The macroscopic description of agent-based models

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Introduction

Most real-life systems are complex:

many units interacting in a non-linear way give rise to not obvious (unexpected) collective behavior.

- Ant colonies
- Human economies
- Social structures
- Nervous systems
- Cells, etc..

How do we study complex systems?

- An Agent-Based Model (ABM) simulates operations of multiple agents to recreate behavior of complex phenomena.
- ABMs describe systems at a *microscopic level.*
- Advantage: ABMs can be used as computer experiments to explore the behavior of a system under a given input.

Some terminology:

- Agent-based models (Sociology, Computer Science, Game Theory)
- Individual-based models (Ecology, Biology).
- Interacting particle systems (Physics).

Examples of interacting particle systems

Ecology:

Species competition.

Invasion processes.

Predator-prey systems (Lotka Volterra).

Biology:

Epidemic spreading (ISI, IRSI).

Allele frequency (genetics).

Bacteria dynamics.

Neural networks.

Tumor growth.

Social Science:

- Opinion spreading.
- Cultural propagation.
- Language dynamics.

Surface Physics/ Chemistry:

- Catalytic reactions.
- Deposition/ reaction-diffusion/ aggregation.

Mathematical description of ABMs

- Analytical treatment of systems provides insight of phenomena.
- But.., systems are composed by a huge number of agents!

  (many degrees of freedom).
- It is hard and unpractical to develop an analytical framework

  (equations) to describe the evolution of each single agent.
- Need to reduce number of degrees of freedom.

how?

- Define a few collective variables that describe the system as a whole.
- Still, a lot of information is obtained from this simplified viewpoint.
- Prediction of macroscopic behavior from a given microscopic dynamics (ABM) becomes relevant.
- Statistical Physics provides a suitable framework that relates micro with macro in systems with many particles/agents.

- Equilibrium statistical physics: thermodynamic relations between macroscopic/measurable variables (P,V,T) are derived from Hamiltonian-Equipartition functions.

- But.., most real-life systems are out of equilibrium.

- Analytical techniques to treat non-equilibrium problems that involve time dependence:
  Master equations, Fokker-Planck equations, Langevin equations, etc.

Field approaches to obtain macro evolution equations

The appropriate type of approach depends on the topology of interactions between agents.

MEAN-FIELD:

- Rate equation for the time evolution of a global quantity.
  Ex: density of particles, spin magnetization, population of species.
- Gives very good estimates on well mixed populations where every agent interacts with any other agent (complete graph or fully connected network).
- Simplest approach, but neglects spatial dependence, correlations and fluctuations.

PAIR APPROXIMATION:

- Rate equation for the evolution of the global density of different types of pairs (neighboring sites).
- Account for nearest neighbor correlations, but neglects fluctuations.
- Used to obtain approximate solutions in square lattices.
- Gives some idea of spatial effects.
- Specially useful in complex networks, with very accurate results.

More refined methods for heterogeneous networks:

- *Node approximation*: group nodes in different degree classes.
- *Heterogeneous pair approximation*: group links in different classes.

LANGEVIN EQUATION:

- Stochastic partial differential equation for the evolution of a continuous field $\Phi(x,t)$.
- Accounts for stochastic fluctuations associated to discreteness effects (agents).
- It contains the mean-field term, a Laplacian term that accounts for spatial dependence and a noise term related to fluctuations.
- Appropriate for spatially extended systems.
- Useful to study stability and and pattern formation.
- Essential in systems driven by noise (patterns, absorbing states).

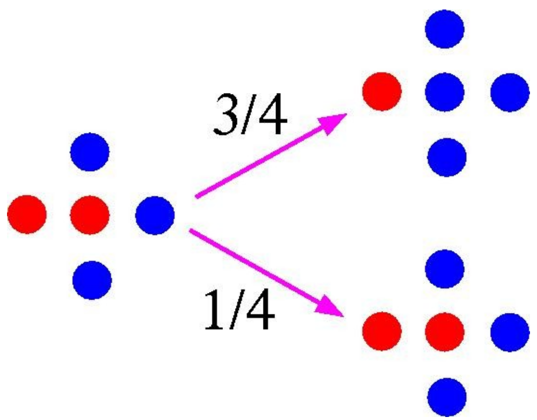
A simple application: The Voter Model [Clifford 1973, Liggett 1975]

- Two possible positions (opinions) {-1=left,1=right} on a political issue.
- Individuals ("voters") blindly adopt the position of a random neighbor.

