Variational principle for the Pareto power law

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A. Chakraborti and M. Patriarca,
Phys. Rev. Lett. 103, 228701 (2009)

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Motivation

- Study of plausible mechanisms for the appearance of power-law distributions

- Examples:
  Pareto's power law of income distribution
  Zipf law for the rate of occurrence of words.

- Heterogeneity is a key ingredient

- Which heterogeneity? Diversity in the number of degrees of freedom.

- Known mechanisms leading to power law distributions are
  - Avalanche processes, e.g. in Self Organized Criticality
  - Multiplicative stochastic process
  - Non-extensive thermodynamics/entropy (C. Tsallis, J.Stat.Phys. 52, 479 (1988); E.M.F. Curado and C. Tsallis, J.Phys.A 24, L69 (1991))
  - Generalized Gibbs distribution (R.A. Treumann and C.H. Jaroschek, Phys. Rev. Lett. 100, 155005 (2008))
  - Superstatistics (C. Beck, Physica A 365, 96 (2006)).

Outline

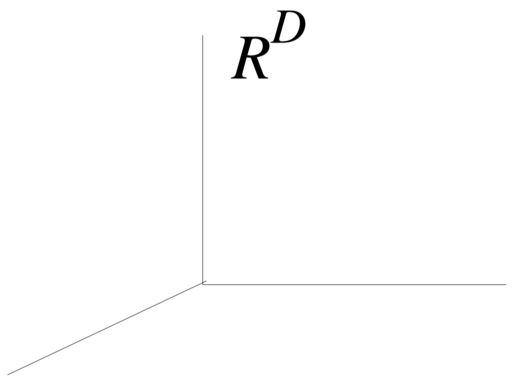
Kinetic theory in *D* dimensions:

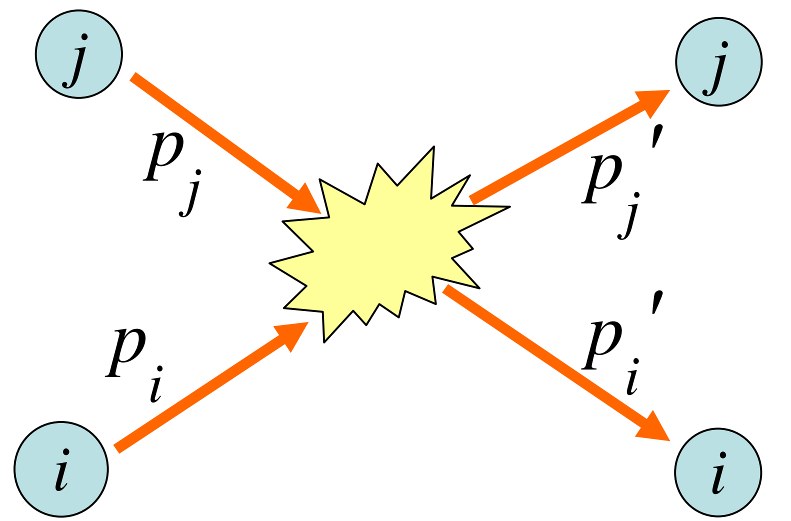
- Study of diffusion in a network
- Kinetic Wealth Exchange Models

Examples:

- Power law in load distribution of scale-free networks: the Zipf's law from the Random walk in the semantic network
- Pareto's Law from heterogeneous Kinetic Wealth Exchange Models

Kinetic Theory in D dimensions

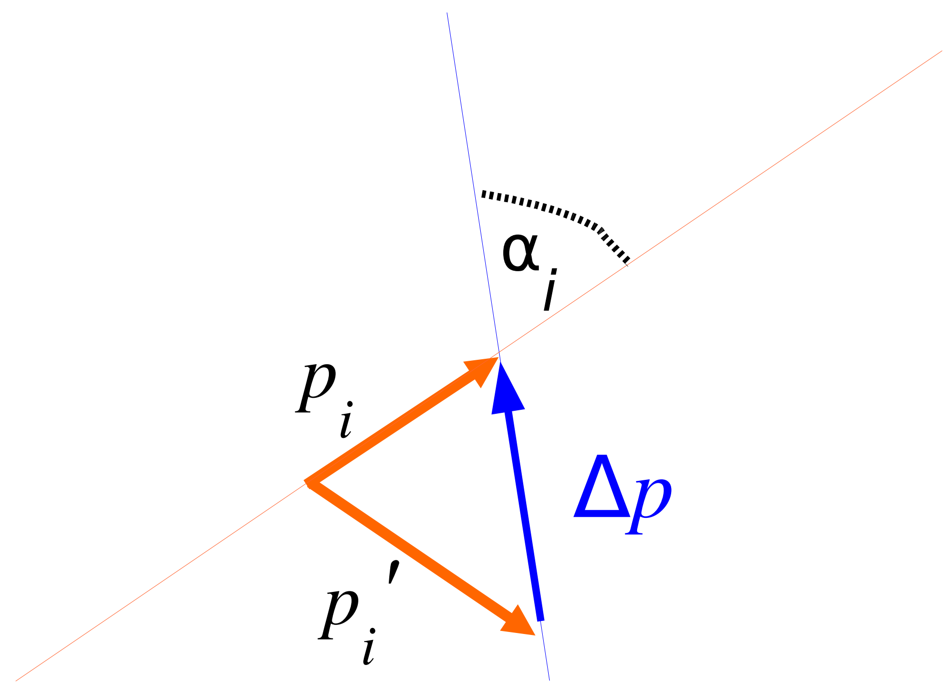




If the initial particles momenta are p_i and p_j, introduce the momentum transfer

$$\Delta p = p_j' - p_j = p_i - p_i'$$

and the angles α_i and α_i respect to the initial momenta p_i and p_j, respectively.



Using energy and momentum conservation, one obtains for the kinetic energies $x_i = ½\,(p_i)^2$ and $x_i = ½\,(p_i)^2$ of particles *i* and *j*

$$x_i \;\rightarrow\; x_i \;-\; \widetilde{\omega}_i x_i \;+\; \widetilde{\omega}_j x_j$$
$$x_j \;\rightarrow\; x_j \;+\; \widetilde{\omega}_i x_i \;-\; \widetilde{\omega}_j x_j$$

where the ω's coefficients are related to the cosines squared,

$$0 \;\leq\; \widetilde{\omega}_i \;=\; (\cos\alpha_i)^2 \;\leq\; 1$$

$$0 \;\leq\; \widetilde{\omega}_j \;=\; (\cos\alpha_j)^2 \;\leq\; 1$$

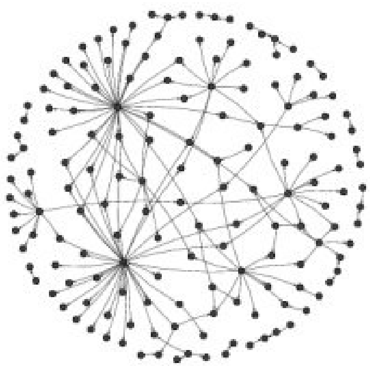
In *D* dimensions it can be shown that assuming initial random directions,

$$\langle \widetilde{\omega} \rangle \;=\; \langle (\cos\alpha)^2 \rangle \;=\; 1/D$$

