Reliability of Lagrangian diagnosis from Finite Size Lyapunov Exponents (FSLE)

Ismael Hernández-Carrasco
Emilio Hernández-García
Cristóbal López
IFISC (CSIC-UIB), Palma de Mallorca, Spain

Antonio Turiel
CMIMA, ICM (CSIC) , Barcelona, Spain

Outline

- Introduction.

- Scale invariance properties of FSLE.
  - FSLE at different spatial scales.
  - FSLE at different spatial resolution of the velocity field.

- Robustness of FSLE.
  - Error in the velocity data.
  - Noise in the particle's trajectories.

- Conclusions.

Introduction

Lyapunov Exponents

Standard Lyapunov Exponents

Total predictability

$$\lambda = \lim_{t\to\infty} \lim_{\delta \mathbf{x}(t_0)\to 0} \frac{1}{t} ln \frac{\delta \mathbf{x}(t)}{\delta \mathbf{x}(t_0)}$$

asymptotic character

Non-asymptotic Lyapunov Exponents

$$\sigma_{t_0}^{T}(\mathbf{x}) = \frac{1}{|T|} \ln \sqrt{\frac{\max_{\delta \mathbf{x}(t_0)} \|\delta \mathbf{x}(\mathbf{t_0}+\mathbf{T})\|}{\|\overline{\delta \mathbf{x}}(t_0)\|}}$$

Finite-time Lyapunov exponents

Local predictability (real data)

Finite-size Lyapunov exponents

$$\Lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \log \frac{\delta_f}{\delta_0}$$

Aurell et al,1997

Lyapunov Exponents

Standard Lyapunov Exponents

Total predictability

$$\lambda = \lim_{t\to\infty} \lim_{\delta\mathbf{x}(t_0)\to 0} \frac{1}{t} ln \frac{\delta\mathbf{x}(t)}{\delta\mathbf{x}(t_0)}$$

Because its asymptotic character, it is limited for practical analyses

Non-asymptotic Lyapunov Exponents

$$\sigma_{t_0}^{T}(\mathbf{x}) = \frac{1}{|T|} \ln \sqrt{\frac{\max_{\delta\mathbf{x}(t_0)} \|\delta\mathbf{x}(\mathbf{t_0}+\mathbf{T})\|}{\|\overline{\delta\mathbf{x}}(t_0)\|}}$$

Finite-time Lyapunov exponents

Local predictability (experimental data)

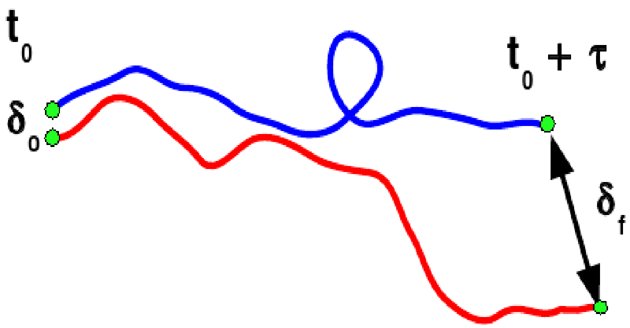
Finite-size Lyapunov exponents

$$\Lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \log \frac{\delta_f}{\delta_0}$$

Aurell et al,1997

Finite Size Lyapunov Exponents (FSLEs)

$$\Lambda(\mathbf{x}, t_0, \delta_0, \delta_f) = \frac{1}{|\tau|} \log \frac{\delta_f}{\delta_0}$$



δ_0 is the initial separation.

δ_f is the final separation.

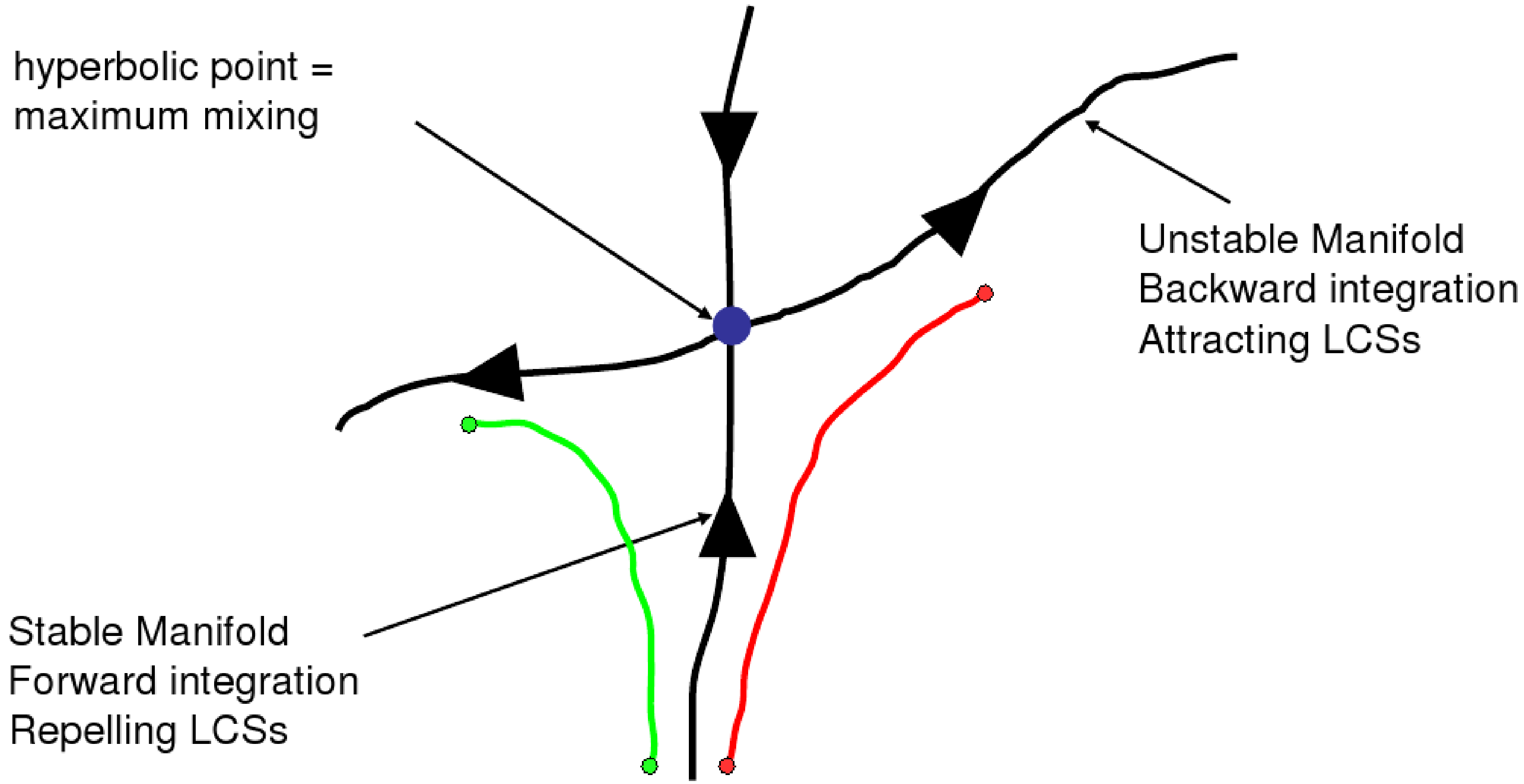
τ is the time needed for two particles initially separated δ_0 , to get separate δ_f

