Diversity Induced Bifurcations

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BioSim NETWORK OF EXCELLENCE

Abstract

Diversity, or inhomogeneity, is ubiquitously present in biological systems as there are no two identical cells – not even in the same tissue of one organism, there are no two completely identical plants even if they reproduce by vegetative cloning and there are probably not even two enzymes of the same chemical composition with identical transition rates (eg. [1]).

We chose prototype models of bifurcations where the bifurcation is induced by some parameter. We couple many of them, all identical in structure but different in the control parameter, to mimic the interaction of "real" biological processes with toy models, simple enough to be studied thoroughly. As tools we use and refine the method of order parameter expansion [2-5] or solve a self-consistency relation as done in [6].

Main Method (Order Parameter Expansion)

$$\dot{x}_j = f(x_j(t), \eta_j, \langle x \rangle) \xrightarrow{\text{Taylor series}} \dot{x}_j = f(\langle x \rangle, \langle \eta \rangle) + f_x \epsilon_j + f_\eta \delta_j + \cdots + f_{xx}\epsilon_j^2 + f_{\eta x}\delta_j \epsilon_j + f_{\eta\eta}\delta_j^2 + \cdots$$

averaging up to second order

$$\epsilon_j(t) = x_j(t) - \langle x \rangle(t), \quad \delta_j = \eta_j - \langle \eta \rangle, \quad \langle \epsilon_j(t) \rangle_j \equiv 0, \quad \langle \delta_j \rangle_j = 0, \quad \langle \delta_j^2 \rangle_j = \sigma^2$$

$$\langle \dot{x} \rangle = f(\langle x \rangle, \langle \eta \rangle) + \frac{1}{2} f_{xx} \langle \epsilon^2 \rangle + f_{\eta x} \langle \epsilon \delta \rangle + \frac{1}{2} f_{\eta\eta} \sigma^2$$

$$\dot{\Omega} = \langle \dot{\epsilon^2} \rangle = 2 f_x \Omega + 2 f_\eta W$$

$$\dot{W} = \langle \epsilon \delta \rangle = f_x W + \sigma^2 f_\eta$$

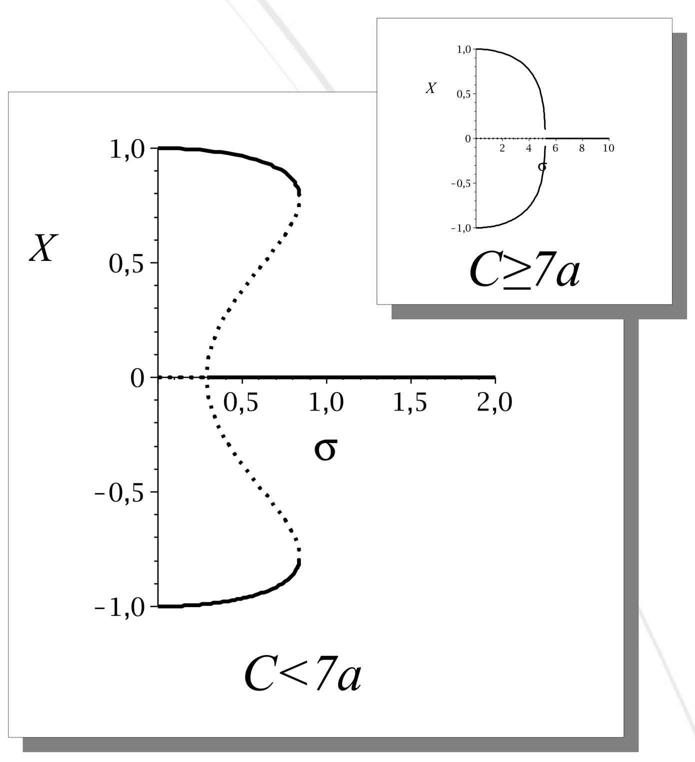
fixed points:

