonpotential effects**

$$v(\mu,\delta)=h(\mu)\,\delta + O(\delta^2)$$

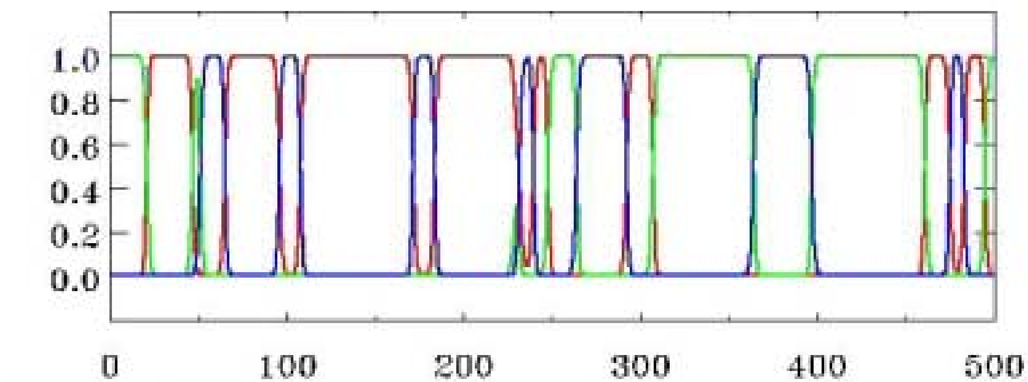




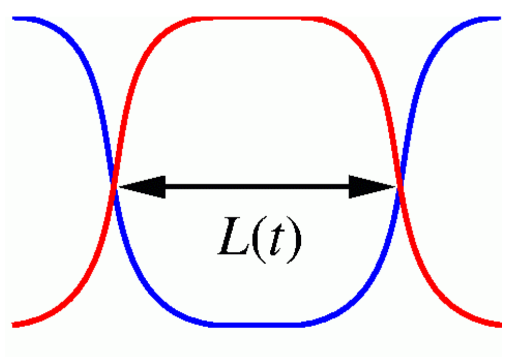
BUSSE-HEIKES MODEL

Multikink Configurations

- **Kink motion ⟶ annihilation ⟶ domain growth**



- **Front motion due to: attractive interaction forces + nonpotential effects**



$$\partial_t L(t) = \pm\, 2v(\mu,\delta) - \gamma \exp(-\mu^{1/2} L(t))$$

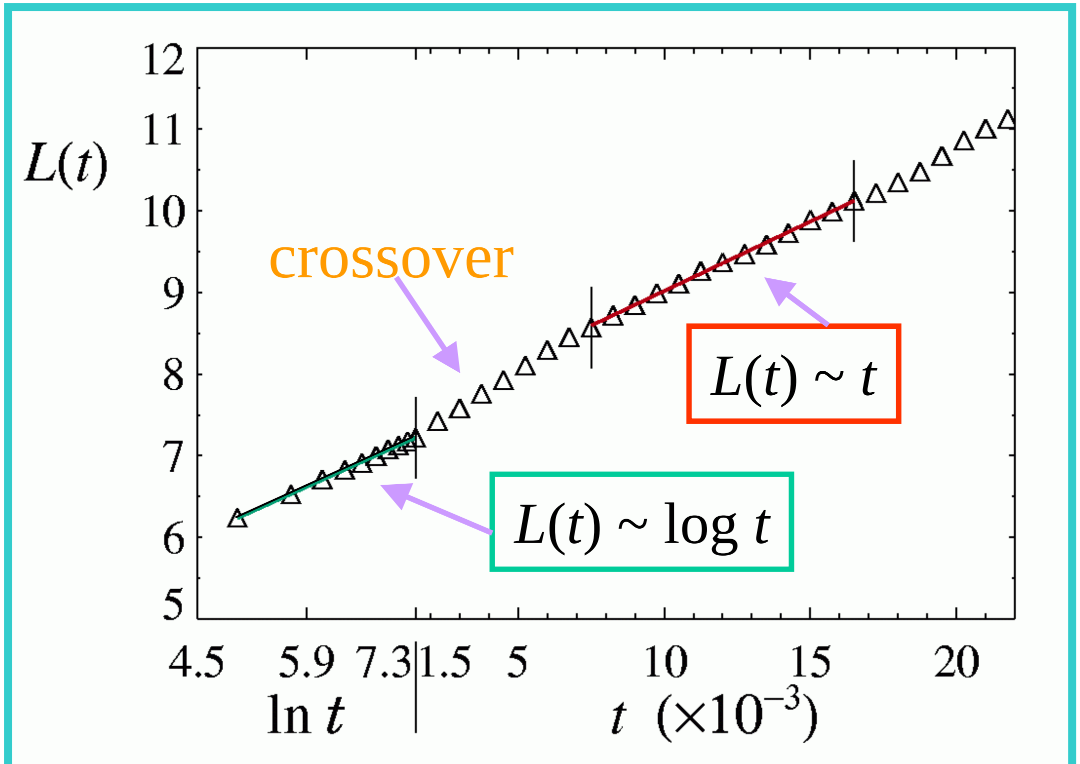