$\mathbf{x}$ are the coordinates

t_0 initial time for the integration

Note: $\boldsymbol{\delta_0}$ ***is the spatial resolution of the FSLE field***

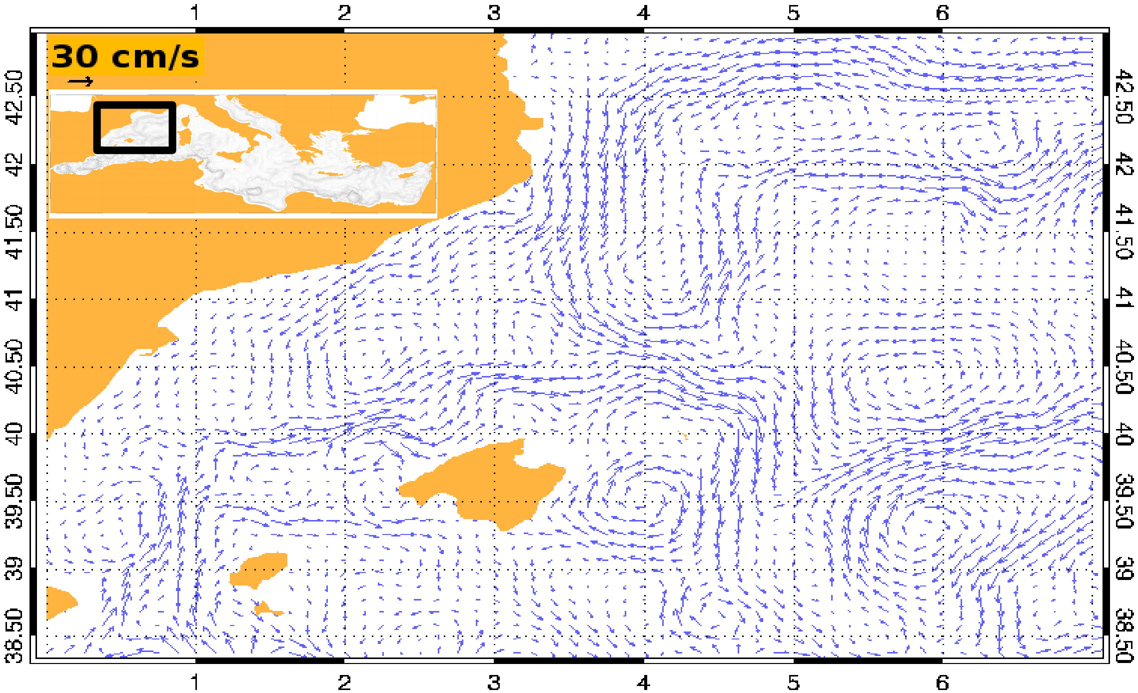
Lagrangian coherent structures (LCS) as Ridges in the FSLE field



Manifolds divide regions of qualitatively different dinamics

Lagrangian diagnosis from marine currents of Balearic Sea

Velocity data from DieCAST model adapted to Mediterranean Sea.



DieCAST : primitive-equation, z-level, finite difference ocean model,
hidrostatics aproximation
incompresible aproximation
rigid lid aproximation

Resolution:

longitudinal resolution $\Delta\phi = \Delta_0 = 1/8^0$
latitudinal resolution $\Delta\lambda = \Delta\phi \cos\lambda$

Equations of motion

$$\frac{d\phi}{dt} = \frac{u(\phi,\lambda,t)}{R\cos\lambda}$$

$$\frac{d\lambda}{dt} = \frac{v(\phi,\lambda,t)}{R}$$

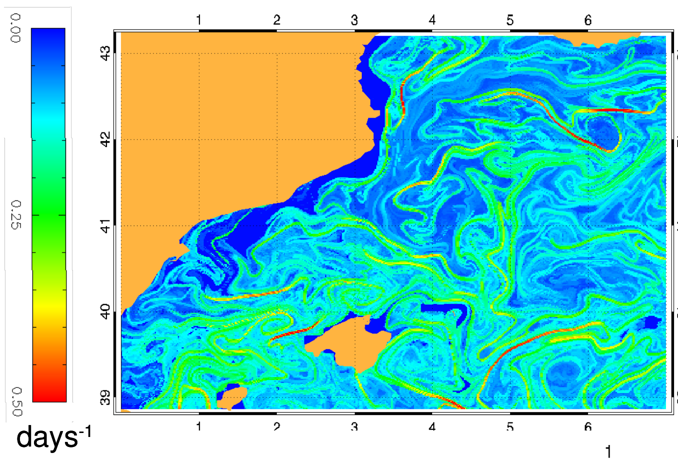
ϕ Latitude coordinate
λ longitude coordinate
R = earth radius
u,v velocity components