Initial state: density σ of – voters and 1-σ of + voters.

Dynamics:

1) Pick a voter i with opinion x_i at random.
2) Pick a neighbor j with opinion x_j at random. i adopts j's opinion ($x_i \rightarrow x_i = x_j$).
3) Repeat ad infinitum.



Final state: -1 consensus with prob. σ
+1 consensus with prob. 1-σ

Complete Graph

σ_- = global density of voters with opinion -1 (spin -1)

σ_+ = global density of voters with opinion 1 (spin 1)

$$m = \sigma_+ - \sigma_- \quad \equiv \quad \text{Global Magnetization}$$

$$1 = \sigma_+ + \sigma_- \quad \text{(Total density of voters is conserved)}$$

$$\rho = \frac{\text{\# links between -1 and +1 spins}}{\text{total \# of links}} \quad \equiv \quad \text{Density of active links}$$

Rate equation for m:

$$\frac{dm(t)}{dt} = \frac{1}{1/N}\left[\sigma_- P(- \to +)\frac{2}{N} - \sigma_+ P(+ \to -)\frac{2}{N}\right]$$

Biased Voter Model:

$$P(+\to -) = \frac{1}{2}(1-v)\sigma_-, \quad P(-\to +) = \frac{1}{2}(1+v)\sigma_+$$

V = bias (preference for one of the opinions)

V > 0 (favor for + opinion), V < 0 (favor for - opinion)

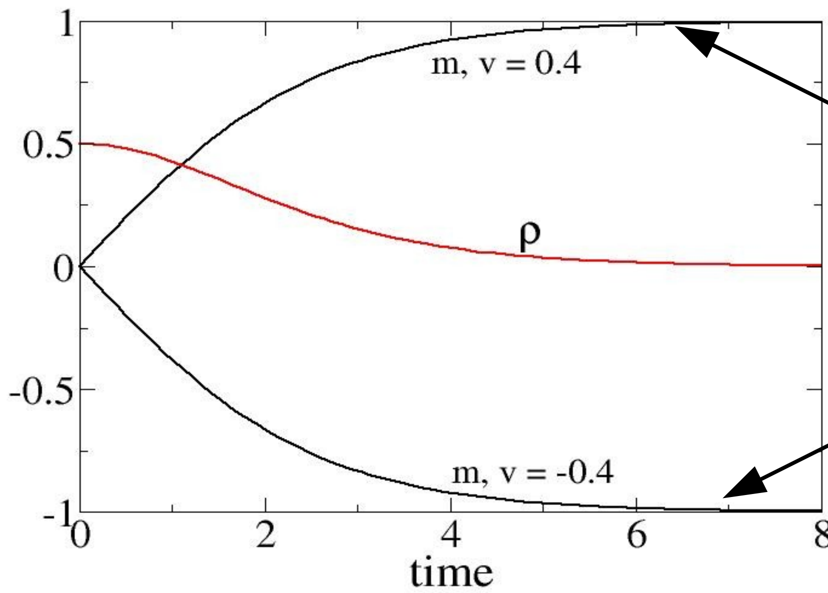
$$\sigma_+ = \frac{1+m}{2}, \quad \sigma_- = \frac{1-m}{2}$$

Mean-field equation for the magnetization:

$$\frac{dm}{dt} = \frac{v}{2}(1-m^2)$$

If m(t=0) = 0 $\rightarrow \quad m(t) = \tanh(vt/2)$

and $\rho(t) = 2\sigma_+\sigma_- = \frac{1-m^2}{2} = \frac{1}{2}\left[1-\tanh^2(vt/2)\right]$



only rightists

only leftists

Symmetric case (v=0):

$$m(t) = 0 = \text{const}, \quad \rho(t) = 1/2 = \text{const}$$

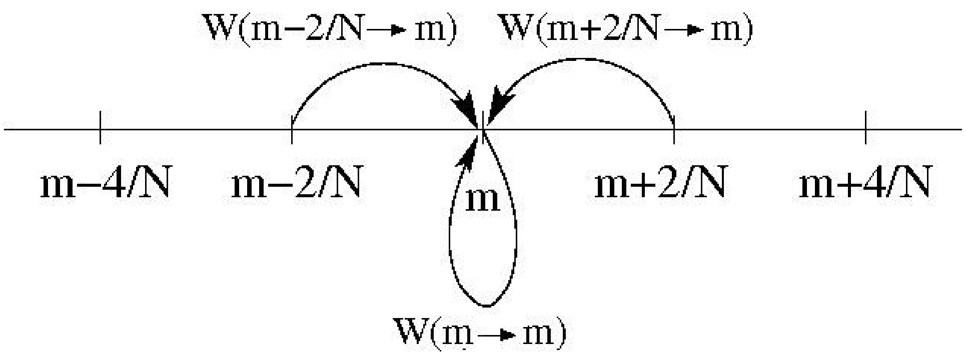
But....we know the system ultimately reaches consensus!