For the equipartition theorem,

$$\langle x_i \rangle \;=\; D\,k_B T/2 \;\sim\; D$$

Random Walk across a network

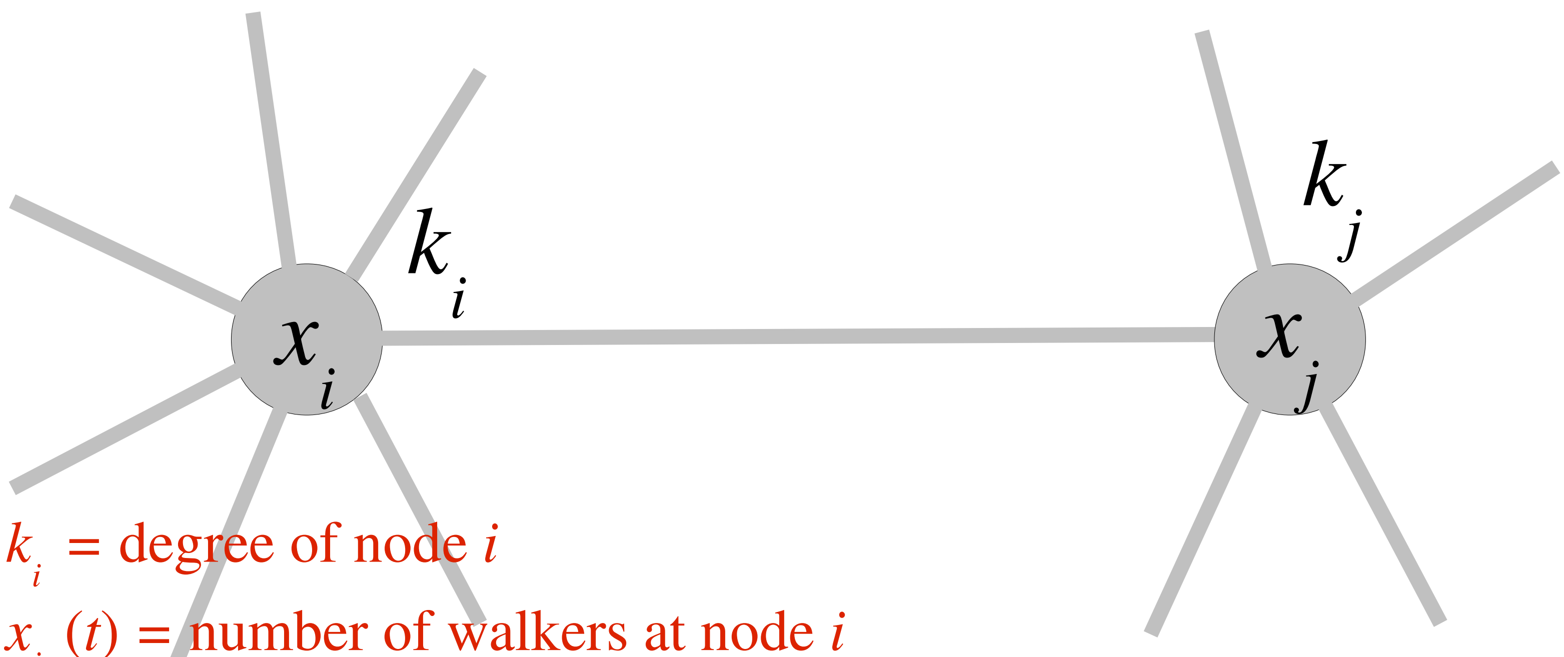


Consider N walkers moving across a network.

A generic node i has k_i links and $x_i(t)$ walkers at time t.

The update rules for the flow between nodes i and j, assuming homogeneous diffusion, is

$$x_i(t+1) = x_i(t) - \varpi_i x_i(t) + \varpi_j x_j(t)$$
$$x_j(t+1) = x_j(t) + \varpi_i x_i(t) - \varpi_j x_j(t)$$



k_i = degree of node i

$x_i(t)$ = number of walkers at node i

Here the ω's are random coefficients in the range

$$O \leq \tilde{\varpi}_i \leq 1$$

$$O \leq \tilde{\varpi}_j \leq 1$$

The average values are

$$\langle \tilde{\varpi}_i \rangle = 1/k_i$$

$$\langle \tilde{\varpi}_j \rangle = \text{١}/k_j$$

It can be shown that in the stationary state

$$\langle x_i \rangle \sim k_i$$

Kinetic wealth exchange model with saving propensity (*)

Definition of the model

- N agents interacting randomly in pairs, characterized by the saving parameters $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $0 < \lambda_i < 1$.
- The state of the system is specified through the agent wealths $(x_1, x_2, \ldots, x_N)$.
- At each time step t two agents i and j are extracted randomly and exchange wealth according to

$$
\begin{aligned}
x_i' &= \lambda_i x_i + \epsilon_1 (1-\lambda_i) x_i + \epsilon_2 (1-\lambda_j) x_j \\
x_j' &= \lambda_j x_j + (1-\epsilon_1)(1-\lambda_i) x_i + (1-\epsilon_2)(1-\lambda_j) x_j
\end{aligned}
$$

(*) - J. Angle, Social Forces 65, 293 (1986)
- A. Chakraborti and B.K. Chakrabarti, Eur. Phys. J. B 17, 167 (2000)

Here ε_1 and ε_2 are uniform random numbers in (0,1), independent or possibly the same random number, depending on the model.