$$\bar{W} = -\sigma^2 \frac{f_\eta}{f_x}$$

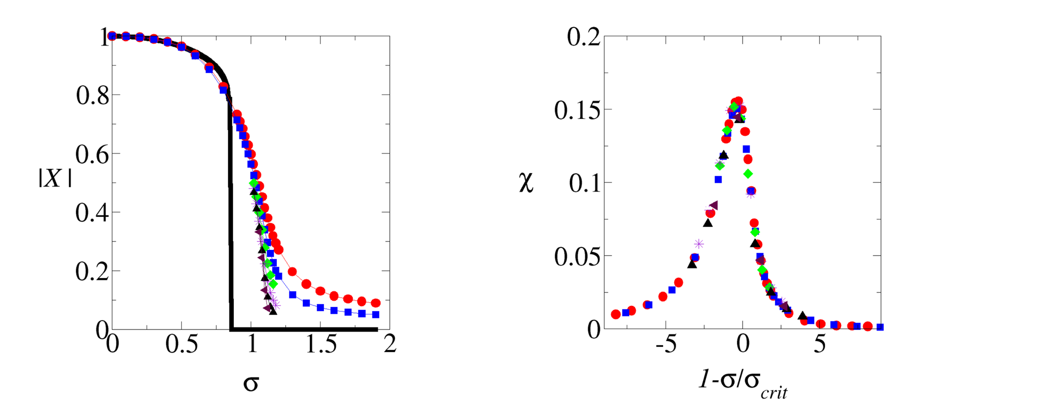
$$\bar{\Omega} = \frac{-f_\eta}{f_x} \bar{W} = \sigma^2 \left(\frac{f_\eta}{f_x}\right)^2$$

Bifurcation prototypes

I - Φ^4-model with additive diversity: $\dot{x}_j = a x_j - x_j^3 + C(\langle x \rangle - x_j) + \eta_j$

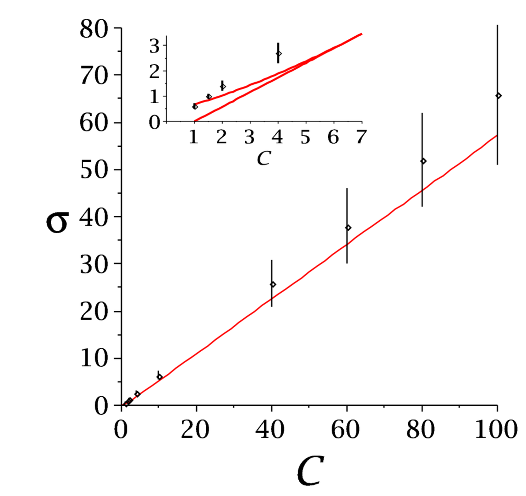


Simulations: C=1.5, a=1 ; N=1000..300000



Mean value goes into zero-state

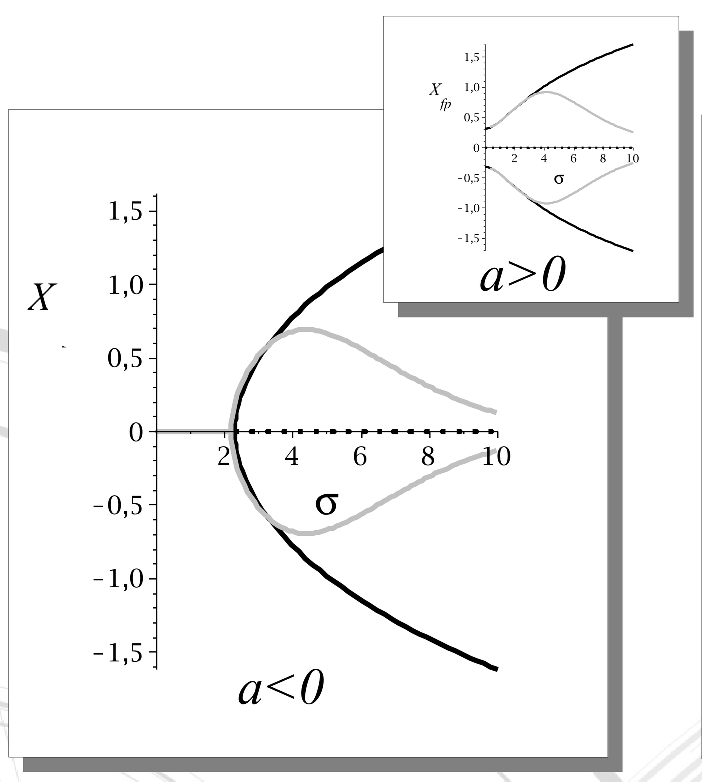
Rescaled fluctuations collapse into one curve $X(N,\sigma) = N^{-0.69} f(N^{0.35}(1 - \frac{\sigma}{1.096}))$