(V. Fernández et al, 2005)

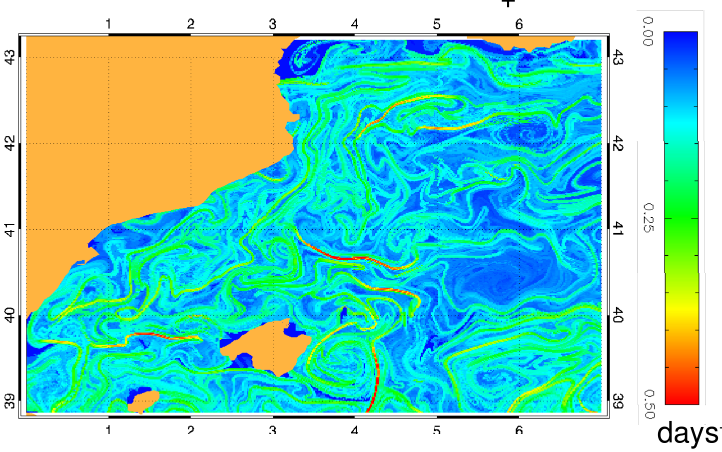
Note: Δ_0 ***is the spatial resolution of the velocity field***

Lagrangian diagnosis from marine currents of Balearic Sea

FSLE backward Λ_-

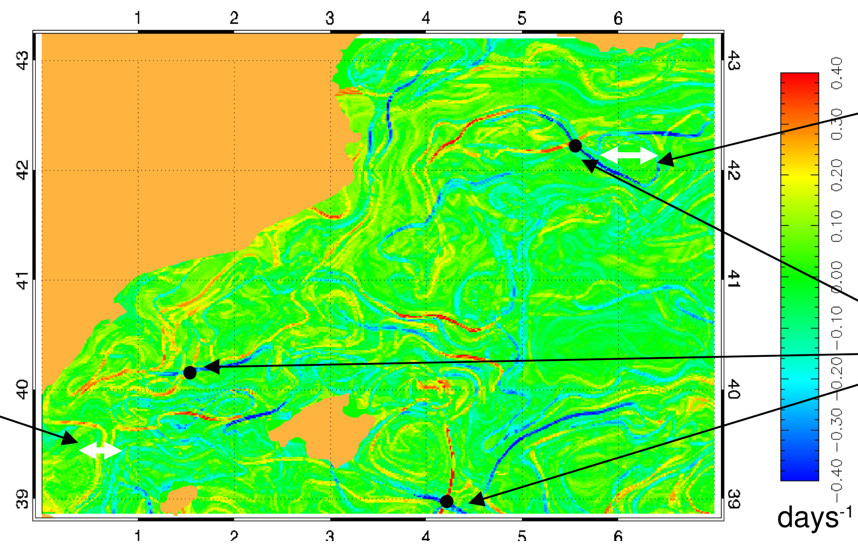


FSLE forward Λ_+



$\Delta_0 = 1/8^0$
$\delta_0 = 1/64^0$
$\delta_f = 1^0$
(mesoscale size)

$\Lambda_+ - \Lambda_-$



Mesoscale eddy

Submesoscale filament

Hyperbolics points. Maximum mixing

→ **Revealing the dynamical structures in the flow which strongly organises fluid motion (eddies edges, filaments, fronts = transport barriers for passive tracers).**

→ **Give additional information of oceanographic interest: characteristics time scales and mixing intensity (localize area with max dispersion rates).**

→ **Revealing oceanic structures under the resolution of the velocity field: *SUBMESOSCALE***

→ **Natural framework to study the interaction: hydrodynamics/tracers.**

→ Revealing the dynamical structures in the flow which strongly organises fluid motion (eddies edges, filaments, fronts = transport barriers for passive tracers).

→ Give additional information of oceanographic interest: characteristics time scales and mixing intensity (localize area with max dispersion rates).

→ **Revealing oceanic structures under the resolution of the velocity field: *SUBMESOSCALE ?   Or introduce a artificiality in the results?***