The update rule can be rewritten as

$$
\begin{aligned}
x_i(t+1) &= x_i(t) - \tilde{\omega}_i x_i(t) + \tilde{\omega}_j x_j(t) \\
x_j(t+1) &= x_j(t) + \tilde{\omega}_i x_i(t) - \tilde{\omega}_j x_j(t)
\end{aligned}
$$

with

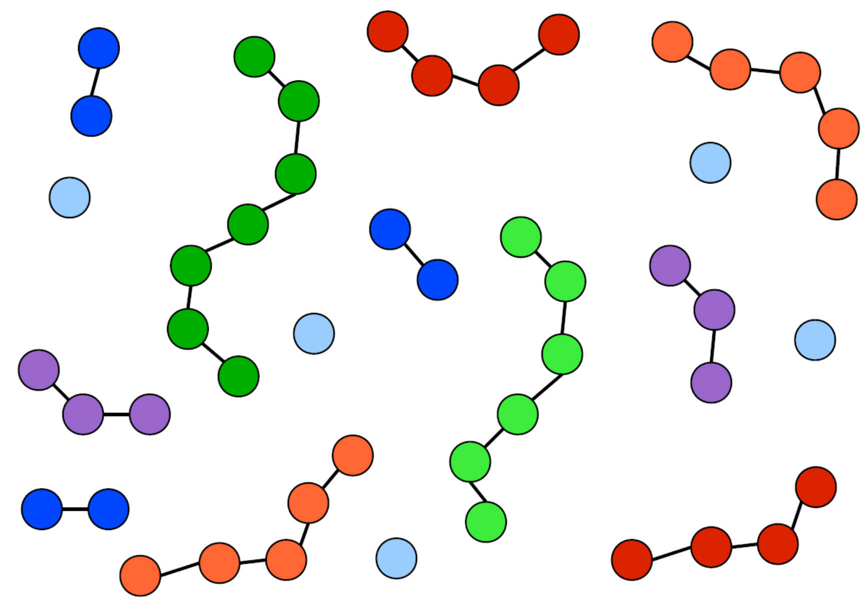
$$
\begin{aligned}
\tilde{\omega}_i &= (1-\epsilon_1)(1-\lambda_i) \equiv (1-\epsilon_1)\omega_i \\
\tilde{\omega}_j &= \epsilon_2(1-\lambda_j) \equiv \epsilon_2\omega_j
\end{aligned}
$$

In a heterogeneous model the average value is $\langle x_i \rangle \sim 1/(1-\lambda_i)$

Model system of a perfect gas with heterogeneous dimensions

The model system can represent a perfect gas with heterogeneous dimensions (each particle lives in a space with a different dimension) or a heterogeneous mixture of polymers, each polymer having a different number of degree of freedoms.

The heterogeneity is described by the probability $P(n)$ that a sub-system has a certain number $D = 2\,n$ of degrees of freedom.



For a fixed n, the equilibrium probability density of a D-dimensional harmonic oscillator is the gamma-distribution of order n,

$$f(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} \mathrm{e}^{-\beta x}$$

Then, for a general $P(n)$, the equilibrium distribution is the aggregate density,

$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x) = \int_1^\infty dn\, P(n) \frac{\beta^n}{\Gamma(n)} x^n e^{-\beta x}$$

This can be obtained by varying the Boltzmann entropy of the heterogeneous system.

Variational principle for heterogeneous dimensions

Given the dimension density $P(n)$, $1 < n < \infty$,
one can define the entropy functional as follows.

Entropy Functional	$S[f] = \int dn\, P(n) \int dx\, f_n(x) \left\{ \ln\left[\frac{f_n(x)}{x^{n-1}} \right] + \mu_n + \beta x \right\}$
Constraints on probability conservation	$I[f] = \int_0^\infty dx\, f_n(x) = 1$
(Single) constraint on energy conservation	$X_{tot}[f] = \int dn\, P(n) \int_0^\infty dx\, x\, f_n(x) = 1$

By variation of S, one obtains the aggregate density, i.e. the probability density to obtain a certain value x of the energy, independently of the corresponding number $2n$ of degrees of freedom,

$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x) = \int_1^\infty dn\, P(n) \frac{\beta^n}{\Gamma(n)} x^n e^{-\beta x}$$

Result for the aggregate distribution

The aggregate density can be rewritten as

$$f(x) = \int dn\, P(n)\beta\gamma_n(\beta x) = \beta \exp(-\beta x)\int dm \exp(-\phi(m))$$

where $m = n - 1$. The integrand function has a maximum at $\beta x \sim 1$.
Then using the Stirling approximation, one can write

$$\phi(m) \approx -\ln[P(m+1)] - m\ln(\beta x) + \ln(\sqrt{2\pi}) + (m + \tfrac{1}{2})\ln(m) - m,$$

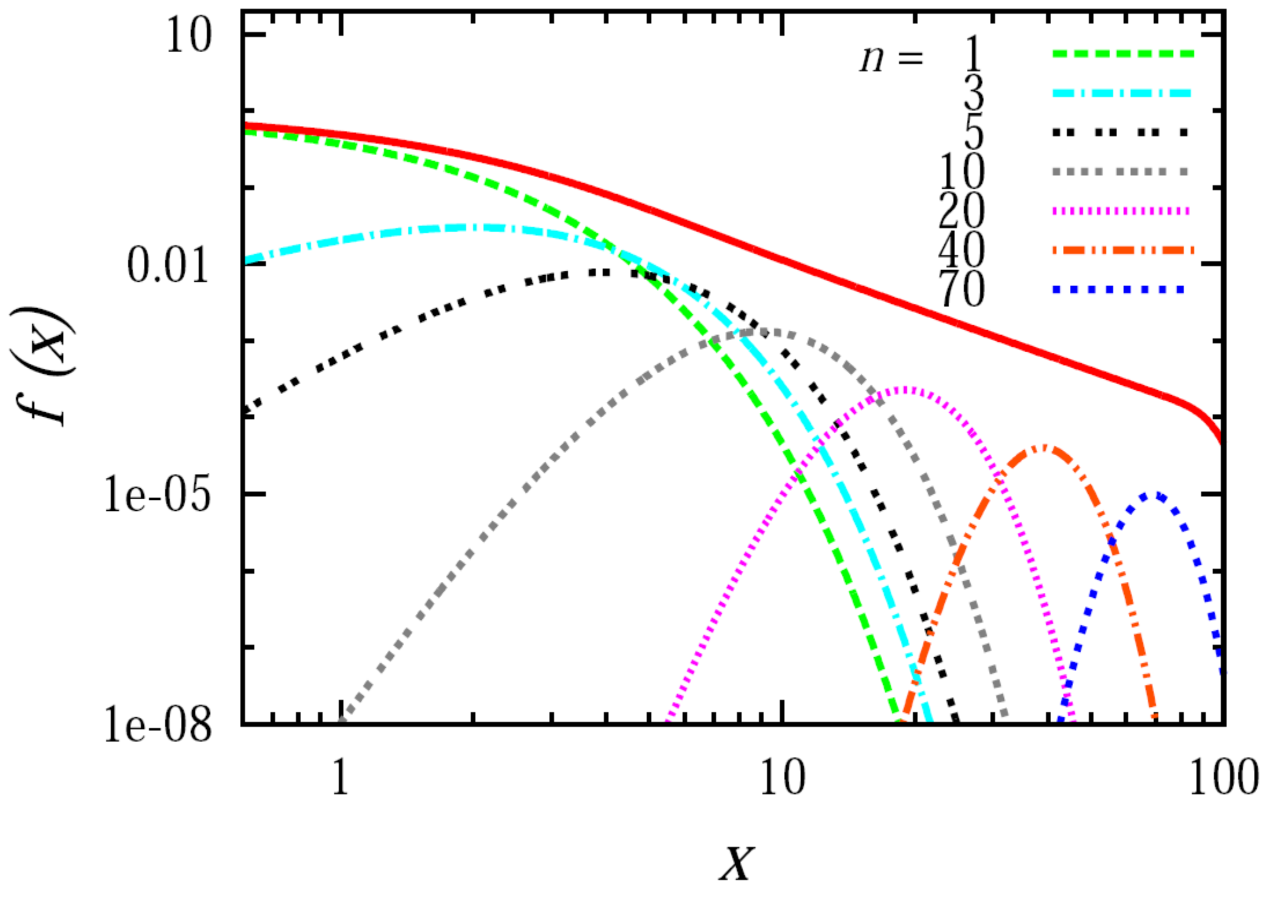
Using the saddle-point approximation, $f(x) \approx \beta \exp[-\beta x - \phi(m_0)]$