or $$m(t=\infty) = \pm 1, \quad \rho(t=\infty) = 0 \;\; !!!$$

Therefore:

- *Fluctuations must lead the system to the ordered state.*
- *Mean-field approach is not enough to describe the system.*

Langevin approach



$$
\begin{aligned}
W(m \to m - 2/N) &= \frac{(1-v)}{8}\left(1-m^2\right) \\
W(m \to m + 2/N) &= \frac{(1+v)}{8}\left(1-m^2\right) \\
W(m \to m) &= 1 - \frac{1}{2}\left(1-m^2\right)
\end{aligned}
$$

Master equation for the magnetization

$$
\begin{aligned}
P(m, t+1/N) &= W(m+2/N \to m)\, P(m+2/N, t) \\
&+ W(m-2/N \to m)\, P(m-2/N, t) + W(m \to m)\, P(m, t)
\end{aligned}
$$

Fokker-Planck equation

$$\frac{\partial P(m,t)}{\partial t} = -\frac{\partial}{\partial m}\left[\frac{v}{2}(1-m^2)P(m,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial m^2}\left[\frac{1}{N}(1-m^2)P(m,t)\right]$$

Langevin equation for the magnetization:

$$\frac{dm}{dt} = \frac{v}{2}(1-m^2) + \sqrt{\frac{(1-m^2)}{N}}\,\eta(t)$$

drift noise

Gaussian white noise: $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$

Solution to the Fokker-Plank equation:

$$P(m,t) = \sum_{l=0}^{\infty} A_l \, C_l^{3/2}(m) \; e^{-(l+1)(l+2)\,t/N}$$

Average density of active links decays to zero:

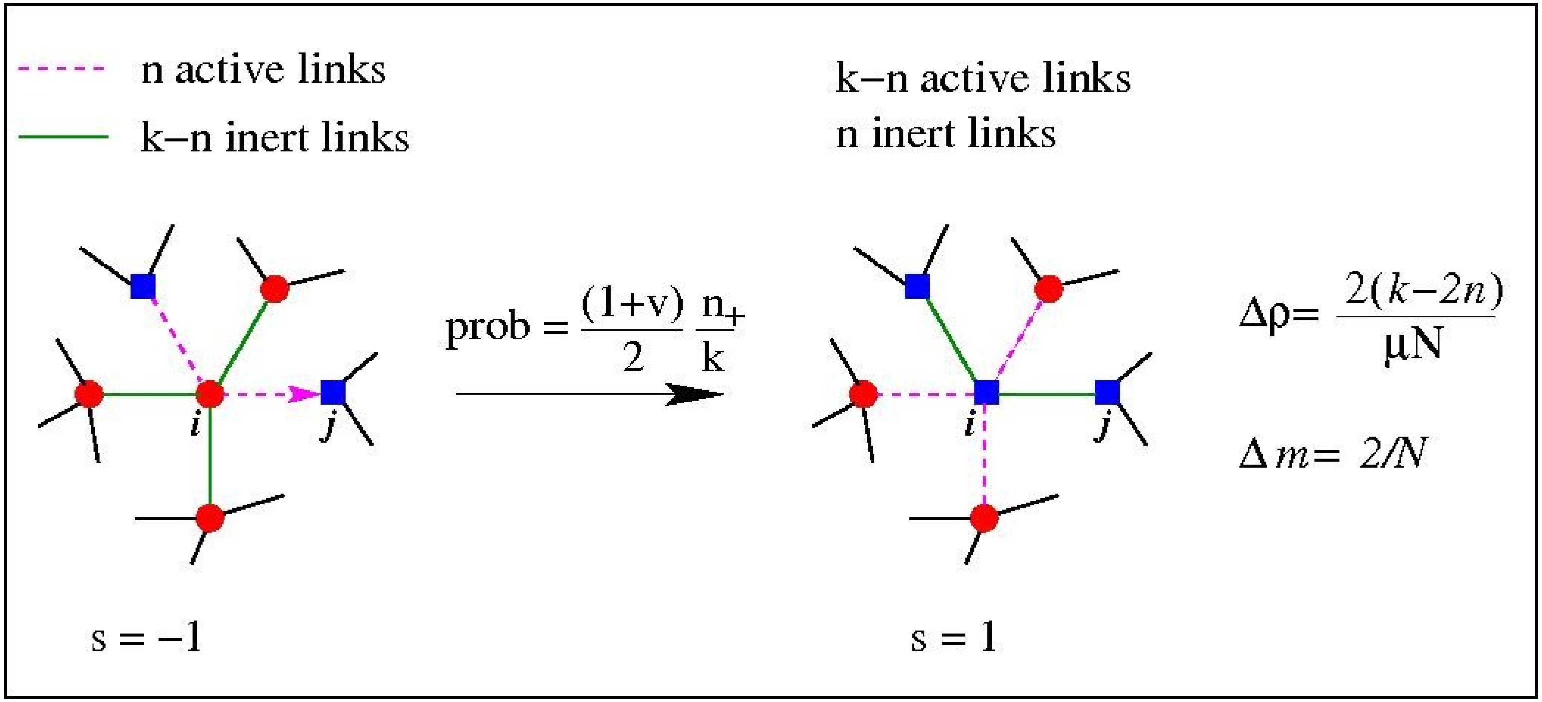
$$\langle \rho(t) \rangle = \frac{1}{2} \langle 1 - m^2(t) \rangle = \frac{1}{2} \int_{-1}^{1} dm \; (1 - m^2) \; P(m,t)$$

