Reliability of Lagrangian diagnosis from Finite Size Lyapunov Exponents.

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Abstract

Due to its inherent turbulent nature, ocean motion possess a great complexity. We can barely describe the main patterns of general circulation at large scale, but the extreme richness of circulation patterns at mesoscale and lower scales makes the assessment of ocean evolution quite complicated. These difficulties are specially relevant when one tries to study problems of Lagrangian character, such as mixing, dispersion and transport of oceanic properties. For that reason, the implementation of appropriate Lagrangian diagnostic tools are in order.A prominent Lagrangian technique which starts to be widely used in oceanography is that of Finite-Size Lyapunov Exponents (FSLE). FSLE is a local measure of particle dispersion is obtained at each point, which serves to characterize Lagrangian structures. Although mathematically appealing, it is rather unclear how robust are FSLE analyses when confronted to real data, that is, data affected by noise and with limited scale sampling. In this study, we analyze the effect of finite scale samplings and of diverse types of noise on FSLE diagnostics. Both effects should be accounted to determine which part of the diagnostics is reliable. Most importantly, scale dependence of FSLE reveals the emergence of a cascade-like hierarchy in Lagrangian structures, which can be used to improve diagnostics and to better understand ocean dynamics.

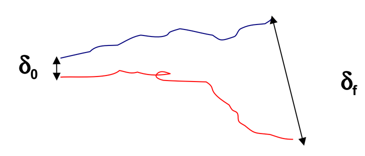
Multifractal character and scale invariance properties of FSLE

FSLE at different spatial scales and at the same resolution of velocity data:

The FSLEs are calculated by the formula

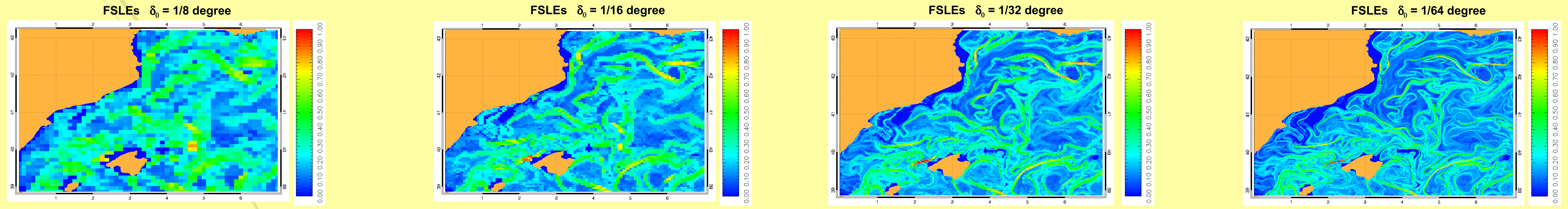
$$\Lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \log \frac{\delta_f}{\delta_0},$$

δ_0 is the initial separation
δ_f is the final separation
τ is the time needed for two particles initially separated δ_0, to get separate δ_f
x_0 are the initial coordinates
t time



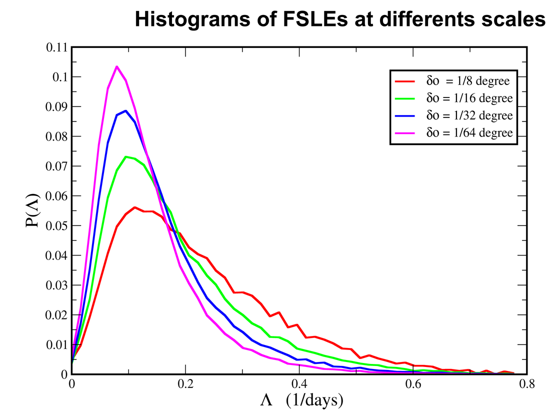
The FSLEs are computed using daily surface velocity data of DieÇAST model applied to the Balearic Sea. In this model the resolution for velocity data is 1/8 degree. We modify the spatial scales at the FSLEs field changing the initial resolution δ_0 . The δ_0 used for this computation are: 1/8 degree, 1/16 degree, 1/32 degree and 1/64 degree.

FSLEs δ_0 = 1/8 degree — **FSLEs δ_0 = 1/16 degree** — **FSLEs δ_0 = 1/32 degree** — **FSLEs δ_0 = 1/64 degree**



We calculated the probability distribution, P (δ_0, Λ), of the FSLEs at the different spatial resolutions δ_0 (1/8, 1/16, 1/32, 1/64 degrees). The histograms have been normalized to the same area, by dividing by the respective maximum values of the FSLEs. The plots shows that when the resolution is finer (smaller δ_0), the probability distribution P(δ_0, Λ) narrows and the values of the peak increase, in a way consistent with a multifractal character of the FSLEs .

