The Voter Model in Complex Networks

KRZYSZTOF SUCHECKI

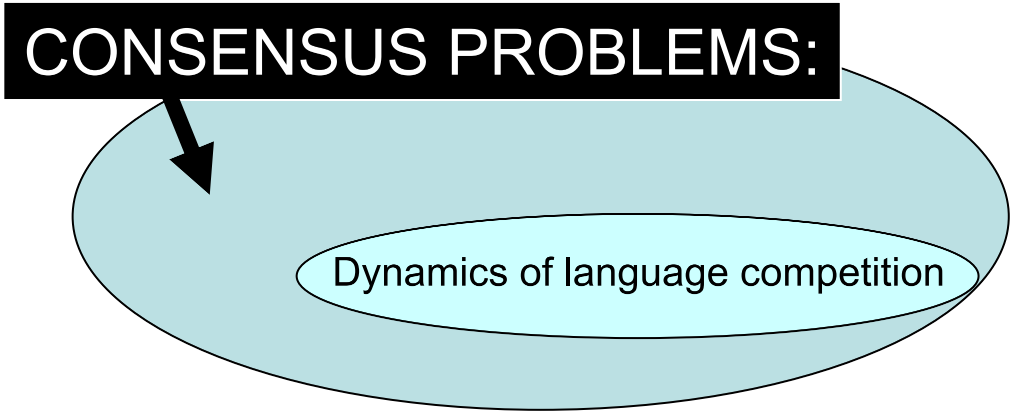
FEDERICO VAZQUEZ

VICTOR M. EGUILUZ

MAXI SAN MIGUEL

IFISC

Determine when and how the dynamics of a set of interacting agents that can choose among several **options** (*political vote, opinion, cultural features*,…) leads to a **consensus** in one of these opinions, or when a state with several **coexisting** options prevails.



INTERACTIONS: Mechanisms ("rule") and Network (with whom)

MODELS

- -VOTER MODEL
- -SPIN FLIP KINETIC ISING MODEL (T=0)
- -AXELROD MODEL
- -GRANOVETTER'S MODEL

↔

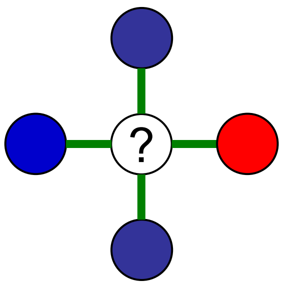
MECHANISMS

- -Imitation
- -Following majority. Social pressure
- -Homophily
- -Threshold for social pressure

- Prototype models ***with two excluding options***:

  - VOTER MODEL

  - SPIN FLIP KINETIC ISING MODEL T=0

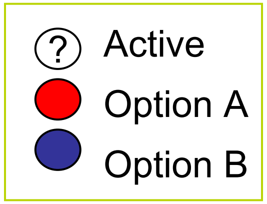


$$p_{?\to B} = 3/4 \qquad p_{?\to A} = 1/4$$

Voter Model — **RANDOM IMITATION**

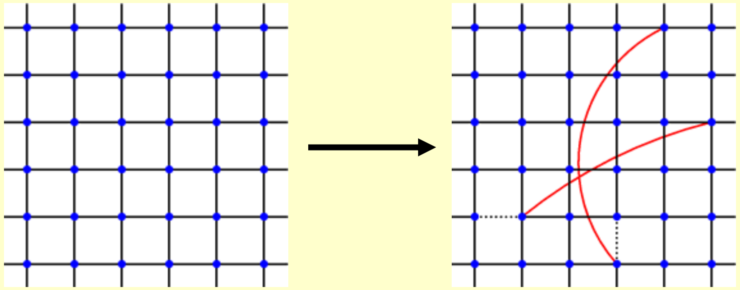
$$p_{?\to B} = 1 \qquad p_{?\to A} = 0$$

