Non-equilibrium transition in a system of active rotators near the excitable regime

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Synchronization phenomena are subject or intense research efforts in physical, biological, chemical, and social systems.

The Kuramoto model is a paradigmatic framework to study synchronization phenomena.

This model has been studied for many different distributions of natural frequencies and results are believed to be qualitatively independent of the specific distribution as long as it is symmetric and unimodal.

Here we consider very similar model in which individual units are excitable units instead of oscillatory.

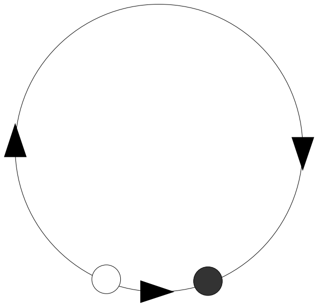
The model can display a non-equilibrium transition depending on the form of the probability distribution.

$$\dot{\phi}_i = \omega_i - \sin(\phi_i) + \frac{K}{N}\sum_{j=1}^{N} \sin(\phi_j - \phi_i)$$

ϕ_i: *Phase of oscillator* i $(\phi_i \in (-\pi, \pi))$

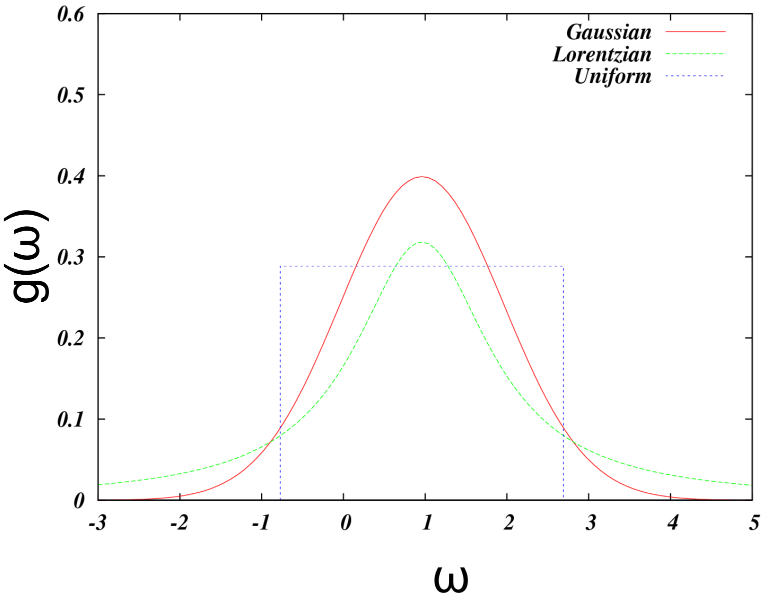
ω_i: *Natural frecuency of oscillator* i*, distributed following a probability density function* $g(\omega_i)$ $(\omega_i < 1, \omega_i \approx 1 \rightarrow$ *excitable unit*$)$

K: *Coupling strength*



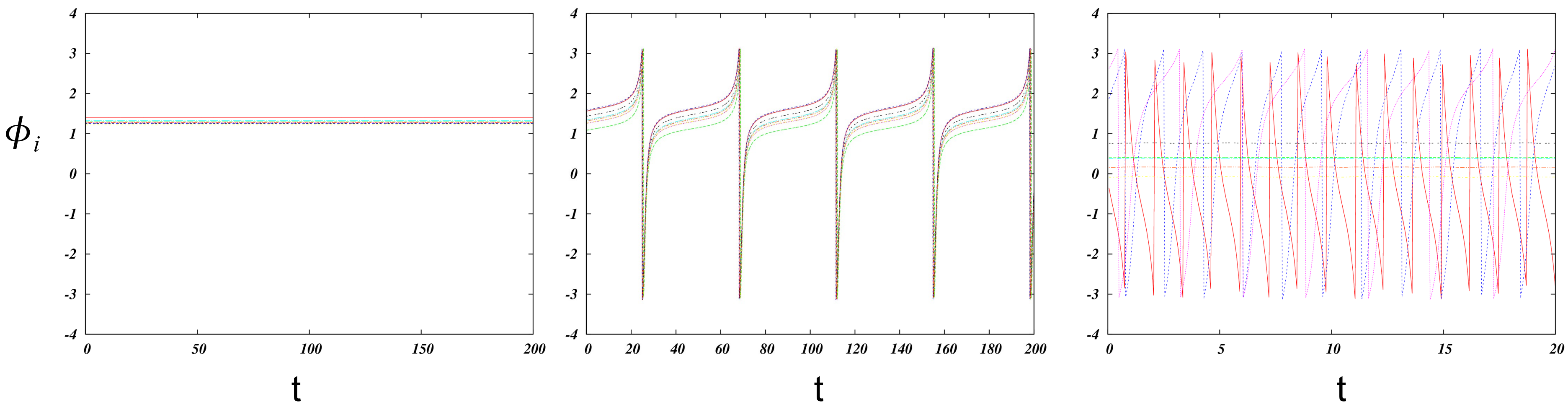
Phase portrait of an individual uncoupled unit with $\omega_i<1$

We consider the following distributions of natural frequencies:



3 Regimes:

Simulations Gaussian g(w), $\overline{\omega}=0.96$, K=4, 8 from 10000 oscillators shown



I: Static
Small σ

II: Synchronized firing
Intermediate σ

III: Desynchronized firing
Large σ

Order parameters: $\rho(t)\, e^{i\Psi(t)} = \frac{1}{N}\sum_{i=1}^{N} e^{i\phi_i(t)}$ $\rho \equiv \langle \rho(t)\rangle$: *degree of synchronization*, $\Psi(t)$: *global phase*

$$\zeta = \langle \left| \rho(t)\, e^{i\Psi(t)} - \langle \rho(t)\, e^{i\Psi(t)} \rangle \right| \rangle$$