Histograms of FSLEs at differents scales



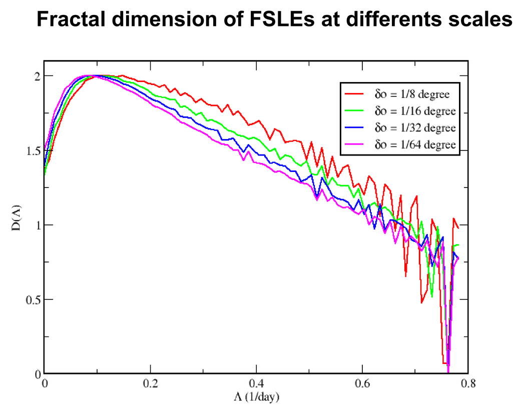
The scale behaviour of these histograms indicates that the distribution of the FSLEs at a scale δ_0 is given by:

$$P(\delta_0, \Lambda) = P(\delta_0, \Lambda_c)\ \delta_0^{d-D(\Lambda)}$$

P(δ_0, Λ_c) is the maximun value of the probality distribution

d in this case is the surface dimension = 2
D (Λ) is the fractal dimension of the set of initial conditions leading to FSLE
δ_0 is the definition scale

Fractal dimension of FSLEs at differents scales



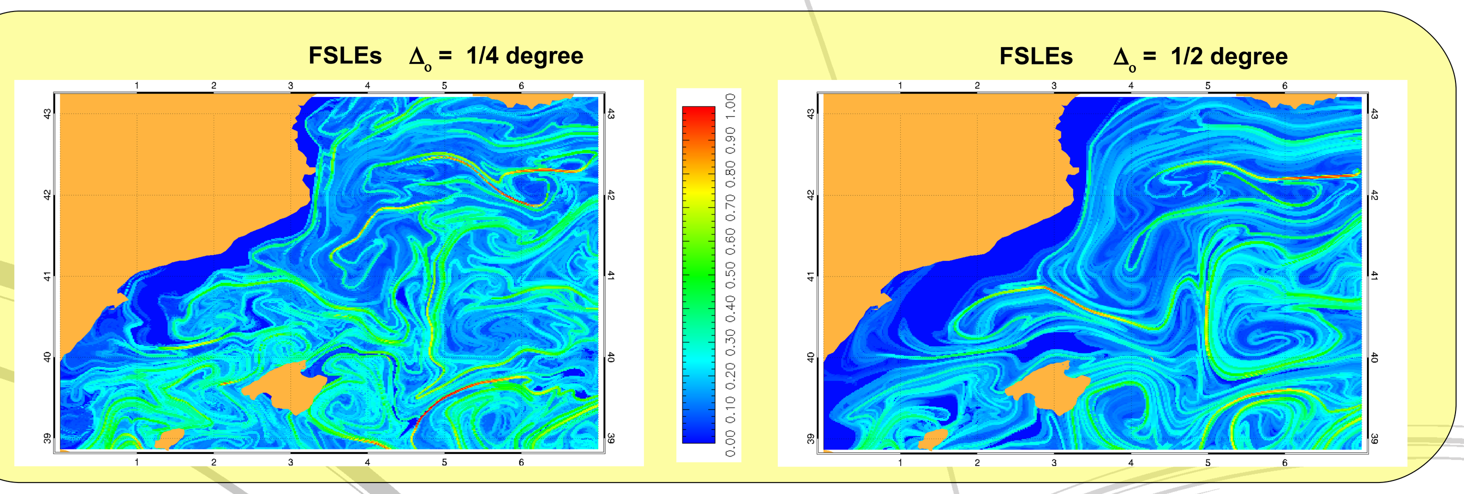
From Eq (1) one obtains a properly normalized expression to computed the fractal dimension at different scales

$$D(\Lambda) = d - \frac{\log \frac{P(\delta_0,\Lambda)}{P(\delta_0,\Lambda_c)}}{\log \delta_0} \qquad (2)$$

The plot of D(Λ) shows a collapse of D (Λ) at the different scales. Because of lost of translational invariance the collapse is not perfect. This effect happens when the resolution and the domain is small. (i.e. It has a few pixels)

FSLE at diferent spatial resolution of the velocity field, and at the same spatial scale of FSLE

FSLEs Δ_0 = 1/4 degree — **FSLEs Δ_0 = 1/2 degree**



Even decreasing the spatial resolution of the velocity data, the structures remain.

We compute the histograms of the spatial distributions of the FSLEs at the same spatial scale δ_0 1/64 degree and a different spatial resolution of the velocity field: Δ_0=1/8 , 1/4 and 1/2 degree Fig (1), and probability distributions (coded as grey level) of FSLEs derived at a coarse velocity resolution (Λ_c, vertical axis) conditioned by a finer velocity grid (Λ_f, horizontal axis)(Fig 2).