$L(t) \sim t$

$L(t) \sim \log t$

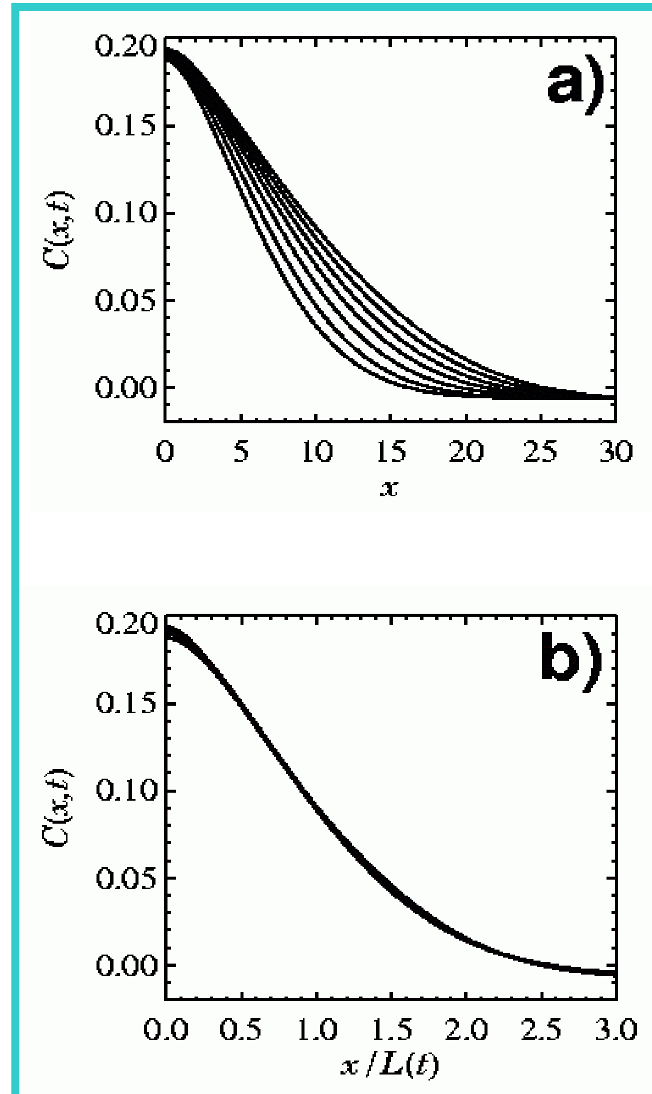
BUSSE-HEIKES MODEL

Domain Growth and Dynamical Scaling

$L(t) \sim \log t + \text{crossover} + t$



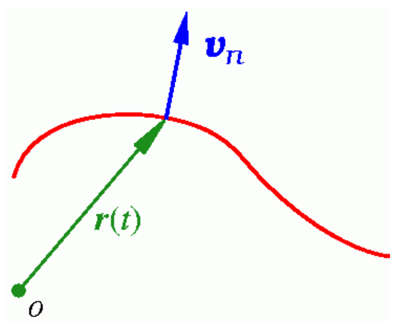
$\eta = 3.5,\ \delta = 10^{-3}$



BUSSE-HEIKES MODEL

TWO-DIMENSIONAL SYSTEMS

- **We take real variables, isotropic diffusion terms and we focus on the region below the KL instability point**
- **Normal front velocity**