$$\langle \rho(t) \rangle = \frac{1}{2} e^{-2\,t/N} \quad \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

We recover the right behavior!

Complex networks

- Each node connected to μ neighbors chosen at random.



$$P(+\to -) = \frac{(1-v)}{2}\frac{n_-}{k}, \qquad P(-\to +) = \frac{(1+v)}{2}\frac{n_+}{k}$$

Equations for m and ρ:

$$\frac{dm(t)}{dt} = \sum_k \frac{\sigma_- P_k}{1/N} \sum_{n_+=0}^{k} B(n_+,k)\frac{(1+v)}{2}\frac{n_+}{k}\frac{2}{N} - \sum_k \frac{\sigma_+ P_k}{1/N} \sum_{n_-=0}^{k} B(n_-,k)\frac{(1-v)}{2}\frac{n_-}{k}\frac{2}{N}$$

$$\frac{d\rho(t)}{dt} = \sum_k \frac{\sigma_- P_k}{1/N} \sum_{n_+=0}^{k} B(n_+,k)\frac{(1+v)}{2}\frac{n_+}{k}\frac{2(k-2n_+)}{\mu N} + \sum_k \frac{\sigma_+ P_k}{1/N} \sum_{n_-=0}^{k} B(n_-,k)\frac{(1-v)}{2}\frac{n_-}{k}\frac{2(k-2n_-)}{\mu N}$$

Pair approximation: neglect 2nd nearest-neighbor correlations.

Coupled equations for m and ρ:

$$\frac{dm(t)}{dt} = v\rho$$

$$\frac{d\rho(t)}{dt} = \frac{\rho}{\mu}\left\{(\mu-2) - \frac{2(\mu-1)(1+v\,m)\rho}{(1-m^2)}\right\}$$

Stationary solution: $\rho(t) = \frac{\xi\left[1-m(t)^2\right]}{\left[1+v\,m(t)\right]}$ $\xi \equiv (\mu-2)/2(\mu-1)$

$$\frac{dm}{dt} = \frac{v\xi(1-m^2)}{(1+v\,m)} \quad \Rightarrow \quad m(t) = \tanh(v\xi t)$$

For v << 1

v =0 $\Rightarrow$ $\rho(t) = \xi\left[1-m(t)^2\right]$ *Like in complete graph!!*

Symmetric case v=0:

Fokker-Planck equation:

$$\frac{\partial P(m,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial m^2}\left[\frac{1}{\tau}(1-m^2)P(m,t)\right]$$