$$\times \int_{-\infty}^{+\infty} d\epsilon \exp[-\phi''(m_0)\epsilon^2/2]$$

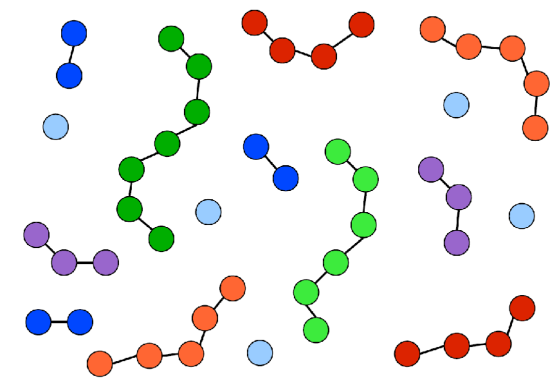
$$= \beta\sqrt{\frac{2\pi}{\phi''(m_0)}}\exp[-\beta x - \phi(m_0)].$$

The asymptotic result is $f(x \gg \beta^{-1}) \equiv f_2(x) = \beta P(1 + \beta x).$

Dimensional decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$



Aggregate distribution of dimensionally heterogeneous systems



Gas in *D* dimensions. For a given dimension D, the equipartition theorem provides an average kinetic energy

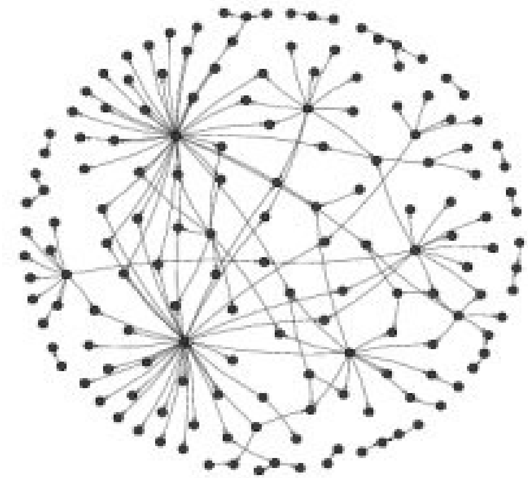
$$x(D) = D\, k_B T/2 \sim D,$$

where T is the temperature of the system.
If $P(D)$ is the dimension density of a heterogeneous system, then for probability conservation, i.e. $f(x)\, dx = P(D)\, dD$, one has

$$f(x) = P(D)\frac{dD}{dx} = \bar{x}^{-1} P(x/\bar{x})$$

$$\bar{x} = k_B T/\text{۲}$$

Complex Networks. In a complex network with degree distribution $P(k)$, the average equilibrium load for the simplest case of free diffusion is



$$x(k) = x_0 k \sim k,$$

where x_0 is a constant (average flux per link and direction).

Then from probability conservation, $f(x)\,dx = P(k)\,dk$, it follows that

$$f(x) = P(k)\frac{dk}{dx} = x_0^{-1} P(x/x_0)$$

In particular, **scale-free networks have a power law load distribution** in the stationary state, $f(x) \sim 1 / x^{\alpha}$.

Zipf's law from the random walk on the semantic network of language

Written text (or spoken language) can be conceived as a walk in the special space of concepts which can be represented by nouns, verbs, etc, the **semantic space.**

A. P. MASUCCI AND G. J. RODGERS

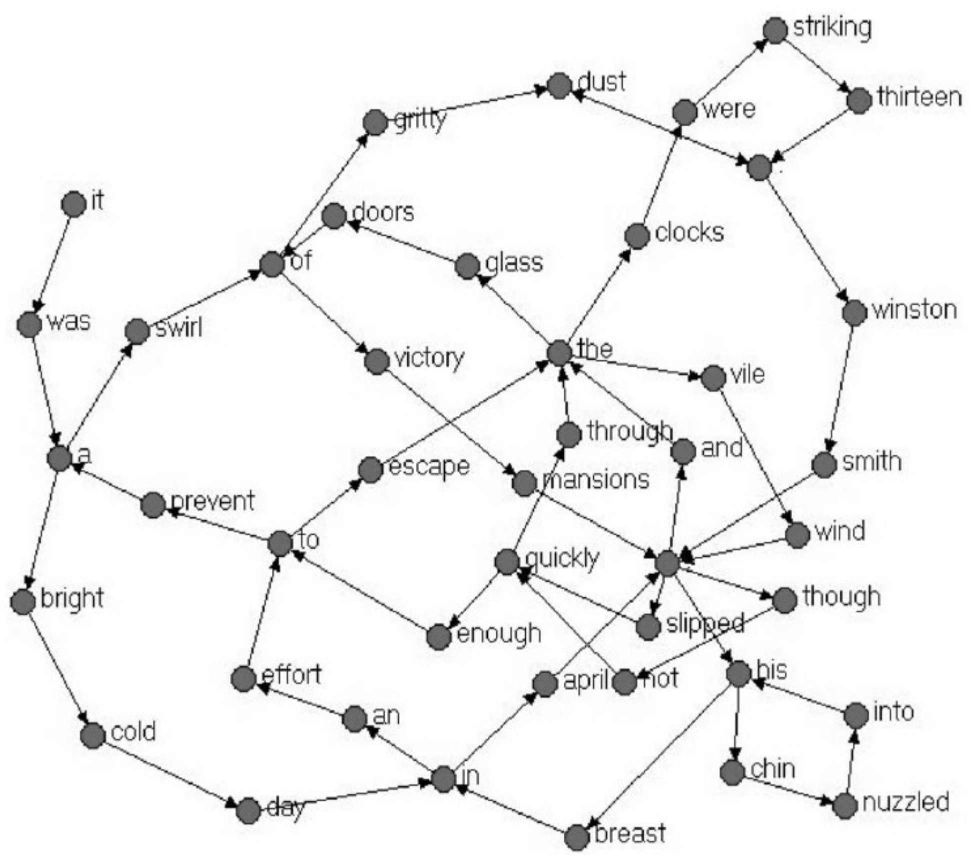


FIG. 1. Illustration of the language network for the first 60 words of Orwell's *1984*.