$$v_n(\mathbf{r}, t\,;\mu, \delta) = -\kappa(\mathbf{r}, t) + v_p(\mu, \delta)$$

$$v_p(\mu, \delta) \sim \delta + O(\delta^2)$$

- **Absence of coarsening** for system sizes large enough

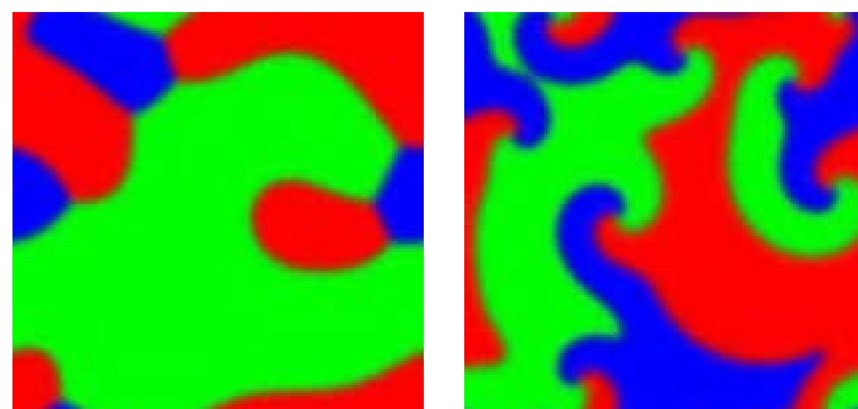
⇑

nonpotential dynamics

+

formation of vertex points

POTENTIAL **NONPOTENTIAL**



$\mu = 3, \delta = 0$ $\mu = 3, \delta = 2$