Spin Flip Kinetic Ising T=0 — **SOCIAL PRESSURE**

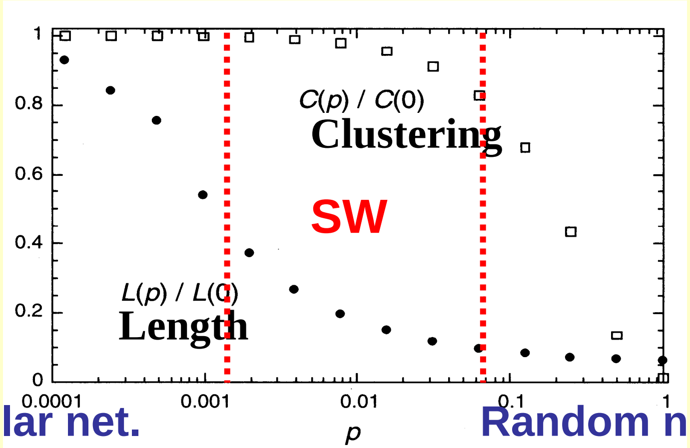


Small World Networks

Watts-Strogatz, Nature ***393****, 440 (1998)*

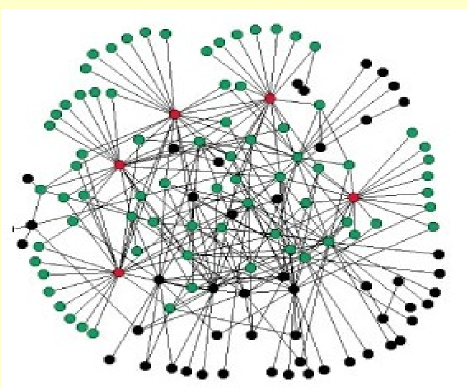


Rewire with prob. *p*



Regular net. **Random net.**

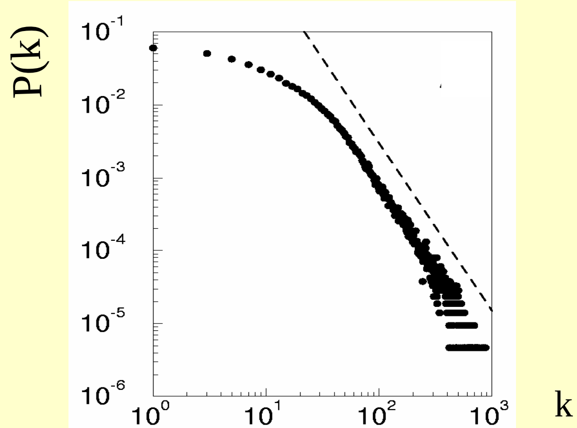
Barabasi-Albert Scale Free networks



Power law for the degree distribution

$$P(k) \sim k^{-\gamma}, \ \gamma=3$$

Importance of hubs



*Albert & Barabasi, Rev. Mod. Phys.****74****, 47 (2002)*

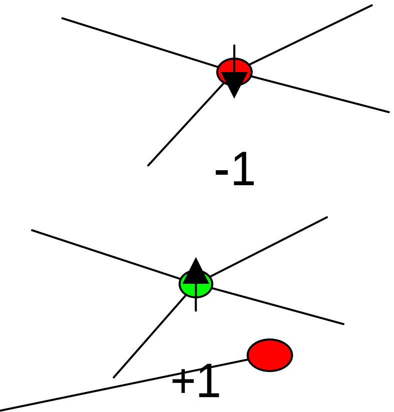
Complex Networks Characteristics

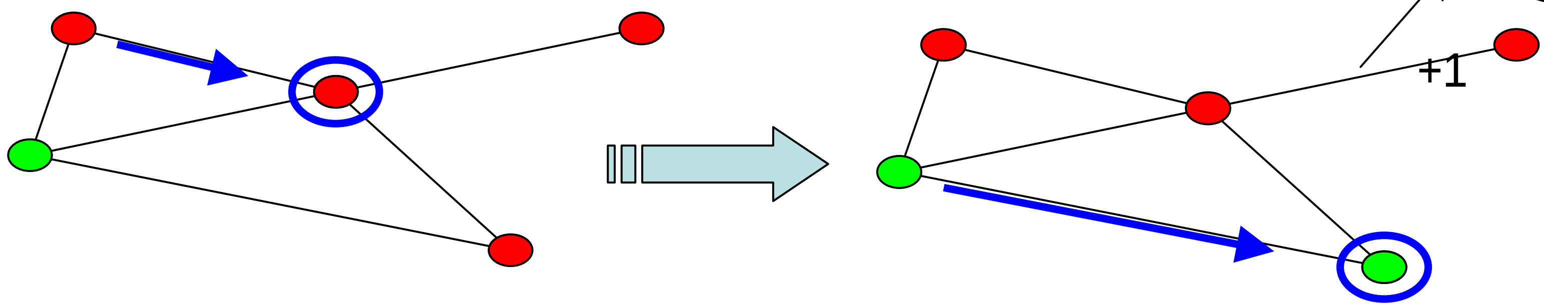
	L Path Length	**C** Clustering	**P(k)** Degree distr.
Regular network	$N^{1/d}$	N^0	$\delta(k-z)$
Random ER	$\ln N$	N^{-1}	Poisson
Small-World	$\ln N$	N^0	Exponential
Barabasi-Albert Scale Free	$\ln N/\ln\ln N$	$(\ln N)^2/N$	k^{-3}
Structured Scale Free: SSF	N^1	N^0	k^{-3}

Voter Model

"Voters" located in the nodes of a network have "opinions" $\sigma_i=1$ or $\sigma_i=-1$.

A randomly chosen voter takes the opinion of one of its neighbors (node update).

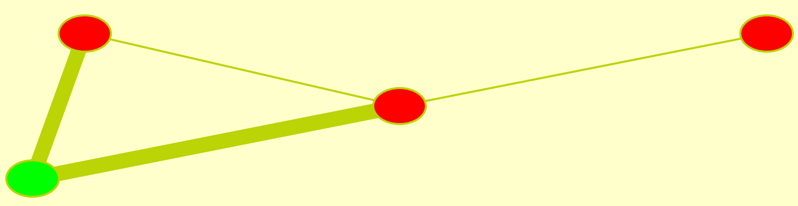




Qs?: When and how one of the two absorbing states (consensus) is reached? Effect of network of interactions?

Poster 13

Order Parameter: Average interface density



$$\rho = \frac{1}{2N\langle k\rangle}\left(1-\sum_{i=1}^{N}\sum_{j\in\nu(i)}\sigma_i\sigma_j\right)$$

ρ=0 in absorbing state

Interface: a link connecting nodes with different states.

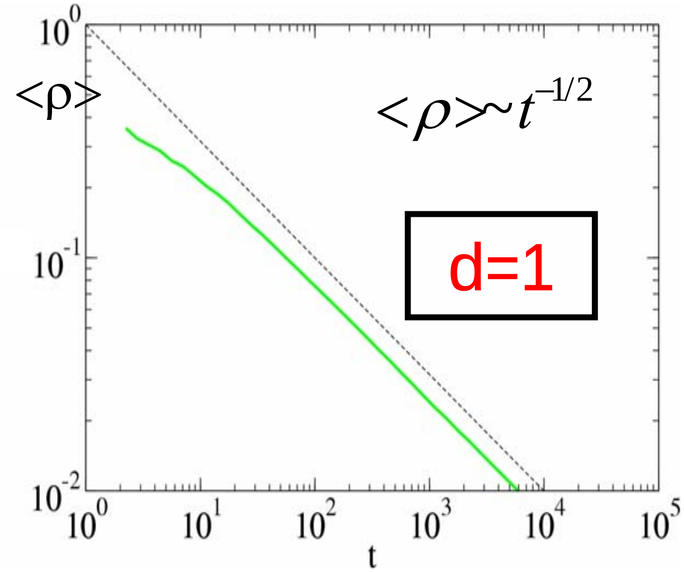
Voter Model in regular networks

$$< \rho > \sim \begin{cases} t^{-1/2}, & d = 1 \\ (\ln t)^{-1}, & d = 2 \\ \xi, & d > 2 \end{cases}$$

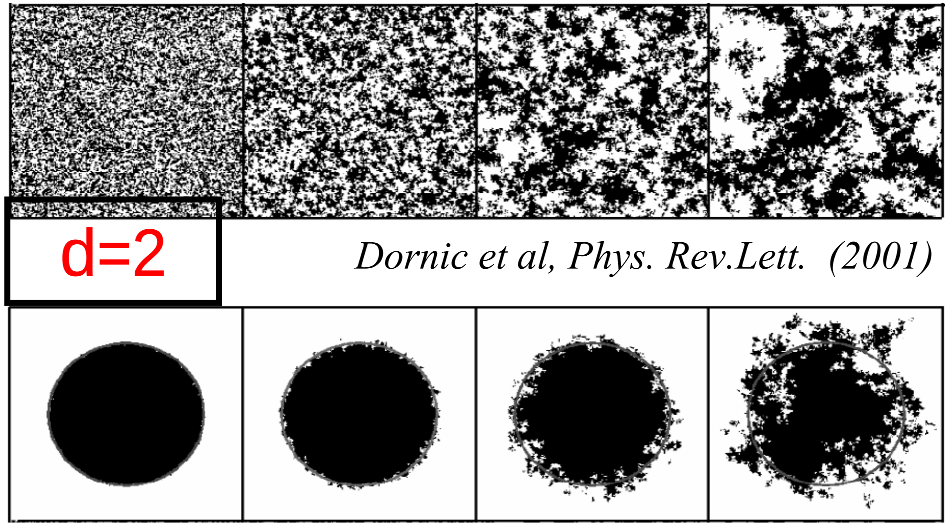
$$\tau \sim \begin{cases} N^2, & d = 1, \text{ time to reach absorbing state} \\ N \ln N, & d = 2, \text{ time to reach absorbing state} \\ N, & d > 2, \text{ survival time of metastable state} \end{cases}$$

d=1,2: Coarsening/Ordering:

Unbounded growth of domains of absorbing states



SF Kinetic Ising T=0



Dornic et al, Phys. Rev.Lett. (2001)

Coarsening without surface tension: Driven by interfacial noise

Voter Model

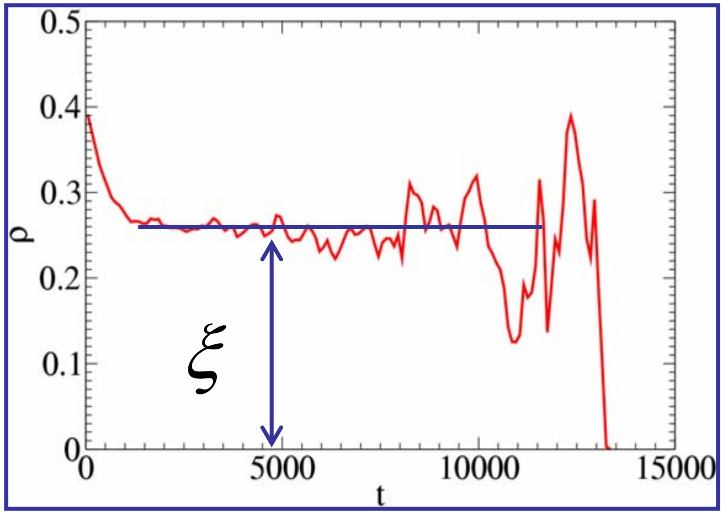
K. Suchecki, V. M. Eguíluz and M. San Miguel Phys. Rev. E **72**, 036132(2005)

d>2 regular and complex networks (Small World, Scale Free):

$< \rho > \sim \xi$ $\tau \sim N$, survival time of metastable state

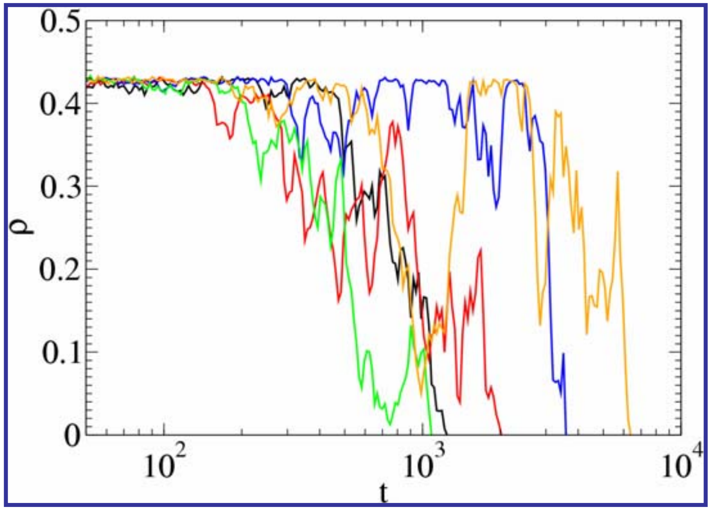
d>2: No Coarsening : Dynamical Metastability

Disordered states.



$l = \xi^{-1}$ *Characteristic size of ordered domain*

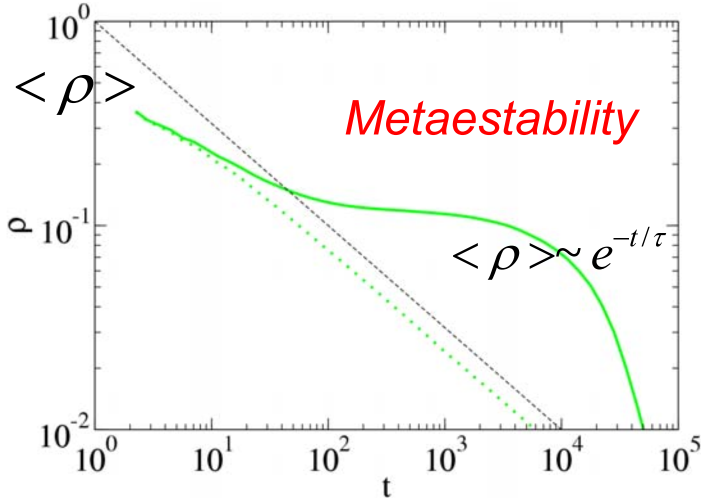
Finite size fluctuations take the system to an absorbing state

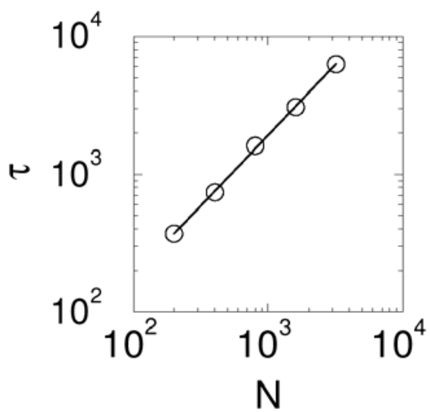


$< \rho > \sim e^{-t/\tau}$ τ *survival time*

Voter Model

Small World Networks



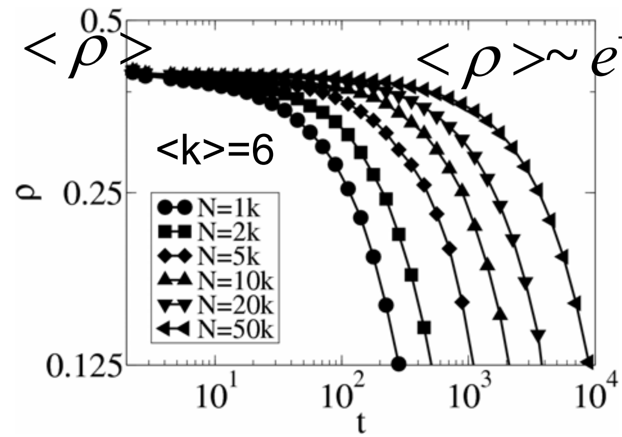


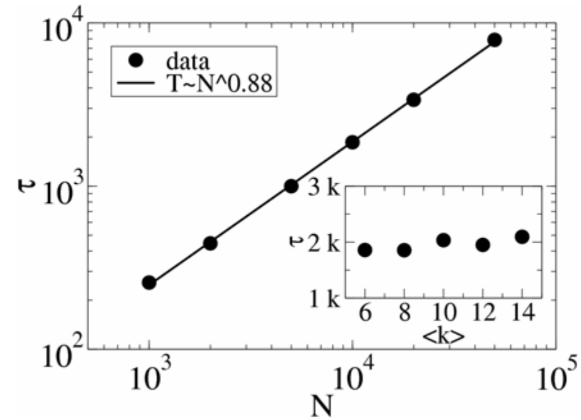
Survival time scales as in regular d>2:

$\tau \sim N$

Castellano et al, Eur. Phys. Lett. 63,153(2003)

Scale Free Barabasi-Albert Nets





$\tau \sim N^{\gamma}$ $\gamma=0.88$

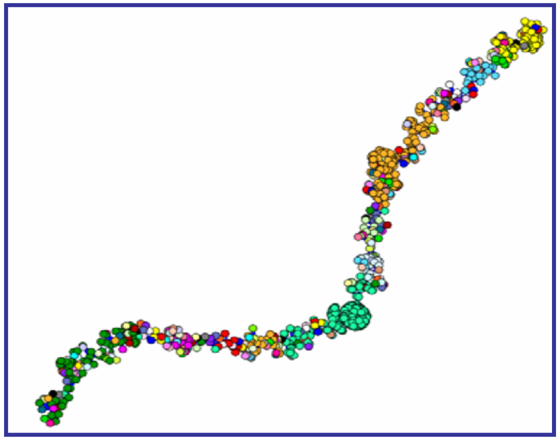
Suchecki et al, Eur. Phys. Lett. ***69****,228(2005)*

Role of dimensionality

1D Scale free net?

Structured SF: ***SSF***

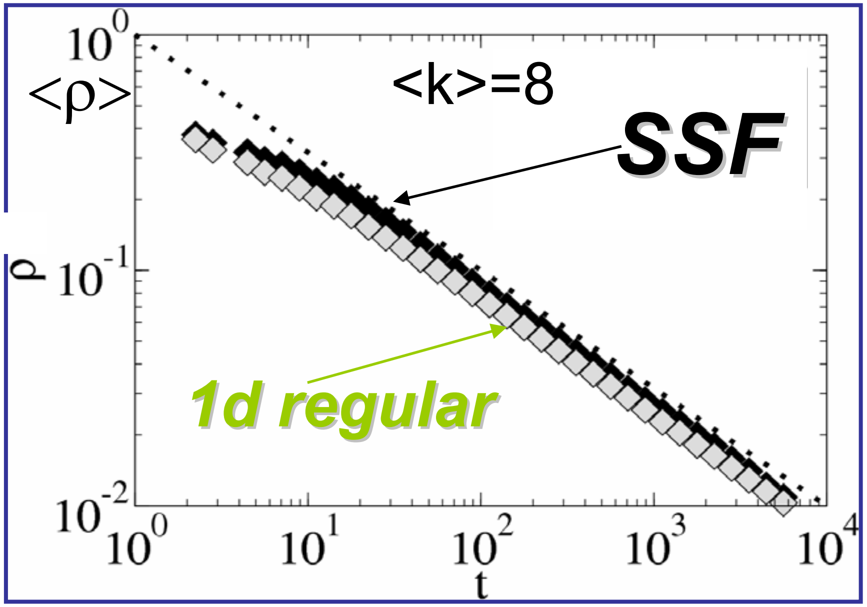
Klemm and Eguíluz,
Phys. Rev. E ***65****,036123 (2002)*



Scale free but
high clustering and 1d

P(k) ~ k $^{-3}$

L ~ N ***C ~ N***^{0}



SSF

$$<\rho> \sim t^{-1/2}$$

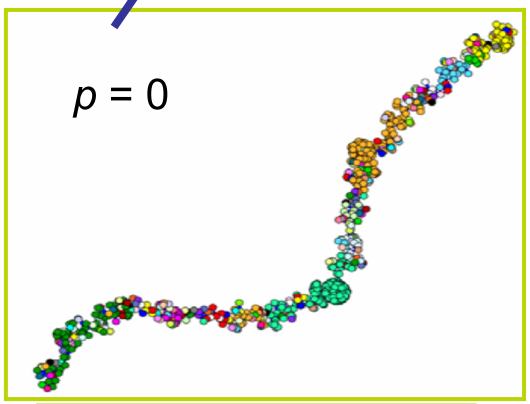
$$\tau_1 \approx N^2$$

Dimensionality *determines when voter dynamics orders the system*

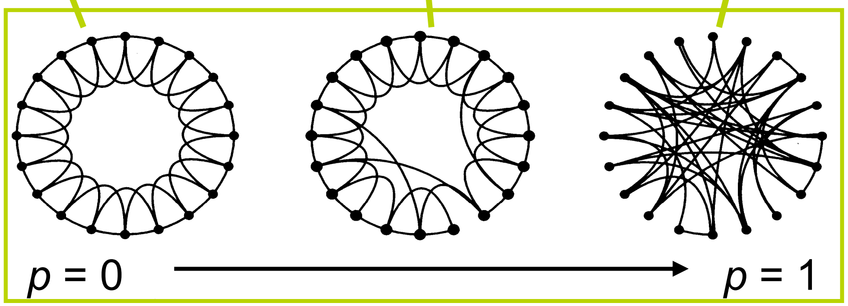
Degree *distribution or network* ***disorder*** *are not relevant*

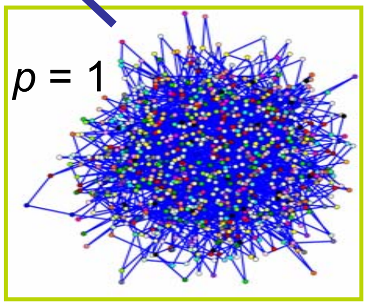
Disorder: *Rewiring parameter 0<p<1. d=1 ⟶ random networks*

$p = 0$	$0 < p < 1$	$p = 1$	
1D regular	*Small-World: SW*	*Random: RN / EN*	<u>*Single SCALE*</u>
Structured SF: SSF	*Small-World SF: SWSF*	*Random SF: RSF / BA*	<u>*HUBS*</u> **$P(k) \sim k^{-3}$**
$d = 1$	$d = \infty$		