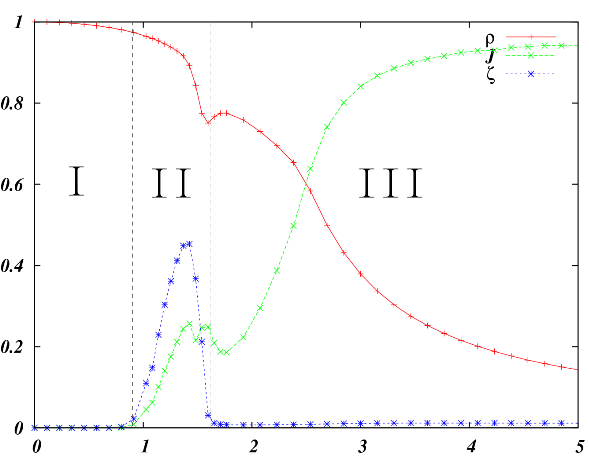
$\zeta \neq 0 \rightarrow$ *nonstationary state, collective firing.*

$\langle\, \rangle$: *time average*

Current: $J = \frac{1}{N}\sum_{i=1}^{N} \langle \dot{\phi}_i(t) \rangle$

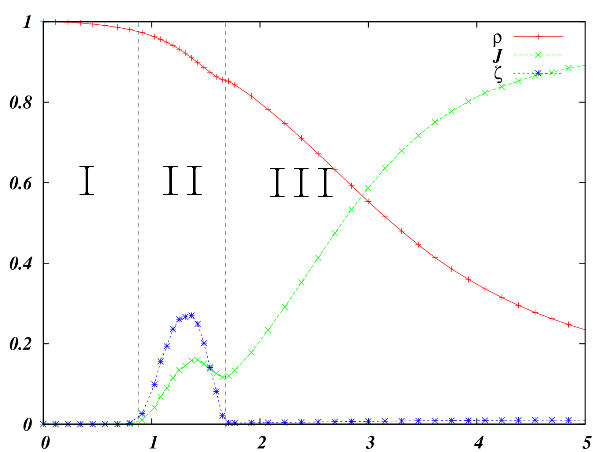
Simulations K=4, $\overline{\omega} = 0.96$

Exact results with Ott's method [1]



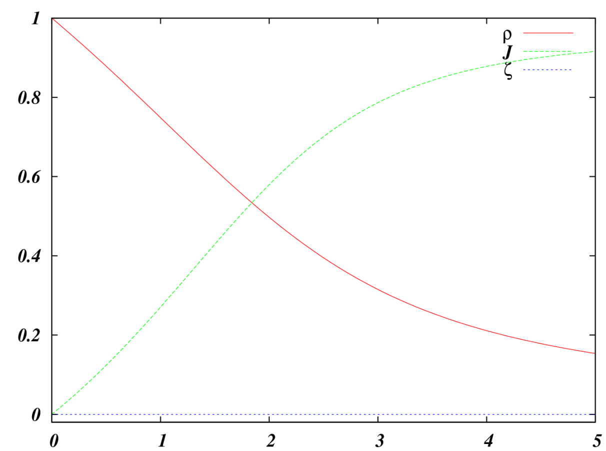
σ

Uniform g(ω)



σ

Gaussian g(ω)



Δ

Lorentzian g(ω), no transition

[1] E. Ott and T.M. Antonsen, Chaos **18**, 037113 (2008)

Discussion

$$b=\sqrt{1+K^2\rho^2+2K\rho\cos(\Psi)}$$

$$\tan(\phi_0)=\frac{K\rho\sin(\Psi)}{1+K\rho\cos(\Psi)}$$

$$\dot{\phi}_i=\omega_i-\sin(\phi_i)+K\rho(t)\sin(\Psi(t)-\phi_i)=\omega_i+b\sin(\phi_0-\phi_i)$$

$N\rightarrow\infty$, *stationary state*:

$$P_s(\phi;\omega,\rho,\Psi)=\begin{cases}\delta\left(\phi-\phi_0-\arcsin\left(\frac{\omega}{b}\right)\right), & if\ |\omega|\leqslant b\\ \frac{\sqrt{\omega^2-b^2}}{2\pi|w+b\sin(\phi_0-\phi)|}, & if\ |\omega|>b\end{cases}$$

$$\rho e^{i\Psi}=\iint P_s(\phi;\omega,\rho,\Psi)e^{i\phi}d\phi\, g(\omega)d\omega$$

$$\rho\sin(\Psi)=\overline{\omega}-\int_{|\omega|>b}\omega\sqrt{1-\frac{b^2}{\omega^2}}g(\omega)d\omega \quad (1)$$

$$K\rho^2+\rho\cos(\Psi)=\int_{-b}^{b}\sqrt{b^2-\omega^2}g(\omega)d\omega \quad (2)$$

$$J=\int_{|\omega|>b}sig(\omega)g(\omega)d\omega$$

If the support of g(ω) is such that g(ω)=0 ∀ |ω|>b, (1)→ No stationary solution for $\rho<\overline{\omega}$
At the critical point $\rho=\overline{\omega},\Psi=\pi/2$. Expanding the integrand of (2) in ω/b, we obtain:

$$\overline{\omega}\simeq 1-\frac{(\overline{\omega})^2-1}{2(\overline{\omega})^2K^2}-\frac{\sigma_c^2}{2(\overline{\omega})^2K^2}\rightarrow\sigma_c=\sqrt{2(\overline{\omega})^2K^2(1-\overline{\omega})+1-(\overline{\omega})^2}$$