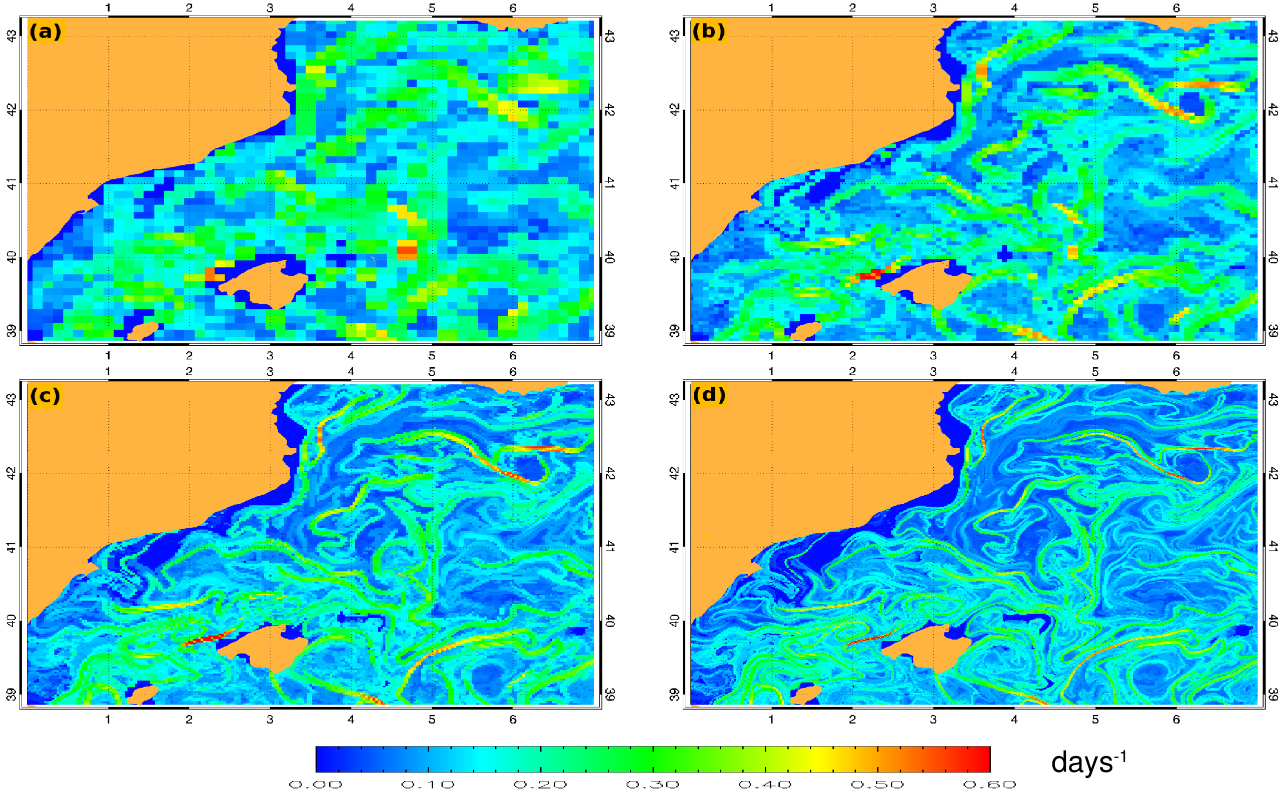
→ Natural framework to study the interaction: hydrodynamics/tracers.

What happen when the FSLE field is computed under the resolution of the velocity data?

Is some artificiality introduced?

Scale invariance properties of FSLE

FSLE at different spatial scales (spatial resolutions, δ_0)



a) $\delta_0 = 1/8^0$ $\Delta_0 = 1/8^0$

b) $\delta_0 = 1/16^0$ $\Delta_0 = 1/8^0$

c) $\delta_0 = 1/32^0$ $\Delta_0 = 1/8^0$

d) $\delta_0 = 1/64^0$ $\Delta_0 = 1/8^0$

Note:
δ_0 = spatial scale of FSLE.
Δ_0 = velocity resolution

Increasing the spatial resolution we improve the identification of the mesoscale and submesoscale structures

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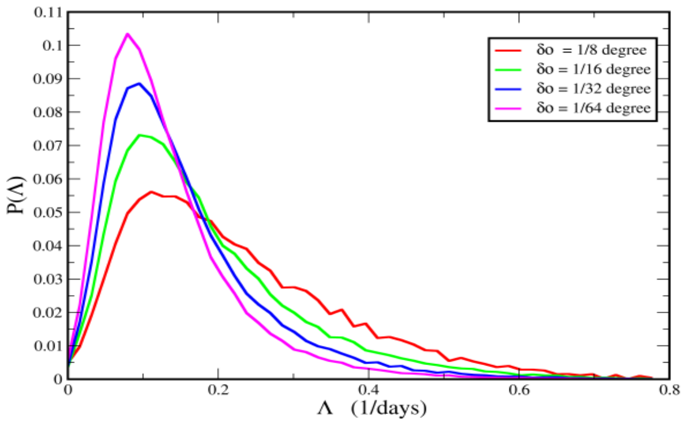
Histograms of FSLEs at differents scales



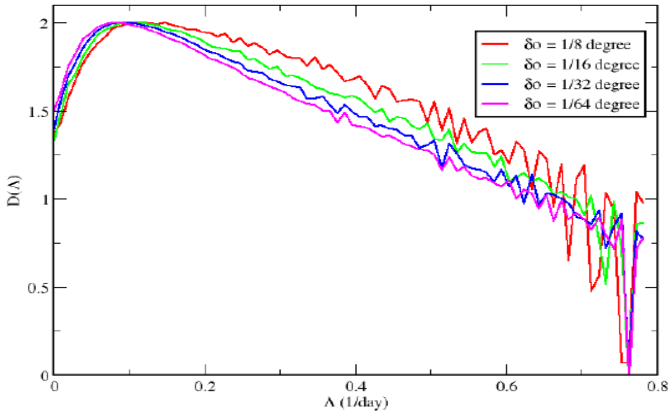
The scale behaviour of these histograms indicates that the distribution of the FSLEs at a scale δ_0 is given by:

$$P(\delta_0, \Lambda) = P(\delta_0, \Lambda_c)\, \delta_0^{d-D(\Lambda)} \qquad (1)$$

$P(\delta_0, \Lambda_c)$ is the maximun value of the probality distribution

d in this case is the surface dimension = 2
$D(\Lambda)$ is the fractal dimension of the set of initial conditions leading to FSLE
δ_0 is the definition scale

Fractal dimension of FSLEs at differents scales



From Eq (1) one obtains a properly normalized expression to computed the fractal dimension at different scales

$$D(\Lambda) = d - \frac{\log \frac{P(\delta_0,\Lambda)}{P(\delta_0,\Lambda_c)}}{\log \delta_0} \qquad (2)$$