$P(k) \sim k^{-3}$; $L \sim N$



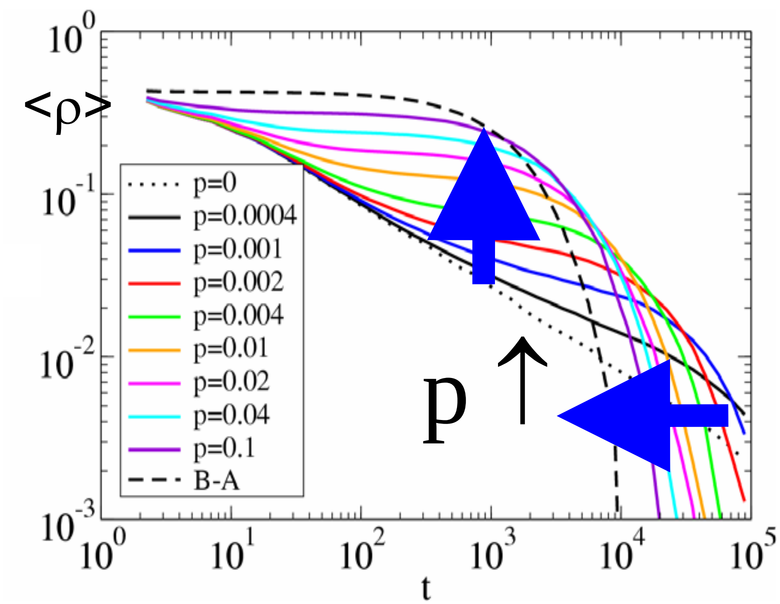


$P(k) \sim k^{-3}$

Role of Network Disorder p

SSF —— SWSF ——→ RSF

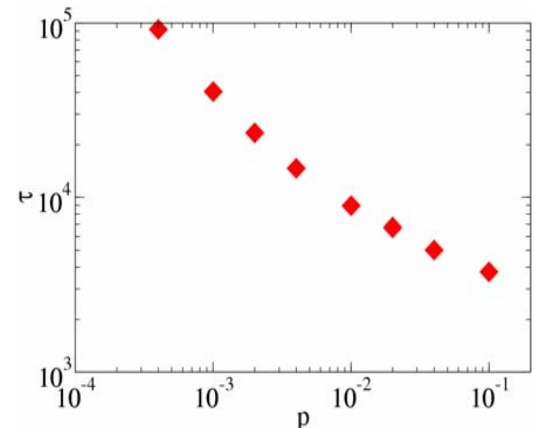
p=0 p=1



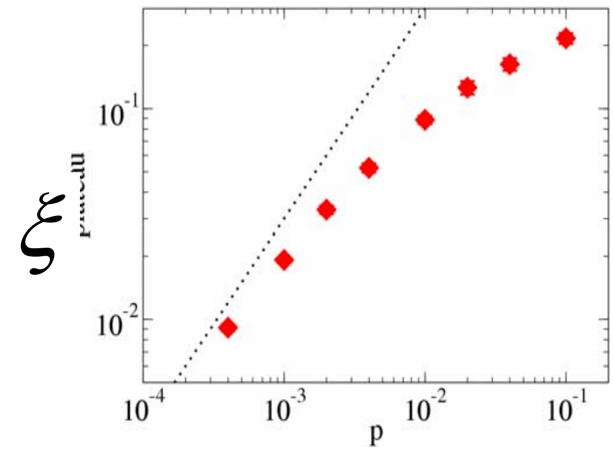
Increasing disorder p:

Higher disordered state

Faster decay

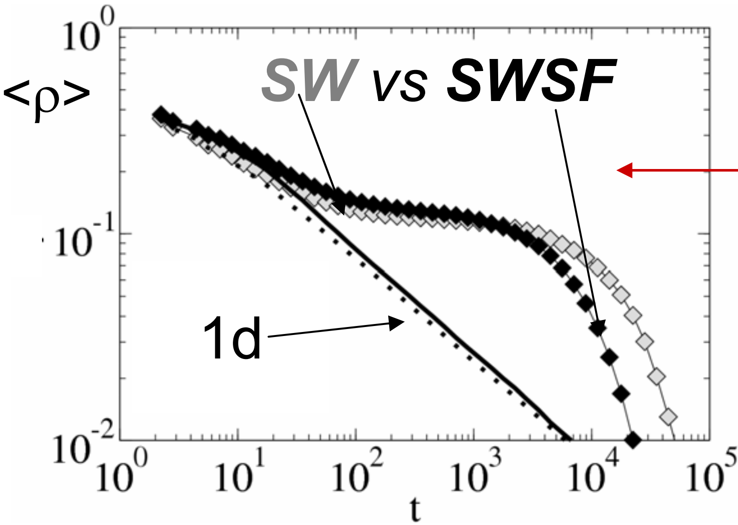


$\tau_{SWSF} > \tau_{RSF}$



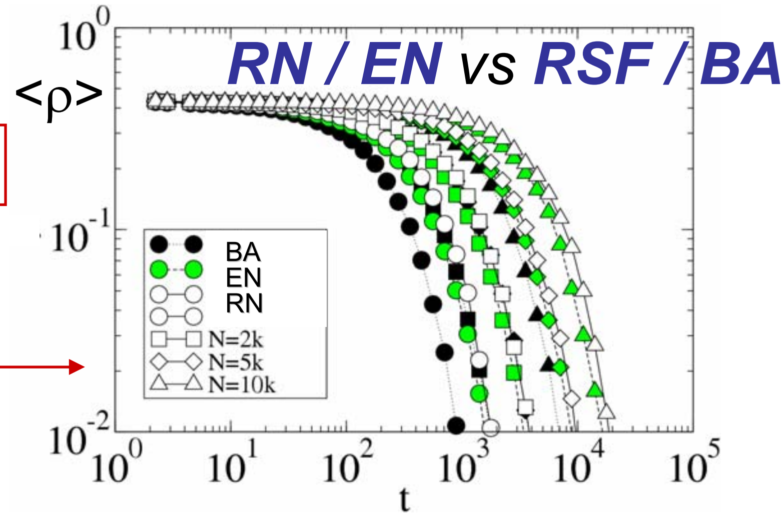
$l = \xi^{-1}: \quad l_{SWSF} > l_{RSF}$