After writing a long text (e.g. a novel) or speaking a long speech, what is the expected rank distribution of words?

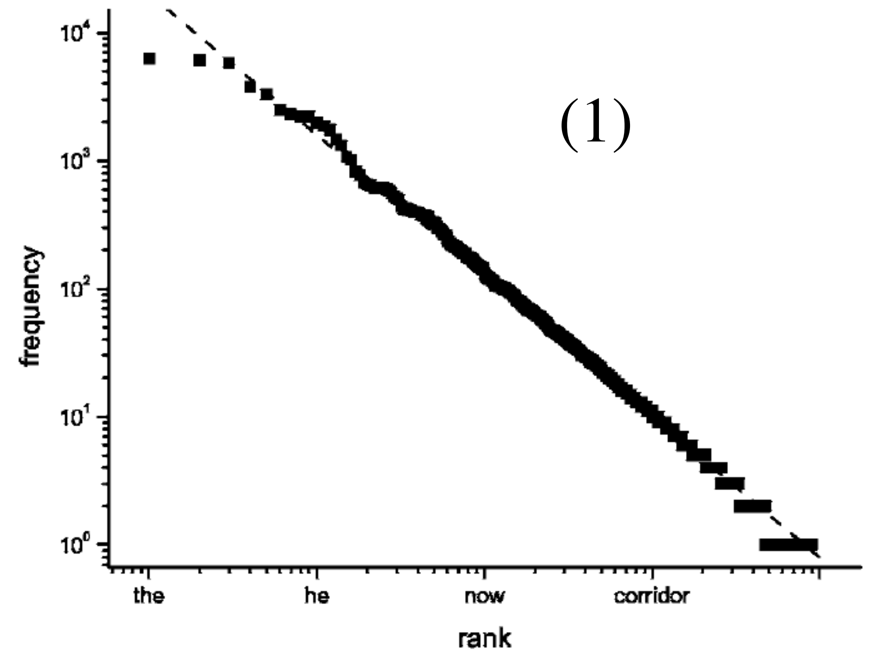
This depends obviously on the correlations between subsequent words, i.e. on the probability that, given a word *w*, another word *w*' will follow.

← From:

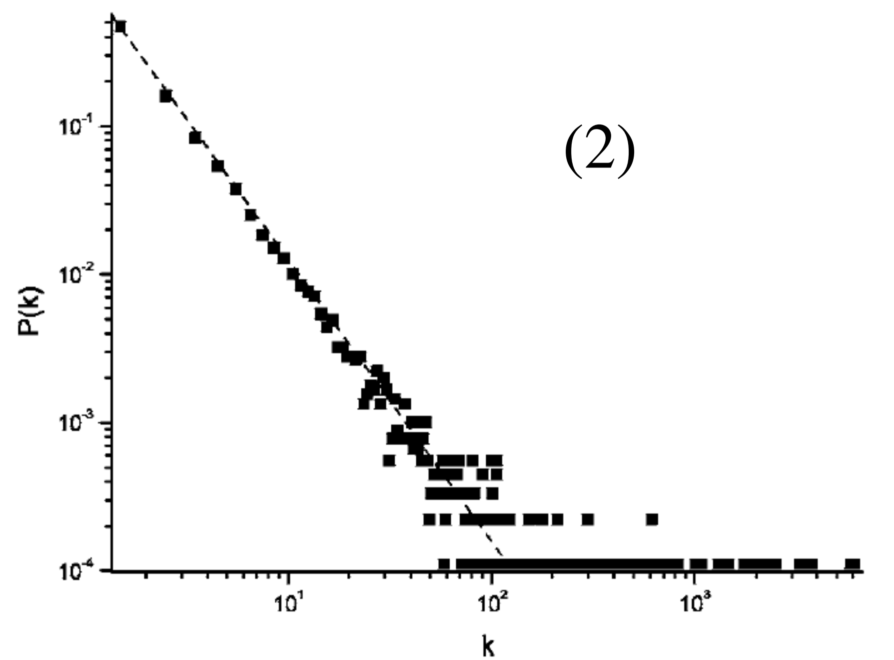
A.P. Masucci and G.J. Rodgers, Phys. Rev. E 74, 026102 (2006)

Measure of Zipf's law on "1984"

(1) **Rank plot**: frequency x versus rank r.
One finds a power law $x \sim r^{-a}$
with slope $a = -1.1$



(2) **Degree distribution** $P(k)$ measured
on the same novel, $P(k) \sim k^{-b}$
The slope found is $b = -1.9$.



From ←
A.P. Masucci and G.J. Rodgers,
Phys. Rev. E 74, 026102 (2006)

From the rank to the probability distribution

From the rank r define the "fraction" variable

$Y = r / N = F(x)$

Where N is the number of items/data.
The variable Y is just the cumulative distribution $F(x)$, since it give the fraction of items with values smaller than x.

Therefore if $x \sim r^{-a}$ and $r \sim x^{-1/a}$
the cumulative distribution is $F(x) \sim x^{-1/a}$

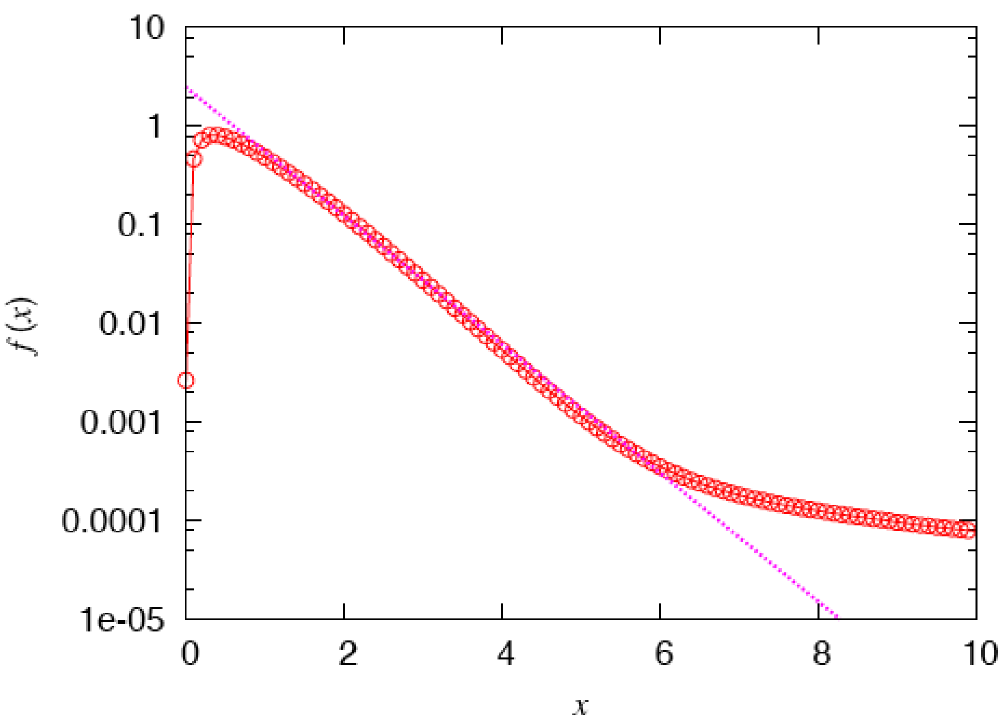
Then the probability density is $f(x) \sim x^{-(1/a+1)}$