BUSSE-HEIKES MODEL

Spiral dynamics

- **Rotation** **angular velocity**

$$\omega(\mu,\delta) \propto v_p^{1/2}\,\kappa_0^{3/2}$$

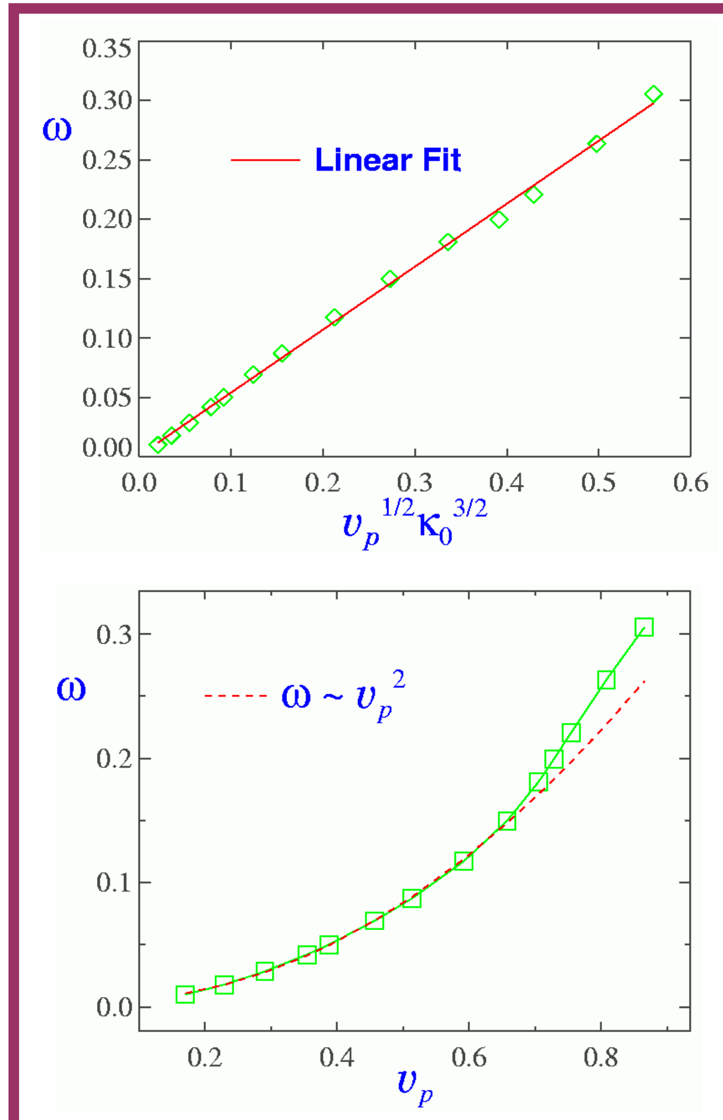
v_p small

↓

$$\kappa_0 \sim v_p(\delta)$$

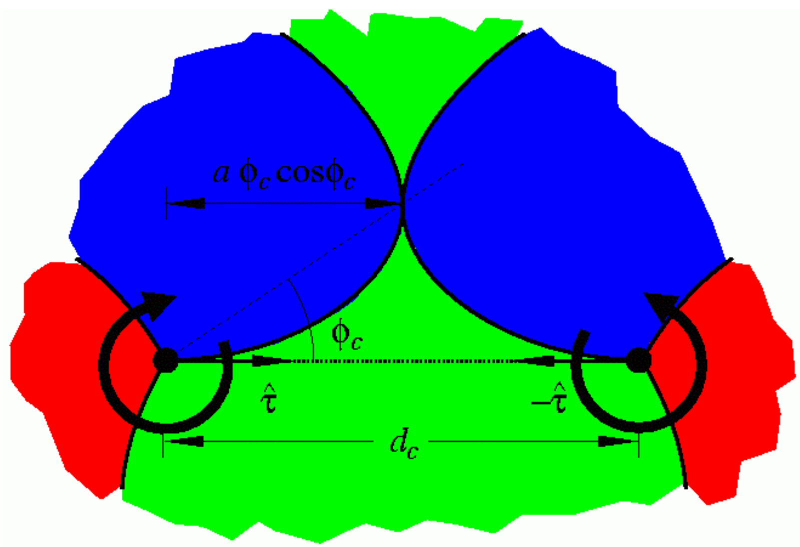
↓

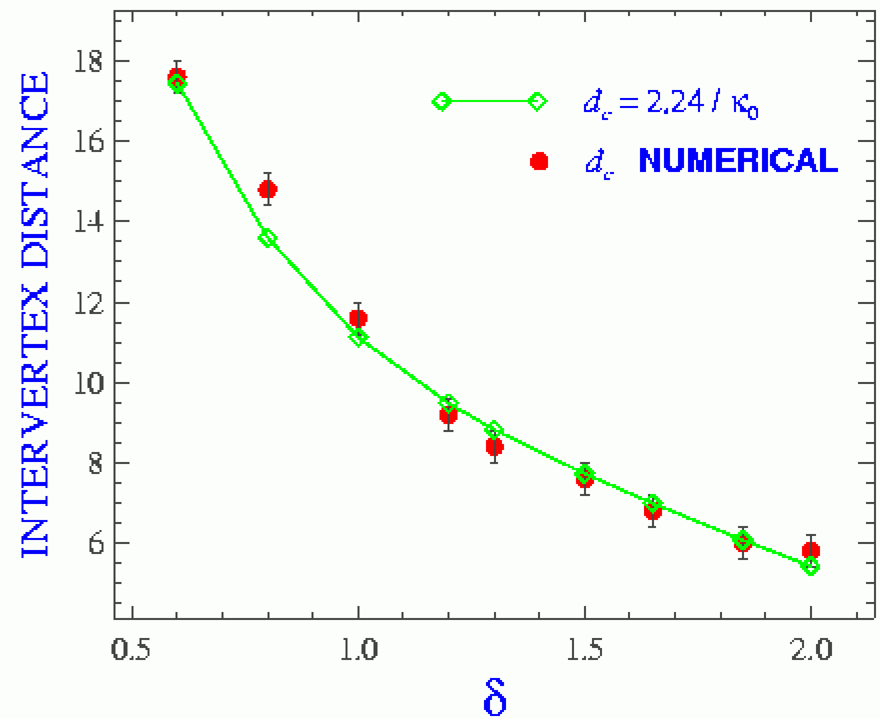
$$\omega \sim v_p(\delta)^2 \sim \delta^2$$