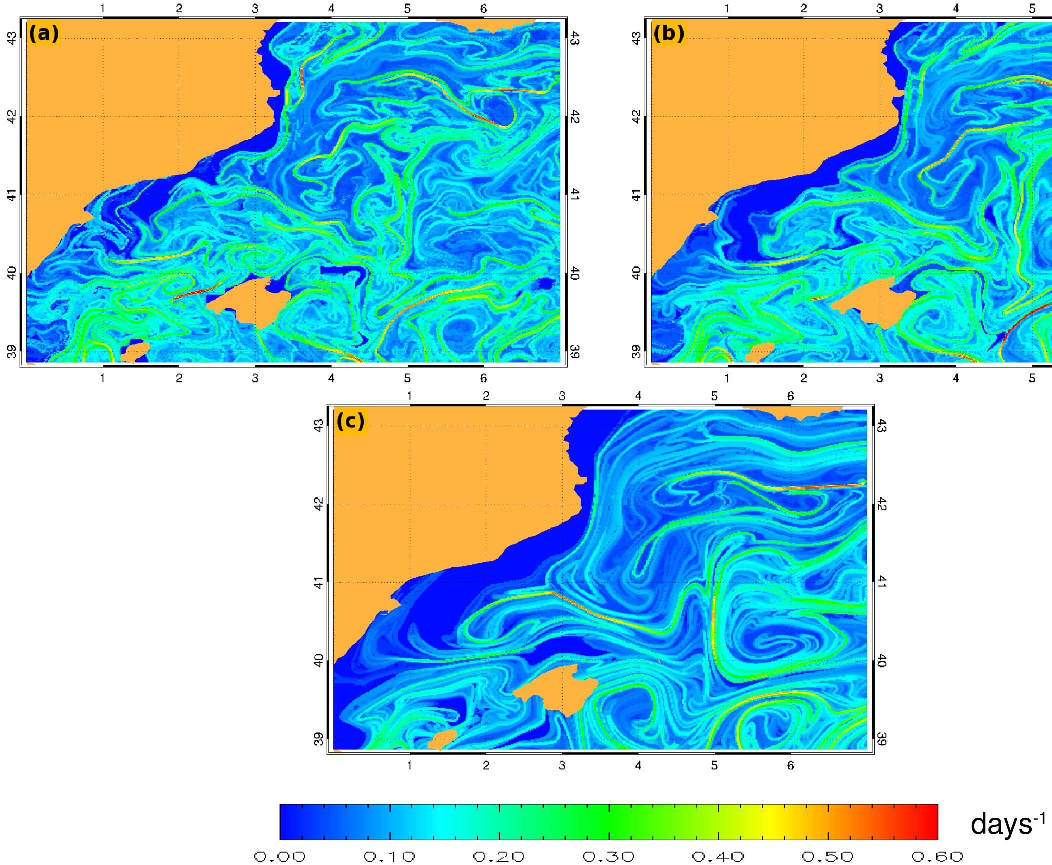
The plot of $D(\Lambda)$ shows a collapse of $D(\Lambda)$ at the different scales.

FSLE display a multifractal character.

What happens if the spatial resolution of the velocity data is decreased?

Can we recover the structures?

FSLE at different spatial resolution of the velocity data



a) $\Delta_0 = 1/8^0$ $\delta_0 = 1/64^0$

b) $\Delta_0 = 1/4^0$ $\delta_0 = 1/64^0$

c) $\Delta_0 = 1/2^0$ $\delta_0 = 1/64^0$

δ_0 = spatial scale of FSLE

Δ_0 = velocity resolution

We use a Gaussian filter in order to reduce the spatial resolution of the velocity field

The main structures remain even when the velocity field resolution is decreased from 10km to 50 km

Robustness of FSLE

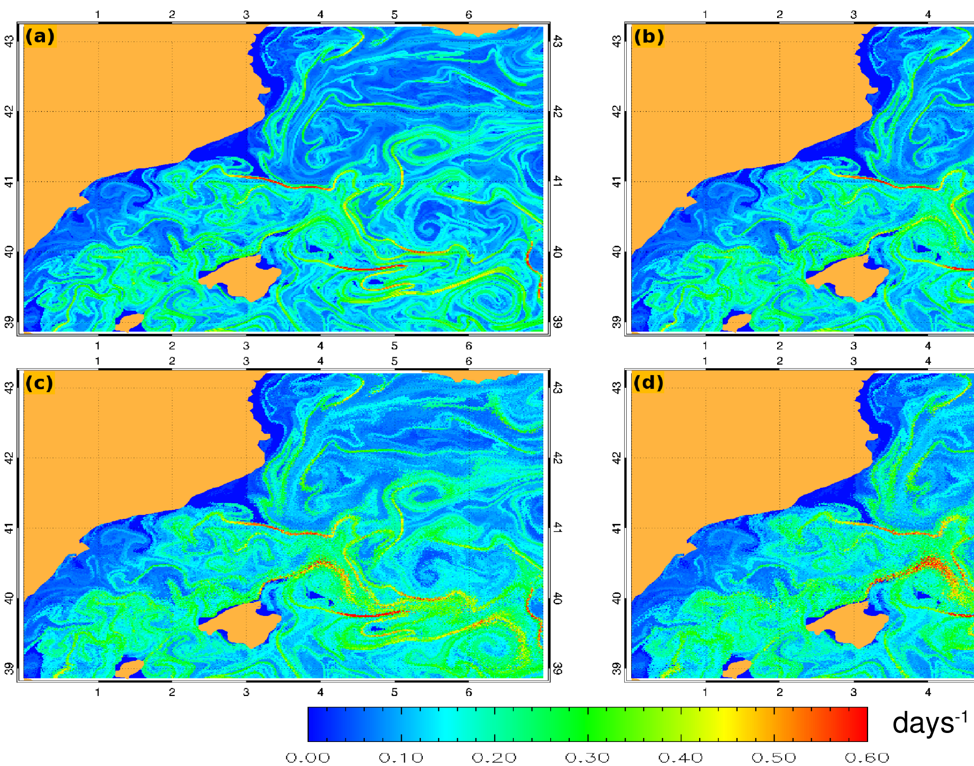
Error in the data

We get a perturbated velocity data (u', v') , by introducing a small random number in the original velocity (u, v).