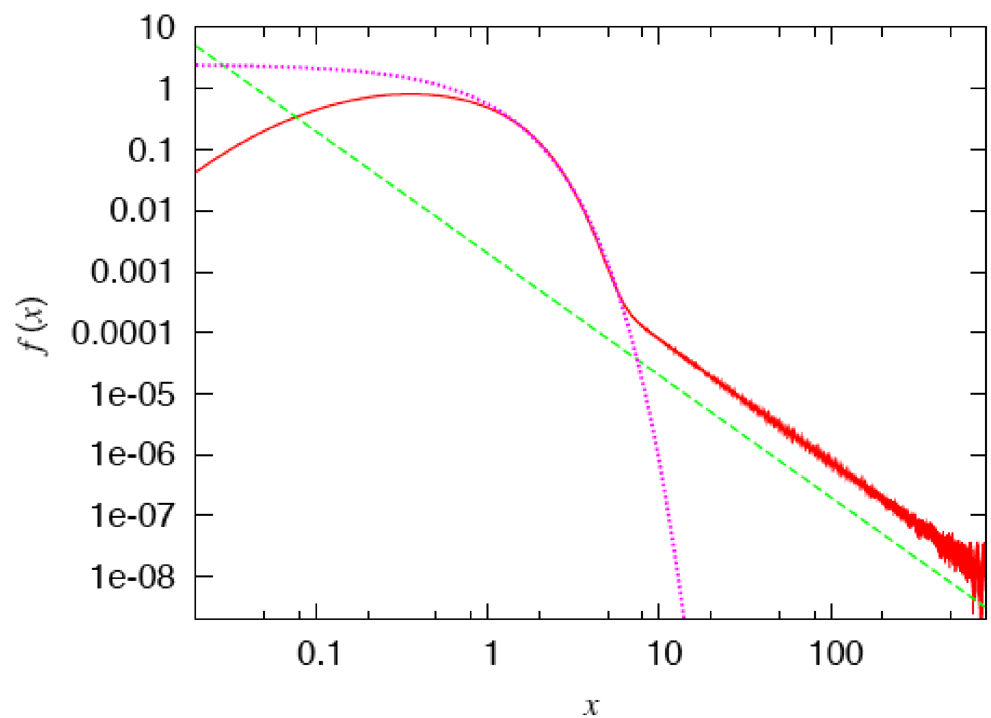
According to the previous considerations, this compares well with the degree distribution $P(k)$, i.e. $b \sim 1/a + 1 \sim 1.1$

Heterogeneous Kinetic Exchange Models

- The analogy with dimensionally heterogeneous systems is based on the similarities discussed above between the models.

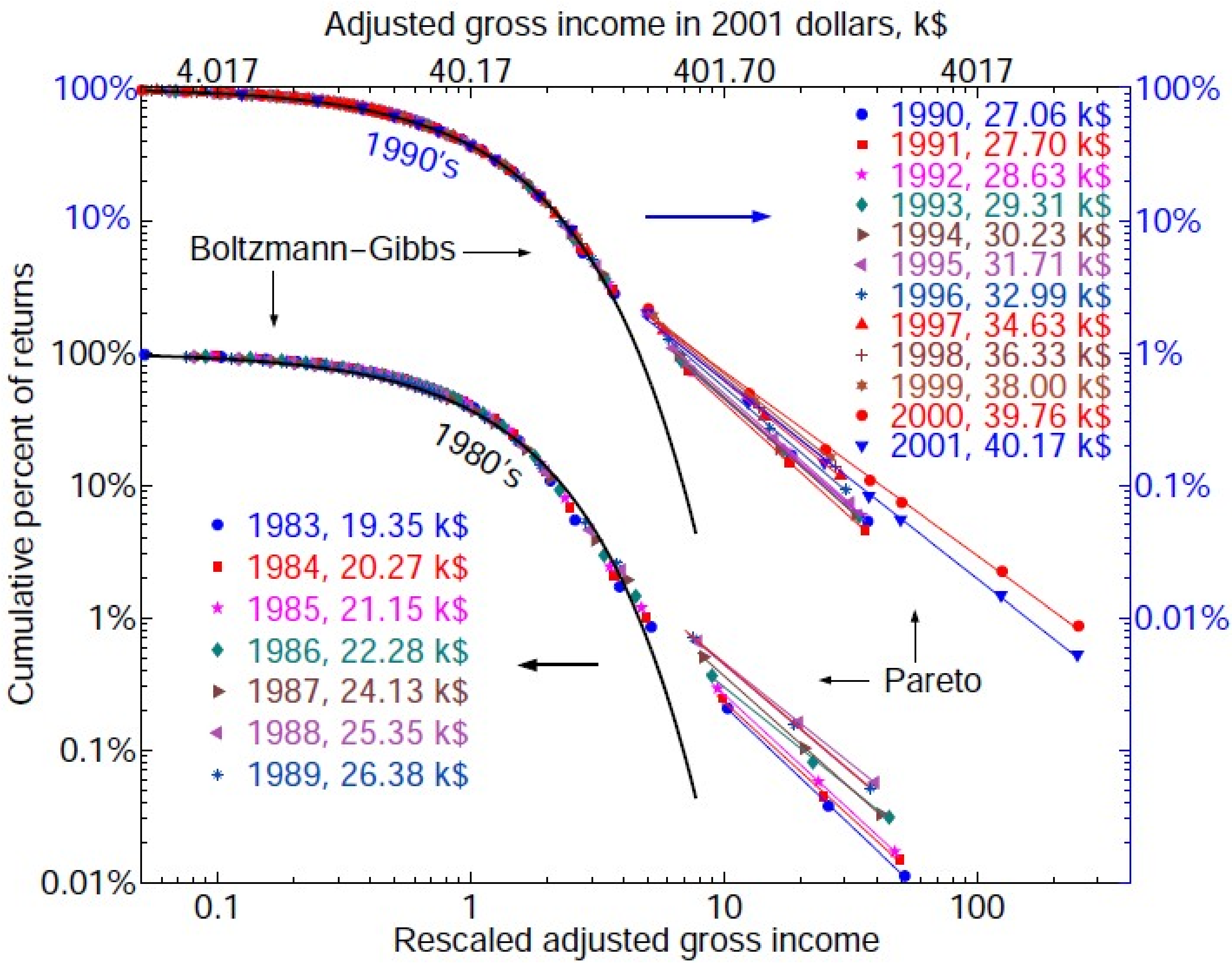
- **Example**: If the saving propensities of the N agents (λ_1, λ_2, ..., λ_N) are for 1% distributed uniformly λ in (0,1),