BUSSE-HEIKES MODEL

- **Critical distance for vertex annihilation**





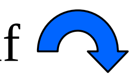
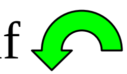
$$d_c \cong 2.24/\kappa_0 \sim \delta^{-1}$$

Consequence: coarsening will occur for system sizes $S \lesssim d_c$

BUSSE-HEIKES MODEL

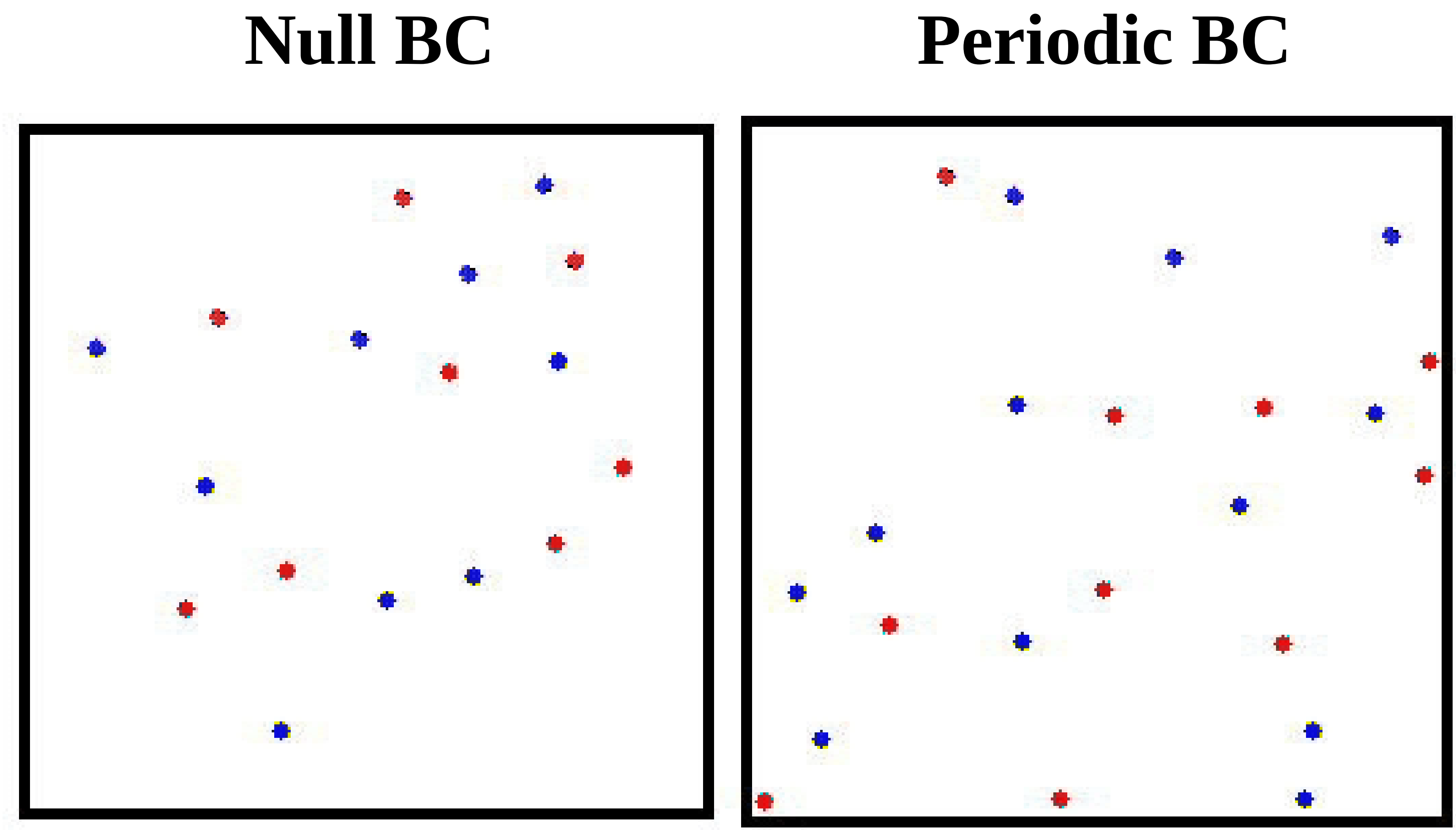
Vertex Motion

- **For long times vertices diffuse through the system**
- **Vertex dynamics affected by the boundary conditions**

- **Periodic BC:**
  - Even number of vertices: half  and half 
  - Vertices disappear by pairs of opposite sence of rotation
  - Correlated motions are observed

- **Null BC:**
  - No restrictions about the number of vertices
  - Vertices may disappear through the egdes of the system
  - No correlated motions observed