Langevin equation:

$$\frac{dm}{dt} = \sqrt{\frac{(1-m^2)}{\tau}}\,\eta(t)$$

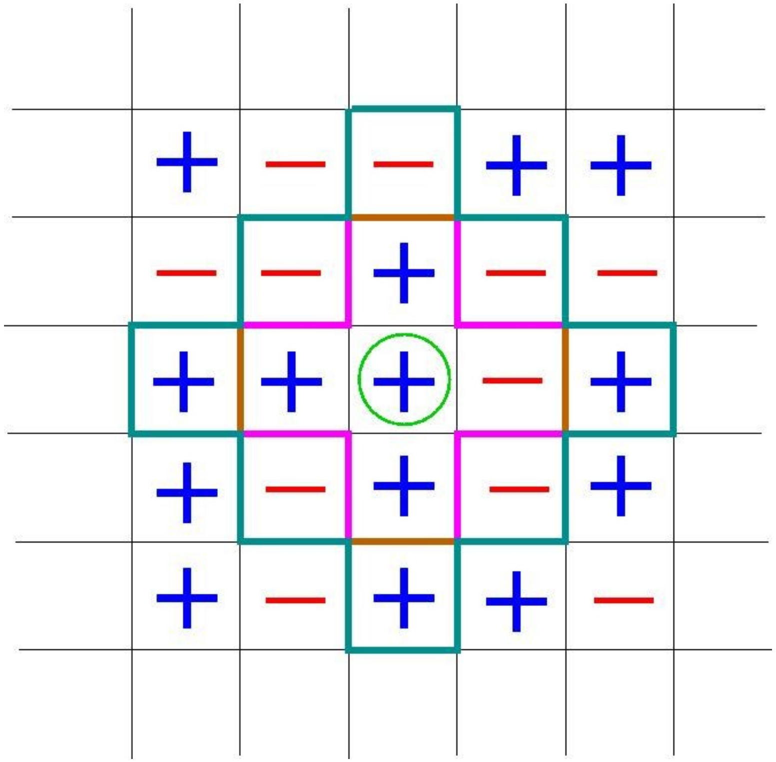
Time decay constant depends on 1st and 2nd moments.

Topology affects relaxation to final state.

$$\langle\rho(t')\rangle = \xi\langle 1-m^2(t)\rangle = \xi\, e^{-2t/\tau} \qquad \tau \equiv \frac{(\mu-1)\mu^2 N}{(\mu-2)\,\mu_2}$$

Sparse networks (μ small) take longer to reach consensus state.

Square lattices



$\Phi_{\boldsymbol{r}}$ = *"opinion field" at site **r**.*

$-1 \leq \Phi_{\boldsymbol{r}} \leq 1$

Continuous field over space/time

σ_A = density of leftists in the near neighborhood.

Prob(+ → -) = ½ (1-v) 1/4 in □, ½ (1-v) 5/8 in □, ½ (1-v) ½ in □

How does the field evolve with time ?

$$P(\mp \to \pm) = \frac{1}{2}(1 \pm v)\left(\frac{1 \pm \psi_{\mathbf{r}}}{2}\right)$$

transition probability

$$\psi_{\mathbf{r}} = \sigma_+ - \sigma_- = 2\sigma_+ - 1$$

$$\psi_{\mathbf{r}} = \phi_{\mathbf{r}} + \Delta\phi_{\mathbf{r}}$$

ψ_r = neighboring field

Equation for the evolution of Φ_r:

$$\frac{\partial \phi_{\mathbf{r}}(t)}{\partial t} = \left[1 - \phi_{\mathbf{r}}(t)\right] P(- \to +) - \left[1 + \phi_{\mathbf{r}}(t)\right] P(+ \to -) + \eta_{\mathbf{r}}(t)$$