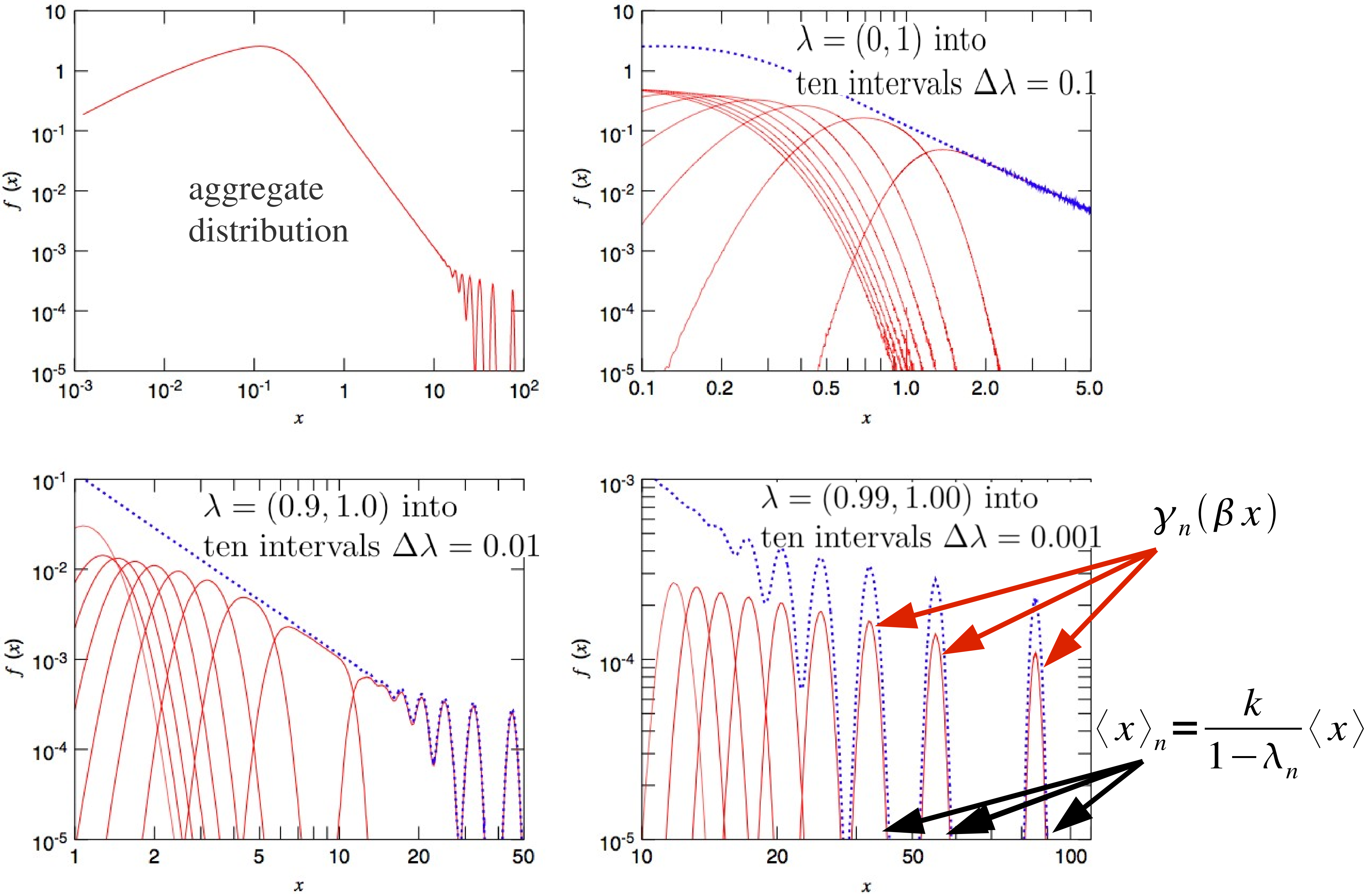
Compare with real data →

Income data overview



Colloquium: Statistical mechanics of money, wealth, and income [arXiv:0905.1518]
Victor M. Yakovenko, J. Barkley Rosser

Decomposition of the aggregate distribution $f(x) = \Sigma_i f_i(x)$ for λ's in (0,1)



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Additional material

Variational principle for one degree of freedom

Variational principle approaches based on the variation of an entropy functional find a natural application in the study of social and economic processes.

Entropy	$S[f]=\int dq\, f(q) \ln[f(q)]$
Probability conservation	$I[f]=\int dq\, f(q)$
Wealth conservation	$X_{tot}[f]=\int dq\, f(q)\, X(q)$

Lagrange method:

$$\delta S_{eff}[f]=\delta\{S[f]+\mu I[f]+\beta X_{tot}[f]\}$$

$$=\delta \int dq\, f(q)\{\ln[f(q)]+\mu+\beta X(q)\}=\cdot \quad \rightarrow \quad f(x)=\frac{\exp(-\beta x)}{\langle x \rangle}$$

Variational principle for N degrees of freedom (dimensions)

Functional $S[f] = \int dq_1 dq_2 \dots f(q_1, q_2, \dots)\{\ln[f(q_1, q_2, \dots)] + \mu + \beta X(q_1, q_2, \dots)\}$

Energy in N-dimensions: $X(q) = \frac{1}{2}\left[q_1^2 + \dots + q_N^2\right]$ (independent particles)

Integrate $N - 1$ angular variables: $S[f_1] = \int dq\, f_1(q)\left\{\ln\left[\frac{f_1(q)}{\sigma_N q^{N-1}}\right] + \mu + \beta X(q)\right\}$

(N –1)-dimensional surface: $\sigma_N = 2\pi^{N/2} / \Gamma(N/2)$

Reduced density in q $f_1(q) = f_N(q) / \sigma_N q^{N-1}$

Move to energy variable $x = X(q^2)$ and apply Lagrange method:

$$\delta S[f] = \delta \int dx\, f(x)\left\{\ln\left[\frac{f(x)}{\sigma_N x^{N/2-1}}\right] + \mu + \beta x\right\} = 0 \quad \rightarrow \quad f(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} e^{-\beta x}$$