Role of Network Degree Heterogeneity

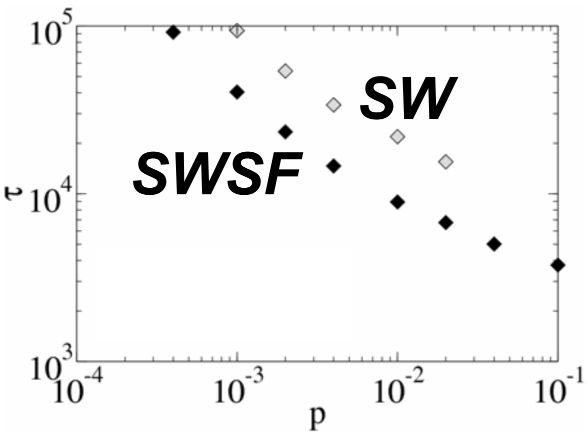


p=0.1

p=1



Hubs cause a faster decay, but do not change size of ordered domains

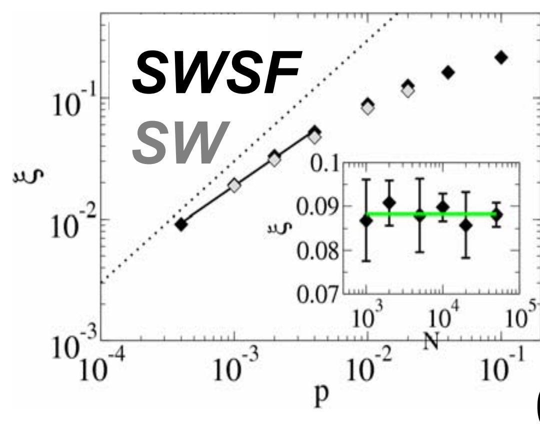


$$\tau_{SW} > \tau_{SWSF}$$

p=1

$$\tau_{RN} > \tau_{RSF}$$

$$(\tau_{SW} > \tau_{RN})$$



$$l_{SW} \cong l_{SWSF}$$

p=1

$$l_{RN} \cong l_{RSF}$$

$$(l_{SW} > l_{RN})$$

Role of Network Degree Heterogeneity: N-dependence

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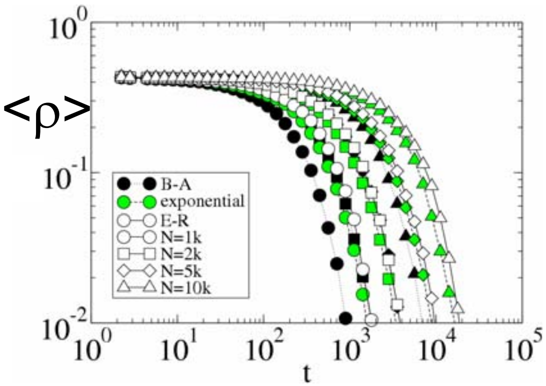
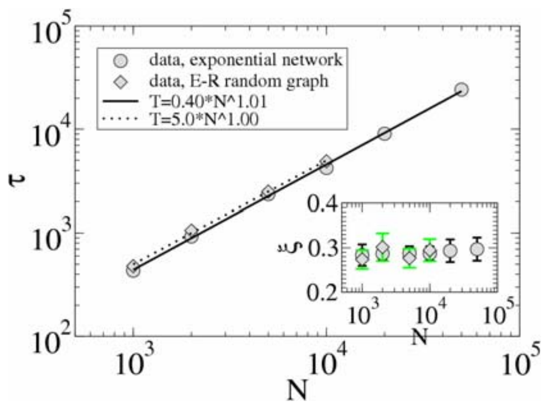
RN / EN



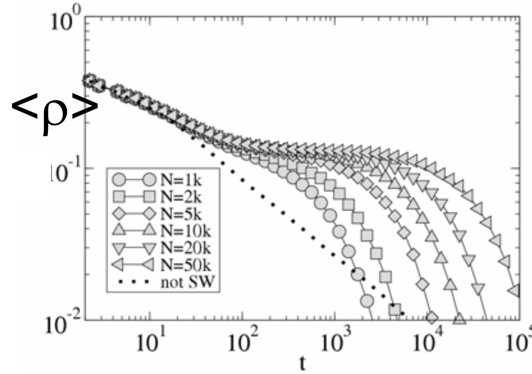


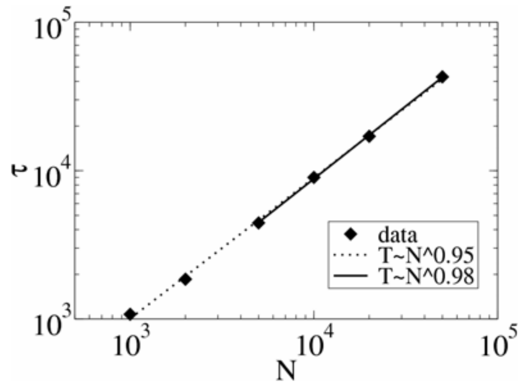
Linear scaling

$$\tau \approx N$$

Sood-Redner, cond-mat/0412599

SWSF



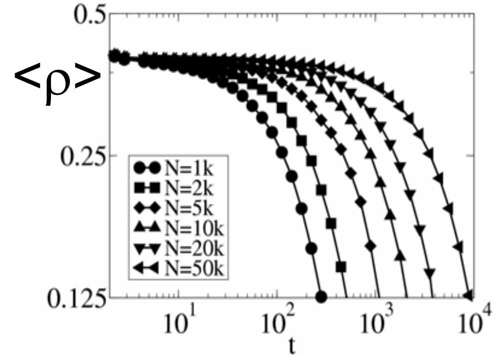


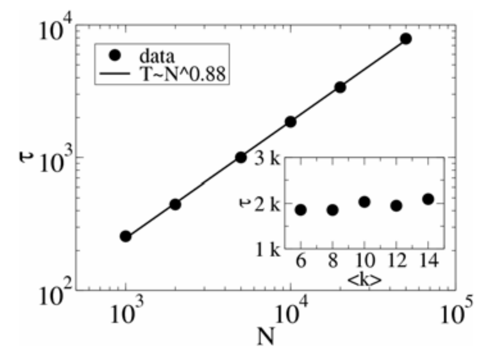
No linear scaling

$$\tau \approx N^{\gamma}; \gamma \neq 1$$

SW: $\gamma = 1$?