$$u'(x,t) = u(\mathbf{x},t)(1+\alpha\eta_x(\mathbf{x},t))$$
$$v'(\mathbf{x},t) = v(\mathbf{x},t)(1+\alpha\eta_y(\mathbf{x},t))$$

$\{\eta_x(\mathbf{x},t), \eta_y(\mathbf{x},t)\}$: noise

Sets of Gaussians random number with mean zero and variance one



α is the relative size of perturbation

a) $\alpha = 0$
b) $\alpha = 2$
c) $\alpha = 6$
d) $\alpha = 10$

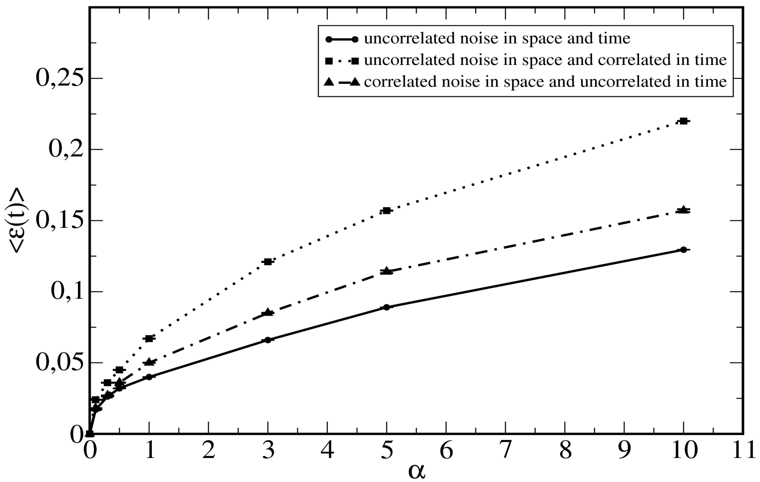
$\Delta_0 = 1/8^0$ $\delta_0 = 1/64^0$

The Lagrangian structures look rather the same despite $\alpha = 10$

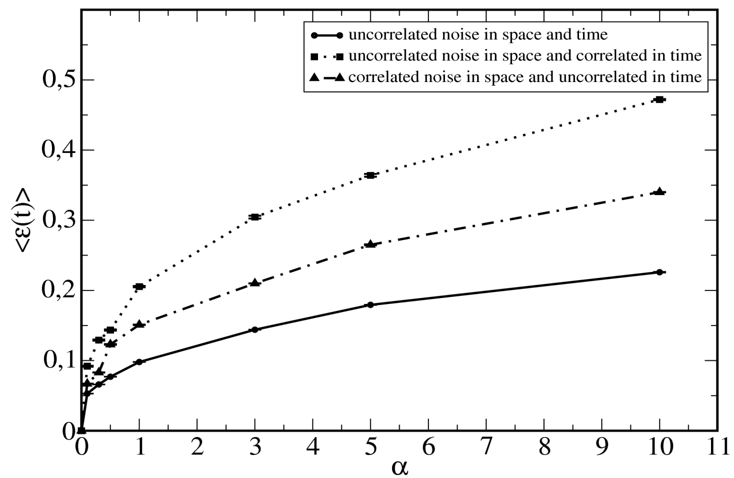
- **Relative error:** $\epsilon(t_i) = \sqrt{\frac{1}{N}\sum_{\mathbf{x}} \frac{|\Lambda^\alpha(\mathbf{x},t_i) - \Lambda(\mathbf{x},t_i)|^2}{|\Lambda(\mathbf{x},t_i)|^2}}, \quad <\epsilon(t)> \equiv \frac{1}{s}\sum_{i=1}^{s}\epsilon(t_i)$