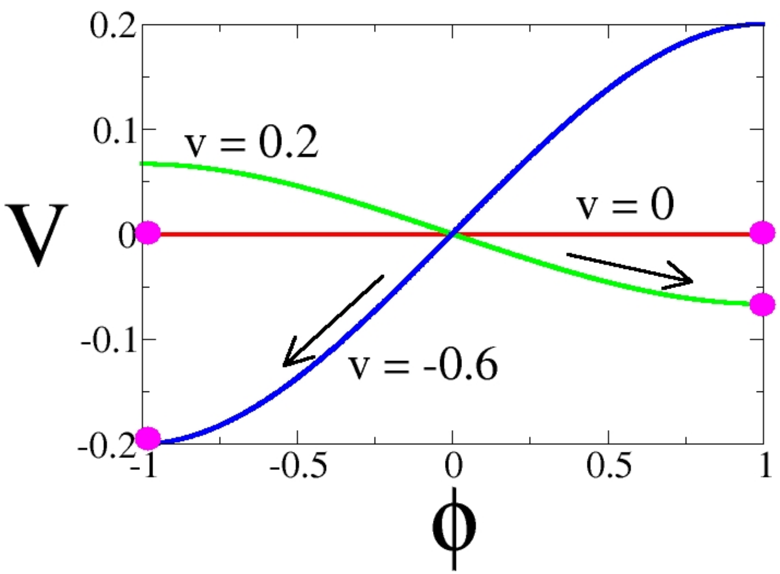
Langevin equation for the opinion field:

$$\frac{\partial \phi_{\mathbf{r}}(t)}{\partial t} = \frac{v}{2}\left[1-\phi_{\mathbf{r}}^2(t)\right] + \frac{1}{2}\left[1 - v\phi_{\mathbf{r}}(t)\right]\Delta\phi_{\mathbf{r}}(t) + \sqrt{1-\phi_{\mathbf{r}}^2(t)}\,\eta_{\mathbf{r}}(t)$$

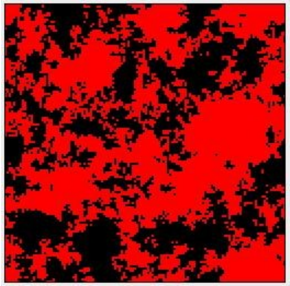
drift diffusion noise

$$\frac{\partial \phi}{\partial t} = D\Delta\phi - \frac{\partial V(\phi)}{\partial \phi}$$

$$V(\phi) = -\frac{v}{2}\left(\phi - \frac{\phi^3}{3}\right)$$



Typical ordering



Summary

- The dynamics of agent based models can be described, at the macroscopic level, by a few collective (global) variables, like density of particles, magnetization, etc.
- Illustration: starting from the microscopic dynamics, we derived equations for the macroscopic evolution of a simple opinion model, in different topologies.
- The type of analytical approach depends on the type of topology.
- Same techniques can be applied to more complicated models (Language dynamics, savanna problem)… see next talks.

$$\frac{dm(t)}{dt} = \sum_k \frac{P_k}{2\,k}\left[(1+v)(1-m)\langle n_+\rangle_k - (1-v)(1+m)\langle n_-\rangle_k\right]$$

$$\frac{d\rho(t)}{dt} = \sum_k \frac{P_k}{2\,\mu\,k}\Big\{(1+v)(1-m)\left[k\langle n_+\rangle_k - 2\langle n_+^2\rangle_k\right] + (1-v)(1+m)\left[k\langle n_-\rangle_k - 2\langle n_-^2\rangle_k\right]\Big\}$$

1st and 2nd moments of Binomial distribution:

$$\langle n_s\rangle_k \equiv \sum_{n_s=0}^{k} B(n_s,k)n_s = P(s|-s)k \simeq \frac{\rho k}{2\sigma_{-s}}$$

$$\langle n_s^2\rangle_k \equiv \sum_{n_s=0}^{k} B(n_s,k)n_s^2 \simeq \frac{\rho k}{2\sigma_{-s}} + \frac{\rho^2 k(k-1)}{4\sigma_{-s}^2}$$

Pair approximation: neglect 2nd nearest-neighbor correlations.

For the symmetric case v=0:

$$\rho(t) = \xi\left[1 - m(t)^2\right]$$

Like in complete graph!!

Uncorrelated networks are mean-field for voter model dynamics

Symmetric RW with steps of length $\delta_k = 2k/\mu N$:

$$\begin{aligned} W(m \to m - \delta_k) &= \frac{\xi}{4}\left(1 - m^2\right) P_k \\ W(m \to m + \delta_k) &= \frac{\xi}{4}\left(1 - m^2\right) P_k \\ W(m \to m) &= \left[1 - \xi\left(1 - m^2\right)\right] P_k \end{aligned}$$

$$\begin{aligned} P(m, t + 1/N) &= \sum_k P_k \Big\{ W(m + \delta_k \to m)\, P\left(m + \delta_k, t\right) \\ &+ W(m - \delta_k \to m)\, P\left(m - \delta_k, t\right) + W(m \to m)\, P(m, t) \Big\} \end{aligned}$$