Example: KWEM Aggregate distribution *f* (*x*) for distributed λ with density Φ(λ)

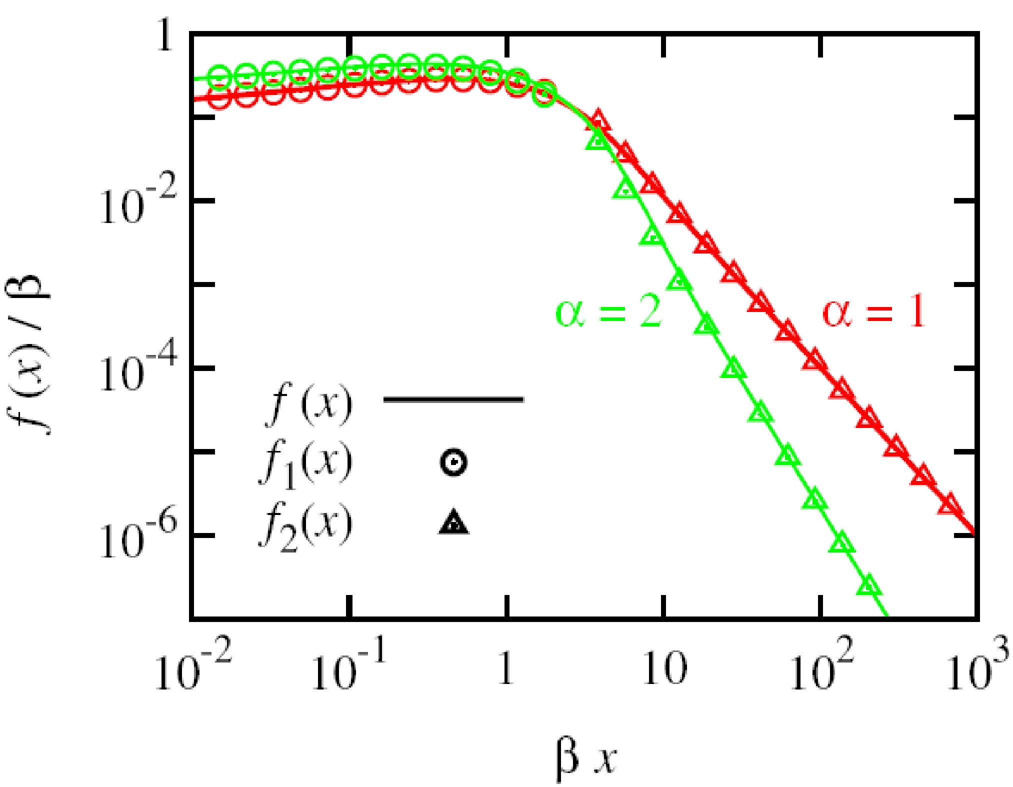
Simple example: the agents have different saving propensities λ_i with a uniform λ-density Φ(λ).

$$\phi(\lambda) = 1 \,, \qquad 0 < \lambda < 1$$

$$\phi(\lambda) = 0 \qquad \text{otherwise}$$

Corresponding form of the *n*-density

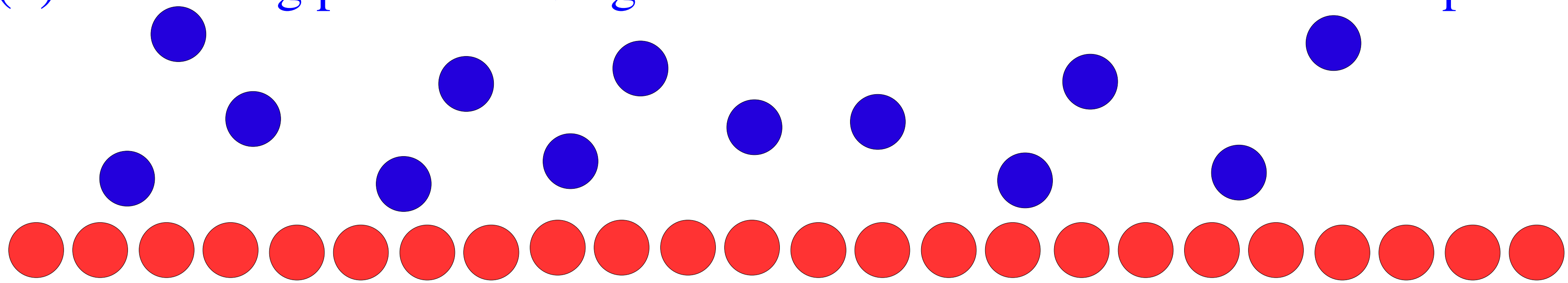
$$P(n) = \frac{d\lambda(n)}{dn}\phi(\lambda(n)) = \frac{3}{(n+2)^2}$$



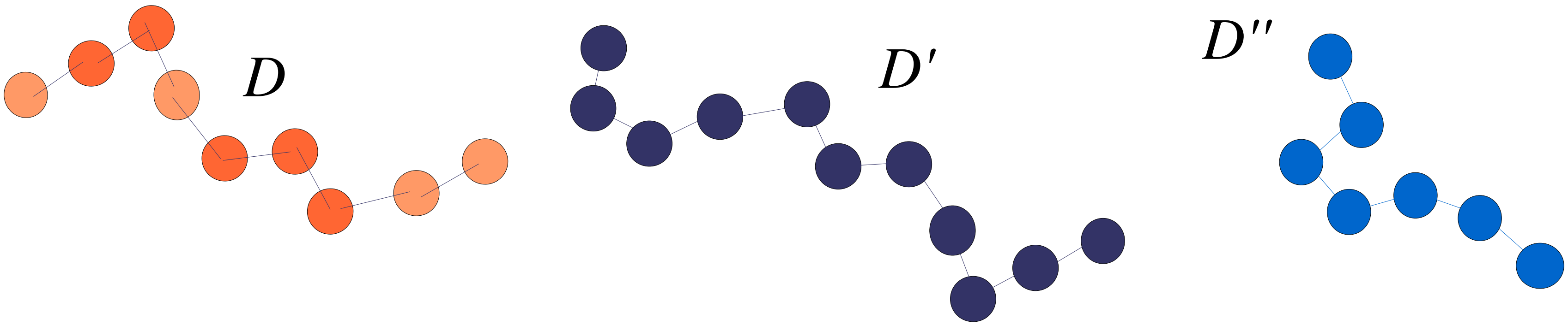
$$f(x) = \int_1^\infty dn\, P(n)\, \beta\, \gamma_n(\beta x)$$

Examples of dimensionally heterogeneous systems

(1) Interacting particles living in different *D*- and *D′*-dimensional spaces



(2) Interacting polymers with different numbers of harmonic degrees of freedom



(3) Heterogeneous Networks