Λ FSLE with data unperturbated
Λ^α FSLE with data perturbated

s = 100 snapshots, and N is the total points in the FSLE field



$\delta_0 = 1/8^0$

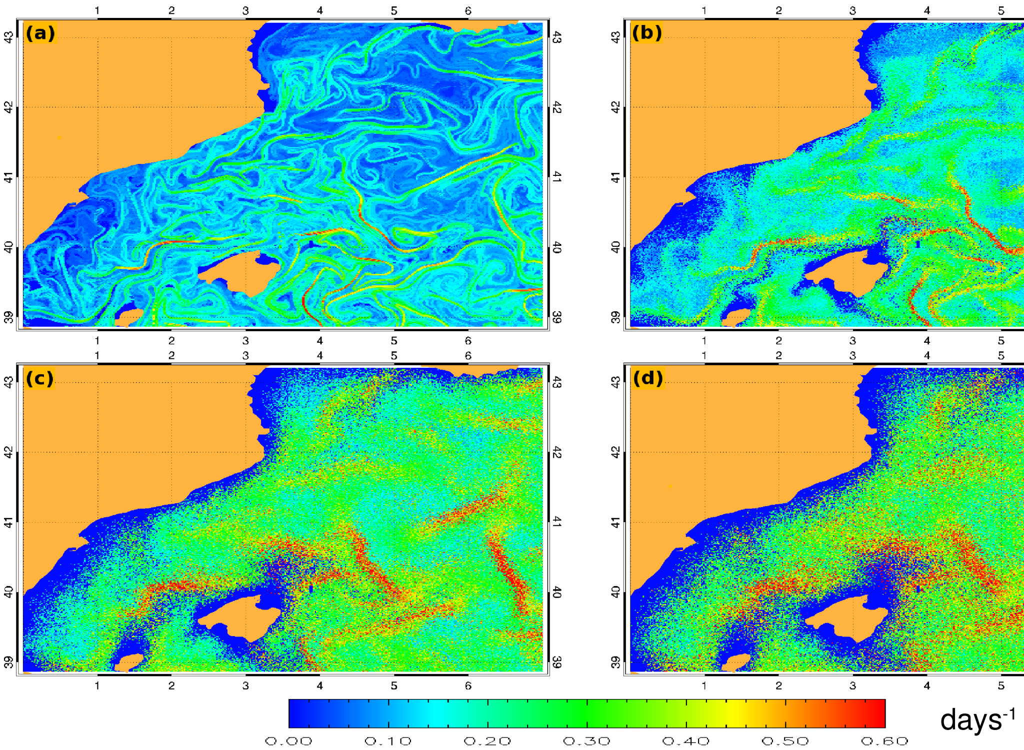


$\delta_0 = 1/64^0$

FSLE are robust to relatively large amount of error in the velocity data.
The average effect produced when computing FSLE by integrating over trajectories, make them robust againts several kinds of uncorrelated noise in the velocity data

Noise in the particle's trajectories

We include unresolved small scales in the computation of FSLE.



$$\frac{d\phi}{dt} = \frac{u(\phi,\lambda,t)}{R\cos(\lambda)} + \frac{\sqrt{2D}}{R\cos(\lambda)}\xi_1(t),$$

$$\frac{d\lambda}{dt} = \frac{v(\phi,\lambda,t)}{R} + \frac{\sqrt{2D}\xi_2(t)}{R}$$

$< \xi_i(t)\xi_j(t') >= \delta_{ij}\delta(t-t')$ Gaussian white noise

Diffusivity: $D(l) = 2.055\ 10^{-4}\ l^{1.15}$

Okubo,1971

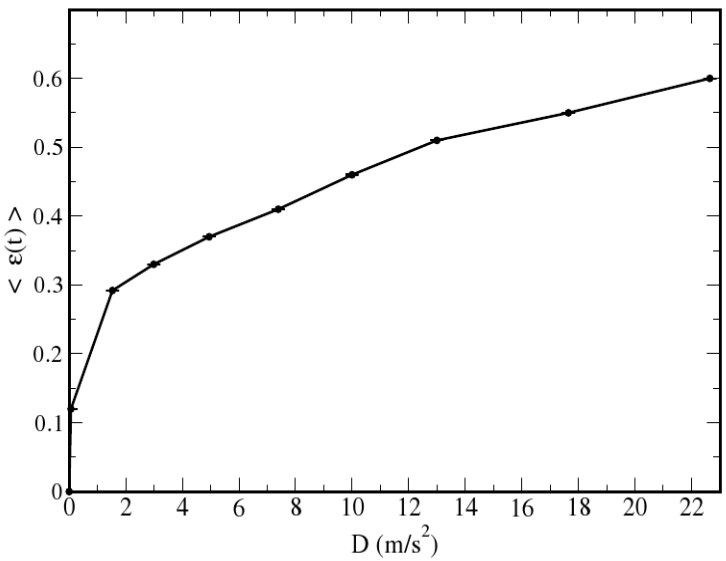
l = lenght scale = spatial resolution

a) D = 0 m^2/s
b) D = 0.9m^2/s
c) D = 10m^2/s
d) D = 17m^2/s

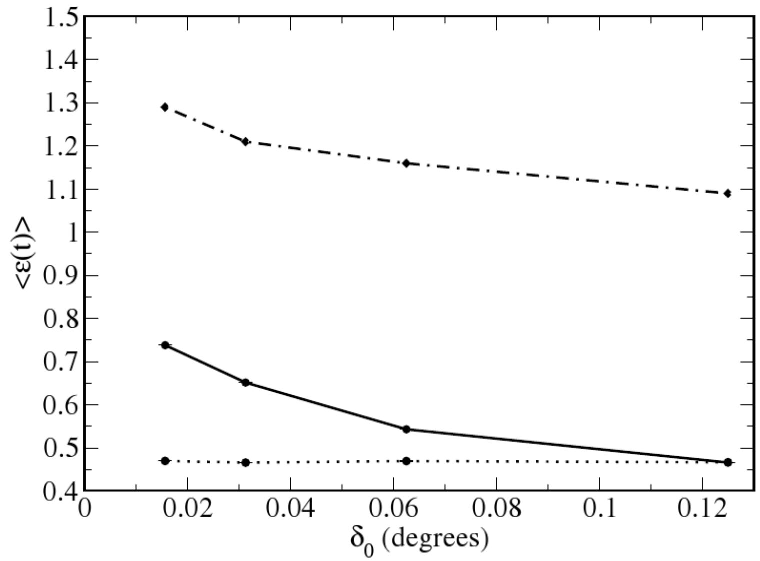
$\Delta_0 = 1/8^0$ $\delta_0 = 1/64^0$

The mesoscale structures are maintained with eddy diffusivity (D)

Relative error



Relative error of the FSLE at different values of D in the particle' trajectories with respect to the D = 0 case.
δ_0 = 1/8°



- **Dotted line**: Relative error of the FSLE at different spatial resolution (δ_0) and at one assigned eddy-difusion: $D=2.055\ 10^{-4}\delta_0^{1.55}$
- **Solid line**: at different spatial resolution, and at the same eddy-diffusion $D_o = 10\ m_2$/s.
- **Dashed-dotted line** is the relative error of the shuffled FSLE at different spatial resolutions (δ_0)

DieCAST: $\delta_0 = \Delta_0$ = 1/8° D = 10 m²/s

δ_0 = 1/16° D = 4.5 m²/s

δ_0 = 1/32° D = 2 m²/s

δ_0 = 1/64° D = 0.9 m²/s

Conclusions

- Increasing the spatial resolution of FSLEs we improve the identification of surface mesoscale structures.