Fig. 1

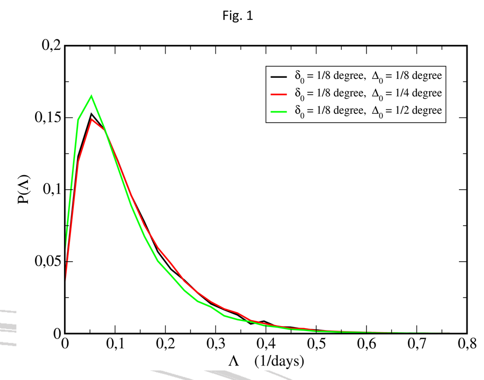
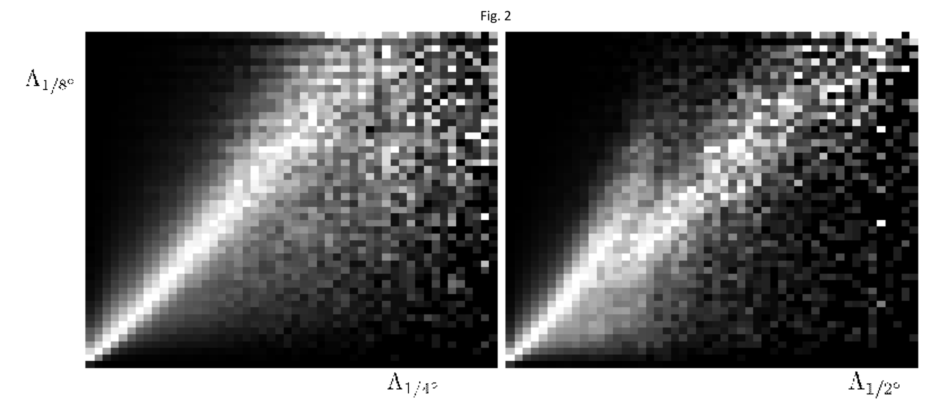


Fig. 2



Linear regression fits:

$$\Lambda_f(\vec{x}) = \Lambda_c(\vec{x}) + \Delta\Lambda_{fc}(\vec{x})$$

$\Lambda_f(\vec{x})$ FSLE at finer resolution
$\Lambda_c(\vec{x})$ FSLE at lower resolution
$\Delta\Lambda_{fc}(\vec{x})$ Contribution not account by the lower resolution

$$\Lambda_{1/8^\circ} = 1.08\Lambda_{1/4^\circ} + 0.05$$
$$\Lambda_{1/4^\circ} = 0.99\Lambda_{1/2^\circ} + 0.04$$

This behaviour suggest a relation of scale between FSLE at different velocity resolutions.
Linel regresion analysis shows a correspondence one to one between values of FSLEs at different velocity resolutions.

Robustness of FSLEs

Error in the data

To study the robustness of the FSLEs, we compute the relative error of FSLE introducing a velocity data perturbated with a small random error, respect to FSLE without perturbation..

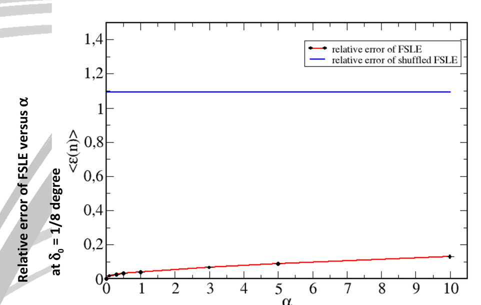
$$V \xrightarrow{\text{perturbation}} V'(perturbated) = V + error = V + \varepsilon \cdot \eta \cdot V = V(1 + \varepsilon \cdot \eta)$$

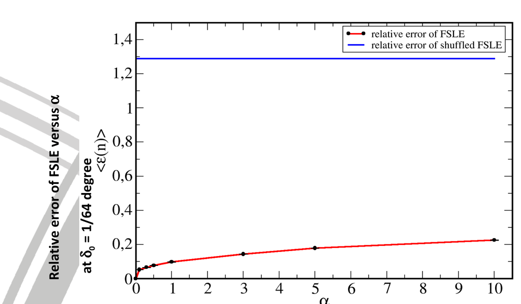
$\alpha = \varepsilon \bullet \eta$ ε is the % of error.
η is a Gaussian distributed random number: mean = 0, variance = 1.
V is the velocity data. V' is the perturbated velocity data