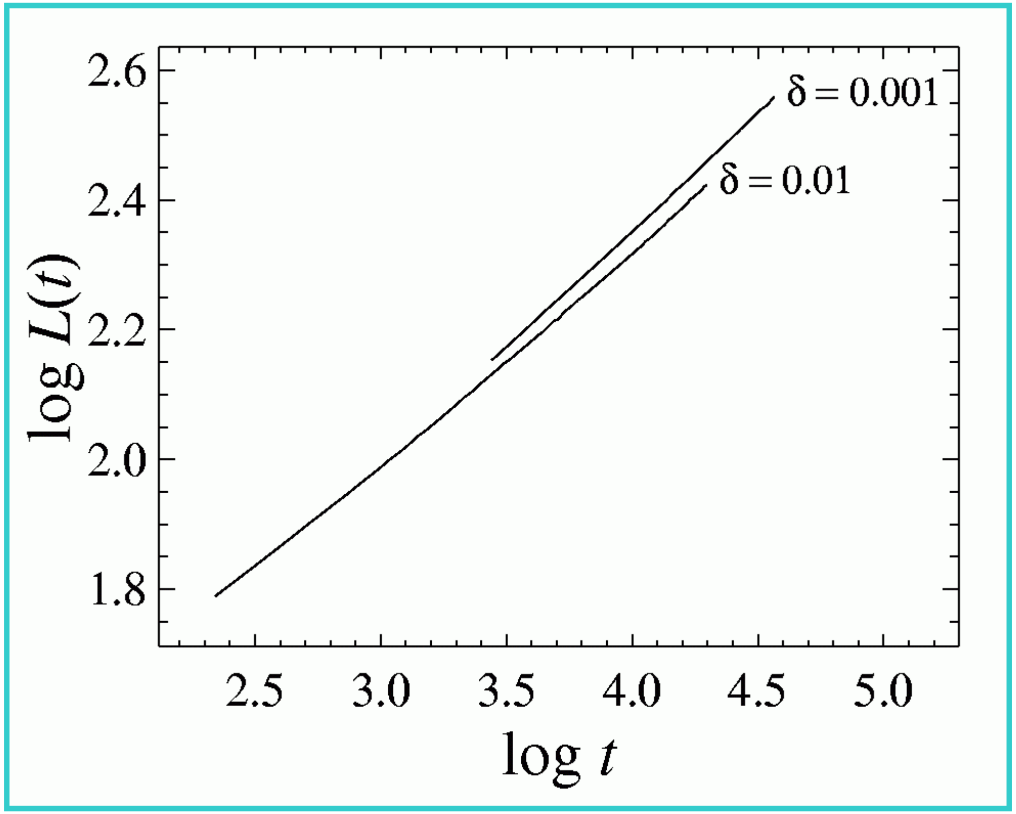
BUSSE-HEIKES MODEL

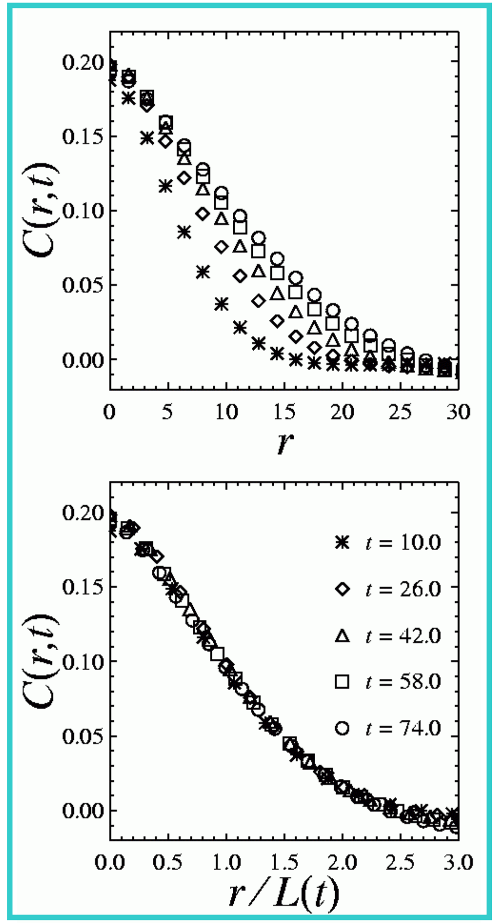


BUSSE-HEIKES MODEL

Domain Growth and Dynamical Scaling

- $\delta = 0$ (**potential limit**) $\Rightarrow L(t) \sim t^{1/2}$ + **dynamical scaling (3 fields)**
- $\delta \neq 0$ (**nonpotential limit**) $\Rightarrow L(t) \nsim t^{1/2}$ + **dynamical scaling (2 fields)**





BUSSE-HEIKES MODEL