- The main surface mesoscales structures in the ocean remain when the spatial resolution of the velocity data decreases.

- The spatial distribution of FSLE displays a multifractal character: different values of Λ occur at sets of points having distinct fractal dimensions $D(\Lambda)$.

- The FSLE are rather robust. The relative error, even for a perturbation of 10 times the velocity data, is smaller than 20 %.

- Mesoscale structures are maintained when the eddy diffusion is included

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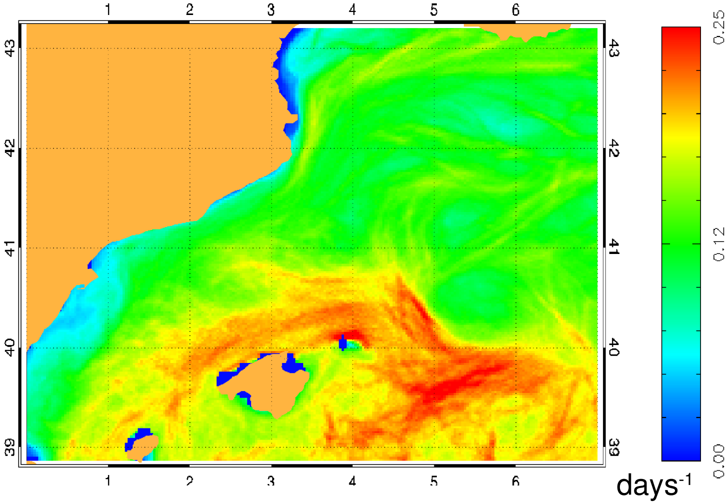
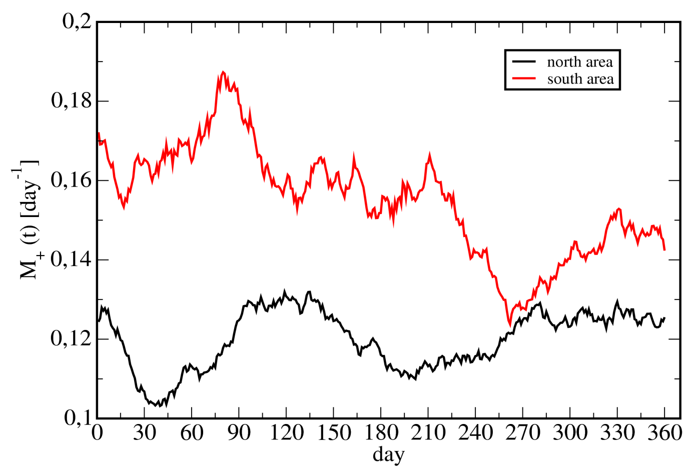
Thank you...

Lagrangian diagnosis from marine currents off Balearic Sea

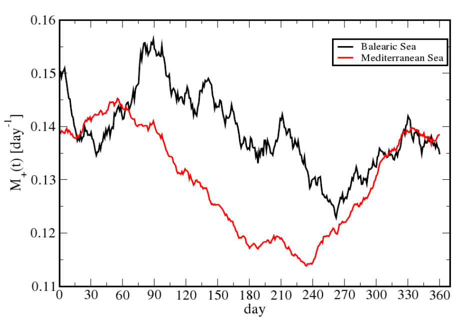
Time average (for the second simulation year) of the FSLEs in the Balearic Sea



Temporal evolution of the mixing measure $M_+(t) = < \Lambda_+ >$



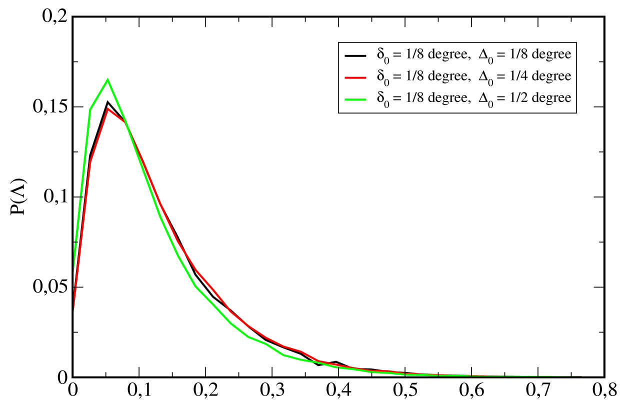
North and South regions are Geographical regions of different mixing activity

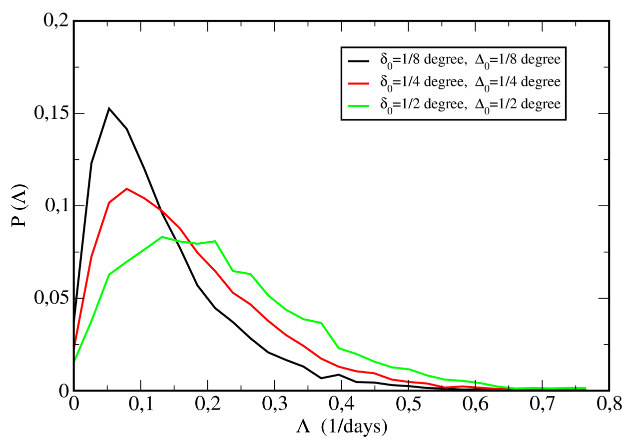


$M_+(t)$ during one year for the Balearic Sea and Mediterranean Sea, show that the Baelaric Sea is a area with low mixing activity for the spring and summer month

(D'Ovidio et al, 2004. (Mediterranean Sea))

Histograms of FSLE at different resolution of the velocity data





Fractal dimension of FSLE at different resolution of the velocity data