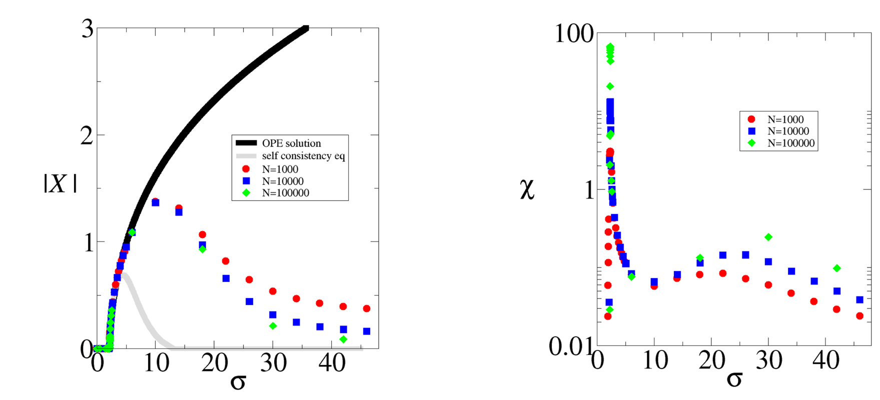
Critical diversity vs. coupling calculated with OPE

- *System goes to zero-state*
- *Bistability is not observed*
- *Fluctuations follow a scaling law*
- *Overall dependence on coupling is predicted to be linear*

II - Φ^4-model with multiplicative diversity: $\dot{x}_j = (a + \eta_j) x_j - x_j^3 + C(\langle x \rangle - x_j)$

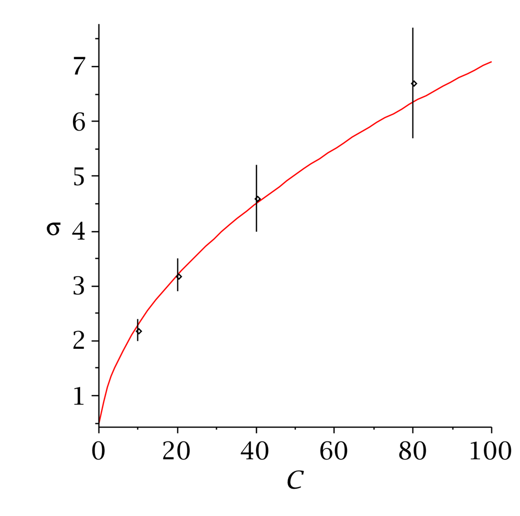


Simulations: C=10.0, a=-0.5



Mean value goes into ordered state and returns to zero-state

Fluctuations diverge at first transition



Critical diversity vs. coupling calculated with OPE

- *Diversity induces ordered state*
- *Re-entrance into zero-state is observed (self-consistency equation solved)*
- *Critical diversity rises slower than linear*

Method (see [6])

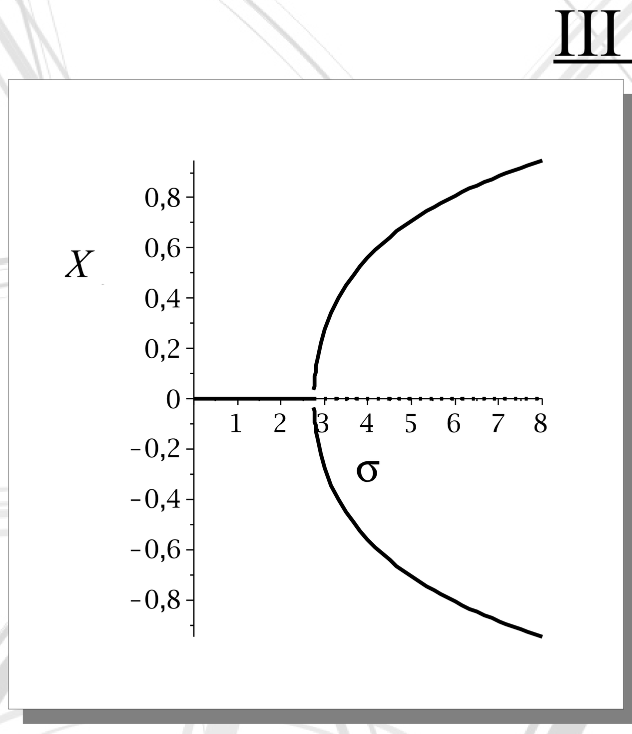
Steady state solution: $x_j(\eta_j, \langle x \rangle) = -\frac{\gamma_j}{u_j} + u_j$ (1)