Spatial-dependent Terms

- **Alternative explanation for period stabilization**

- **Two kinds of differential operators**

$$\mathfrak{L}_j,\ j=1,2,3 \begin{cases} \nabla^2 & \textbf{ISOTROPIC} \\ (\hat{\mathbf{e}}_j\cdot\nabla)^2 & \textbf{ANISOTROPIC } (\leftarrow \textbf{NWS, GOS terms}) \end{cases}$$

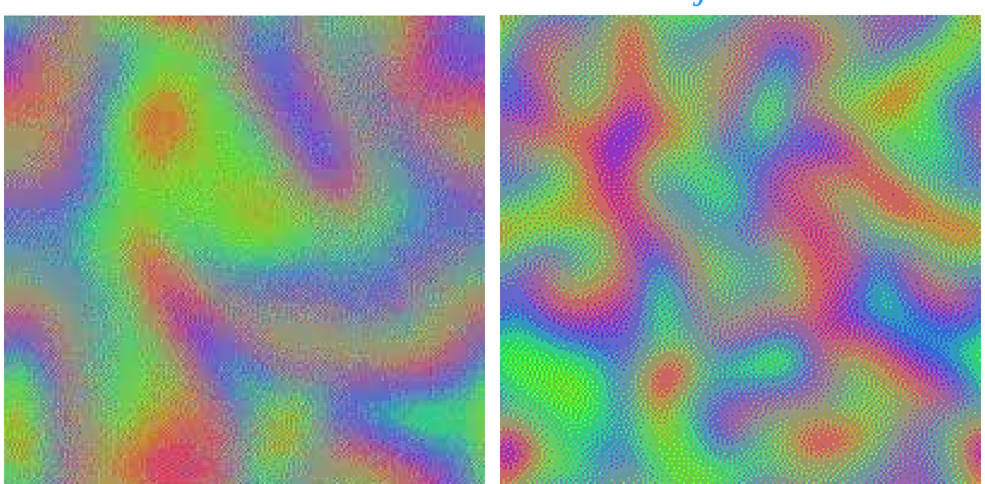
- **We focus on the region beyond the KL instability point**