We computed the relative error of FSLEs by the formula:

$$\epsilon(n) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \frac{|\Lambda_i^\epsilon(n) - \Lambda_i(n)|^2}{|\Lambda_i(n)|^2}}$$

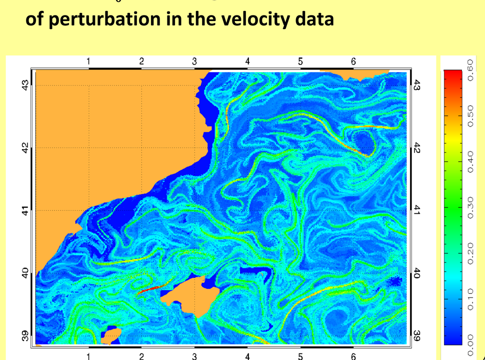
Λ are the values of FSLEs without error in the velocity data.
Λ^ε are the values of FSLEs with error in the velocity data.
N is the total number of points in the FSLEs field.





Even with a perturbation 10 times the velocity data, the error remains smaller than 0.2

FSLEs at δ_0 = 1/64 degree with 20% of perturbation in the velocity data



The mesoscales structures are mainted

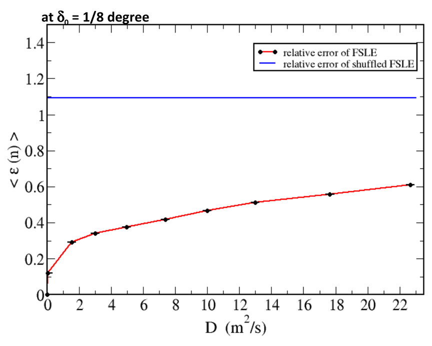
Noise in the particle's trajectories

We have computed the FSLEs introducing white noise in the particles trajectories. We have solved the following equations:

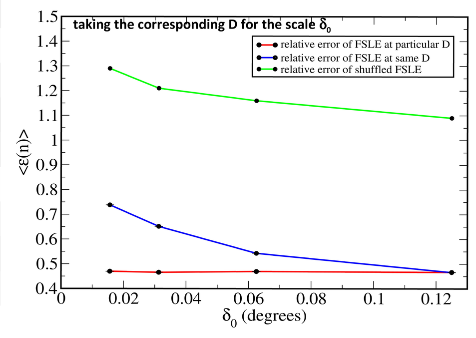
$$\frac{d\phi}{dt} = \frac{u(\phi,\lambda,t)}{R\cos(\lambda)} + \frac{\sqrt{2D}}{\cos(\lambda)}\xi_1(t)$$
$$\frac{d\lambda}{dt} = \frac{v(\phi,\lambda,t)}{R\cos(\lambda)} + \sqrt{2D}\xi_2(t)$$

u , v **are the components of the velocity.**
Φ is the longitudinal coordinate.
λ is the latitudinal coordinate.
D **is the eddy diffusivity computed by the Okubo's formula (OKubo, 1971)**
ξ(t) **is white noise** $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$ $\langle \xi(t)\rangle = 0$

Relative error of FSLE vs eddy diffusivity (D) at δ_0 = 1/8 degree

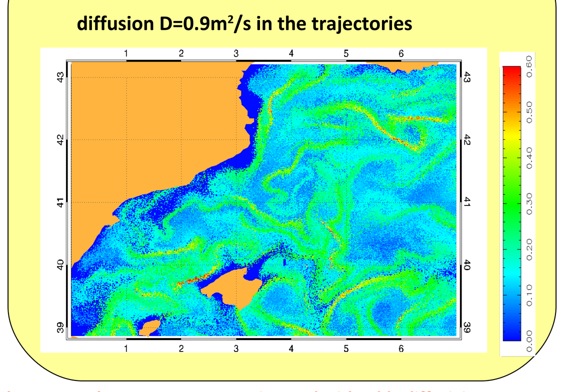


Relative error of FSLE vsδ_0.
One plot using constant D and another taking the corresponding D for the scale δ_0



Even with 10m2/s of diffusion, the relative error remains smaller y than 0.5

FSLEsδ_0 = 1/64 degree with the diffusion D=0.9m²/s in the trajectories



The mesoscales structures are maintaned with eddy diffusivity

Conclusions

- **Increasing the spatial resolution of FSLEs we improve the identification of surface mesoscale structures.**
- **The surface mesoscales structures in the ocean remain when the spatial resolutions of the velocity decreases**
- **All the dependence of FSLEs on scale parameters reveal a multifractal structure. What is diagnosed at the coarser scales is still valid when scale is refined.**
- **The FSLEs are rather robust. The relative error, even for an error of 10% of the velocity data, is smaller than 20 %.**
- **Mesoscale structures are maintained when the eddy diffusion is included.**

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