$\gamma_j = \frac{C - a - \eta_j}{3}$, $u_j = \left(\frac{C\langle x \rangle}{2} + \sqrt{\gamma_j^3 + \frac{C^2 \langle x \rangle^2}{4}}\right)^{1/3}$

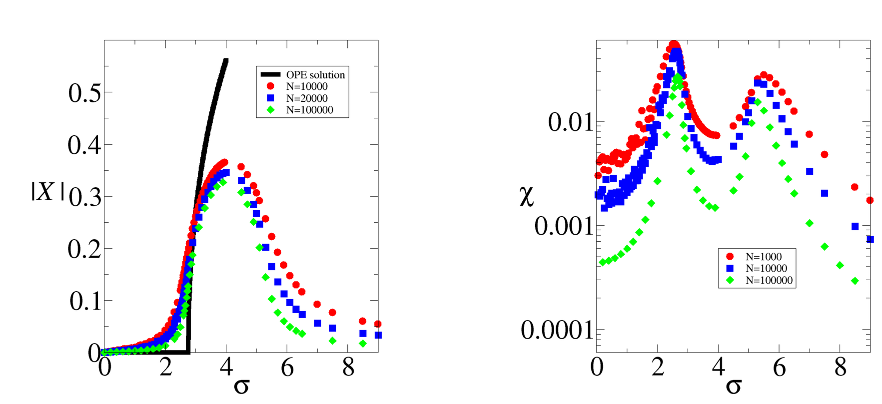
eq. (1) must fulfil self consistency equation (2):

$\langle x \rangle = \int d\eta\, g(\eta)\, x_j(\eta_j, \langle x \rangle)$ (2)

III - Canonical model for noise-induced phase transitions: $\dot{x}_j = -x_j(1 + x_j^2)^2 + (1 + x_j^2)\eta_j + C(\langle x \rangle - x_j)$

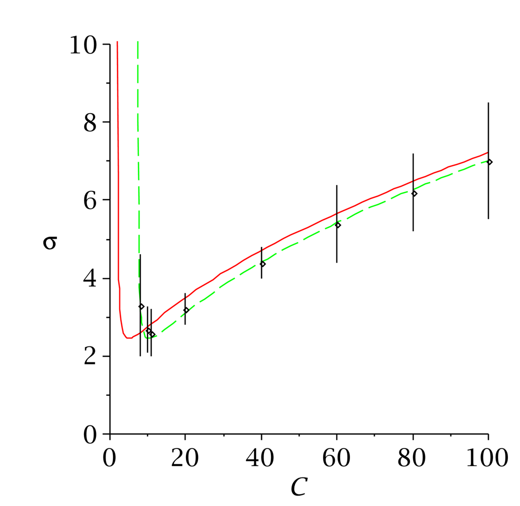


Simulations: C=10.0



Mean value goes into ordered state and returns to zero-state

Fluctuations peak at transitions



Critical diversity vs. coupling calculated with OPE

- *Diversity induces ordered state*
- *Re-entrance into zero-state is observed*
- *Critical diversity depends non-monotonously on coupling*
- *Qualitative prediction is good apart from a systematic error ΔC*

Conclusions

Diversity can induce transitions from the disordered to **the ordered state**. In the present work the bifurcations are approximated with the method of **order parameter expansion**, where the equations of motion are expanded around the mean value of the dynamical variable and the varying parameter. The fixed points of the resulting equations are easily calculated and **critical values** for the transitions can be derived as long as they are small values. In ranges of larger diversity the method fails to reproduce exact results such as the re-entrance into the zero-state. A second method to calculate the steady state was used, with it the return to the zero-state is predicted qualitatively (in example II).

References

[1] Xie, Single Mol. 2 (2001) 4, 229-236
[2] de Monte et al., Europhys. Lett. (2002), 58:21-27
[3] de Monte et al., Phys. Rev. Lett (2003), 90(5)
[4] de Monte et al., Phys. Rev. Lett. (2004), 92(25)
[5] de Monte et al., Physica D (2005), 205:25-40
[6] Toral et al., Eur. Phys. J. Special Topics (2007), 143:59-67

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