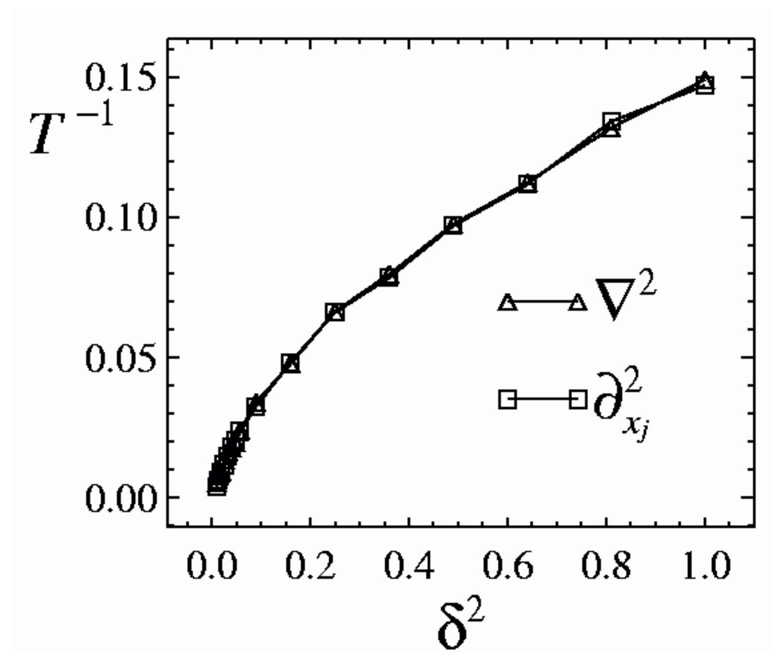
- **Dynamics depends on the type of spatial derivatives and on the size of μ**

BUSSE-HEIKES MODEL

- μ **small**
  - Similar morphology of domains
  - Alternating period dominated by the KL instability
  - Intrinsic KL period stabilizes to a statistically constant value with both kinds of terms

∇^2 $(\hat{\mathbf{e}}_j \cdot \nabla)^2$

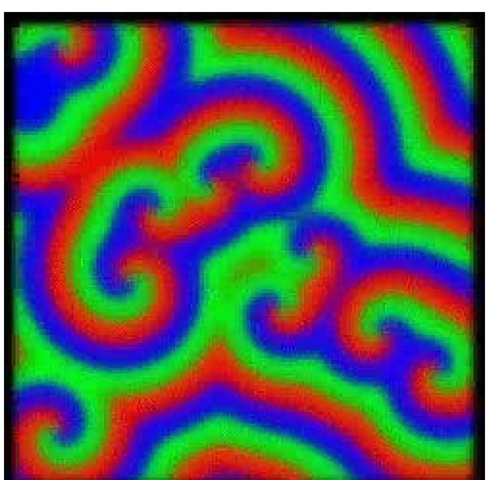




BUSSE-HEIKES MODEL

- μ **large**
  - Different morphology of domains
  - Intrinsic KL period $\begin{cases} \nabla^2: & \text{diverges with time} \\ (\hat{\mathbf{e}}_j \cdot \nabla)^2: & \text{saturates to a constant value} \end{cases}$
  - Different alternating periods

∇^2



$(\hat{\mathbf{e}}_j \cdot \nabla)^2$