RSF





No linear scaling

$$\tau \approx N^{\gamma}; \gamma = 0.88$$

Sood-Redner: $\tau \approx N / \ln N$

Summary

Dimensionality: d=1 (SSF) $< \rho > \sim t^{-1/2}$; d = ∞ $< \rho > \sim e^{-t/\tau}$

Network Disorder:

SSF, 1d — p=0 → p=1 — SWSF, SW — RSF, RN

Shorter lifetimes: $\tau_{SWSF} > \tau_{RSF}$ $\tau_{SW} > \tau_{RN}$

Smaller size of domains: $l_{SWSF} > l_{RSF}$ $l_{SW} > l_{RN}$

Degree Heterogeneity: **SW** vs **SWSF; RN** vs **RSF**

Fluctuations more efficient with hubs: $\tau_{SW} > \tau_{SWSF}$ $\tau_{RN} > \tau_{RSF}$

Hubs do not affect size of domains: $l_{SW} \cong l_{SWSF}$ $l_{RN} \cong l_{RSF}$

Hubs change scaling law: $\tau \approx N^{\gamma}$ **SW, RN:** $\gamma = 1$ **; RSF,SWSF:** $\gamma \neq 1$

$<\sigma>$ is not conserved in node-update dynamics

Mean Field Voter Model

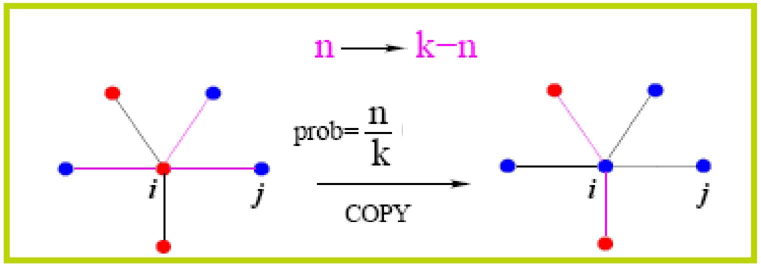
F. Vázquez, V. M. Eguíluz and M. San Miguel, Phys. Rev. Lett. ***100****, 108702 (2008)*

Mean Field Node Dynamics:

$$\frac{d<\sigma>}{dt}=0$$

Mean Field Link Dynamics:

ρ = global density of active links
n = active links
k-n = inert links
k= degree of node *i*



$$\Delta\rho=\frac{2(k-2n)}{<k>N}$$

Node i of degree k:

$$\left.\frac{d\rho}{dt}\right|_k=\frac{1}{1/N}\sum_{n=0}^{k}B(n,k)\,\frac{n}{k}\,\frac{2(k-2n)}{<k>N}$$

$B(n,k)=$ Prob. that node i has n active links

$$B(n,k)\approx\frac{k!}{n!(k-n)!}\rho^n(1-\rho)^{k-n}$$

⬅ **Mean Field**: $\rho\sim$ prob that a link from node *i* is active

$$\frac{d\rho}{dt}=\sum_k P_k\left.\frac{d\rho}{dt}\right|_k=\frac{2\rho}{<k>}\left[(<k>-1)(1-2\rho)-1\right]$$

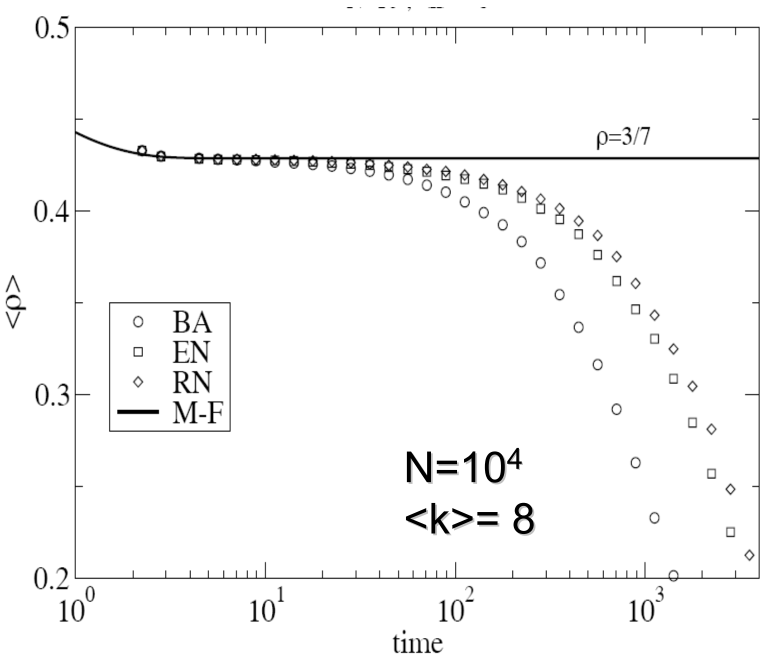
$$\Rightarrow\quad \rho^s=\frac{<k>-2}{2(<k>-1)}$$

Mean Field Link Dynamics:

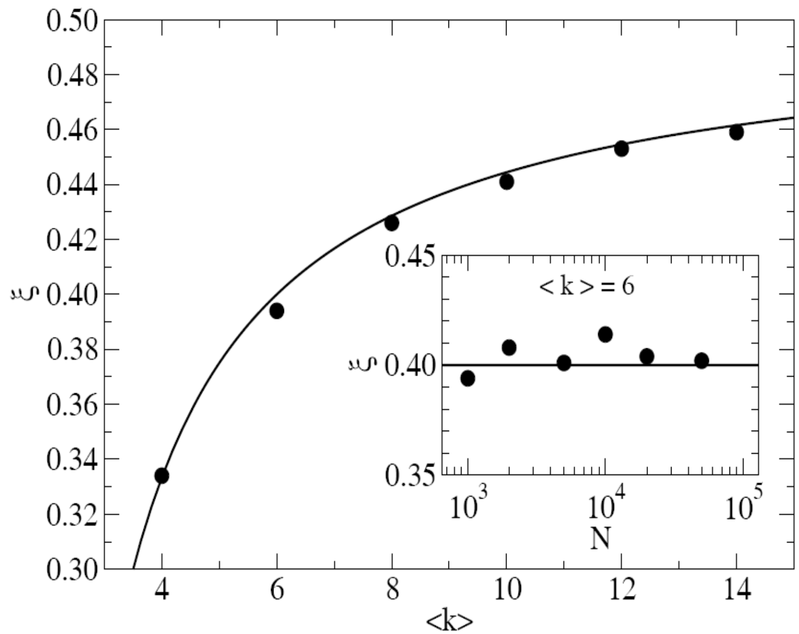
Single parameter theory

$$\rho^{s} = \xi = \frac{<k> -2}{2(<k> -1)}$$

Network topology independence



Barabasi-Albert Scale Free Networks

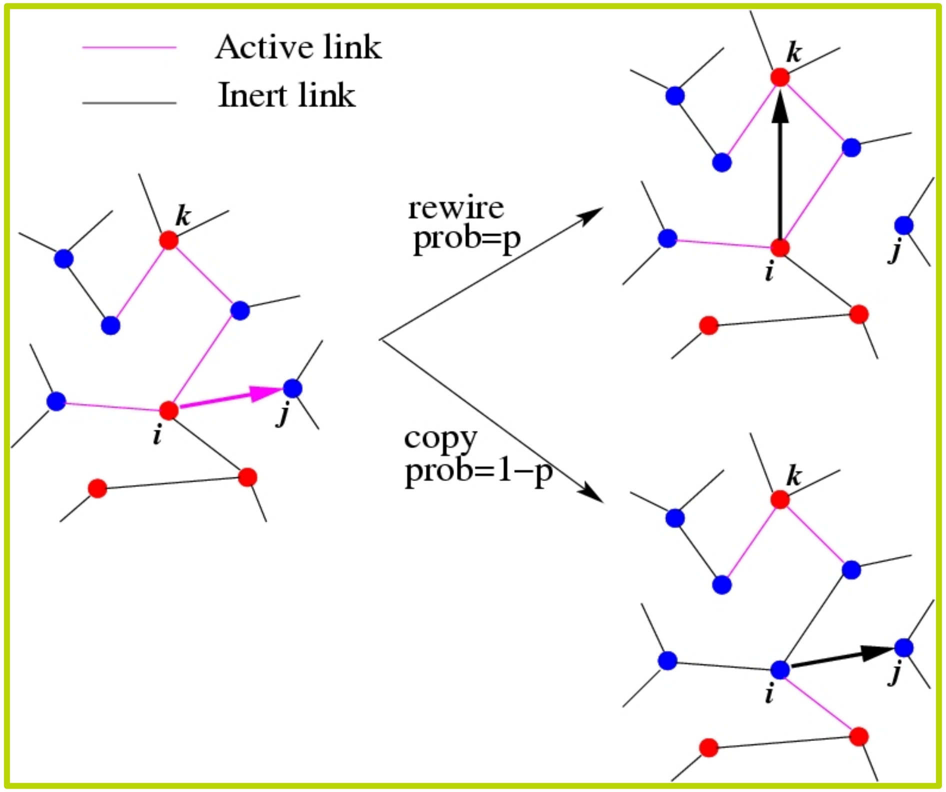


Coevolution Voter Model

F. Vázquez, V. M. Eguíluz and M. San Miguel, Phys. Rev. Lett. ***100****, 108702 (2008)*

Initial: Degree-regular random graph with μ neighbors.
Nodes take state $S = -1$ *or* $S = +1$ with the same probability 1/2.

1. Pick a node i and a neighbor j at random.
2. If $S_i = S_j$ nothing happens.
3. If $S_i \neq S_j$ then:
   - Network dynamics: *rewire*
     with probability *p* delete link $i-j$
     and create link $i-k$ $(S_i = S_k)$.
   - State dynamics: *copy*
     with probability *1-p* set $S_i = S_j$.
4. Repeat ad infinitum.



- **Agents select interacting partner according to their state**
- **p gives a ratio of time scales of evolution of state of nodes and network**

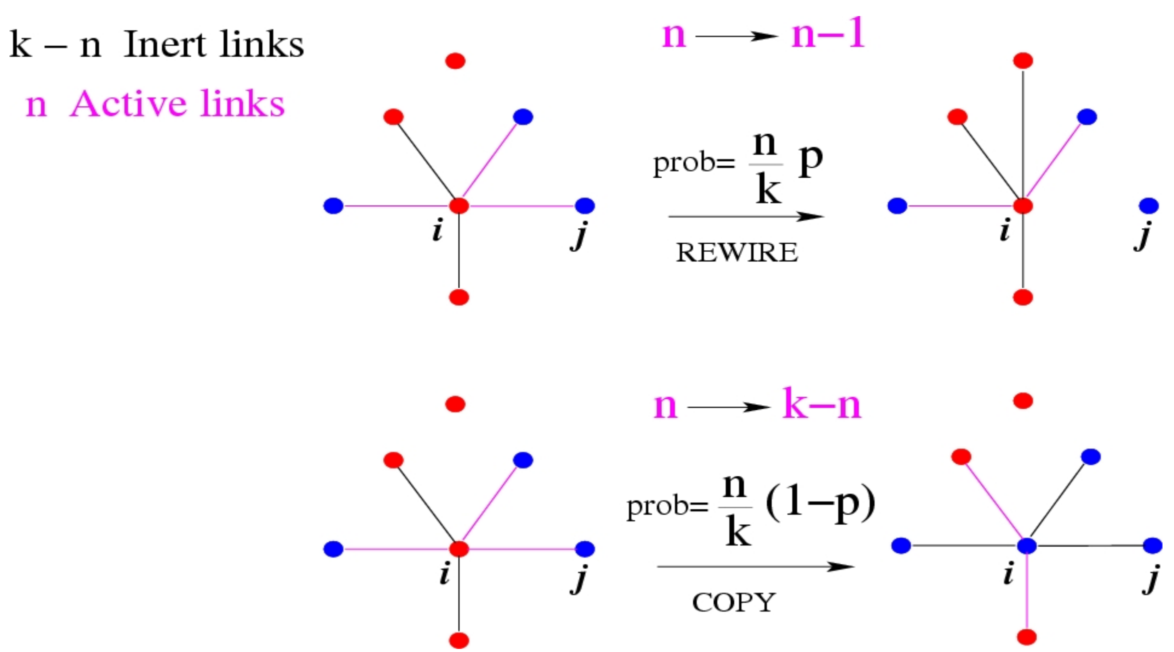
Link dynamics: Mean-Field approach

Active links: + -   Inert links: + +, - -

P_k ≡ fraction of nodes with k neighbors.

$\mu = \langle k \rangle = \sum_k k P_k(t)$ ≡ average node degree.

ρ ≡ global density of active links.



$$\Delta\rho = \frac{\Delta n}{\mu N/2} = \frac{-2}{\mu N}$$

$$\Delta\rho = \frac{2(k-2n)}{\mu N}$$

$$\frac{d\rho}{dt} = \sum_k \frac{P_k}{1/N} \sum_{n=0}^{k} B_{n,k} \frac{n}{k} \left[(1-p)\frac{2(k-2n)}{\mu N} - p\frac{2}{\mu N} \right]$$
$$= \sum_k P_k \frac{2}{\mu k} \left[(1-p)\left(k\langle n\rangle_k - 2\langle n^2\rangle_k \right) - p\langle n\rangle_k \right],$$

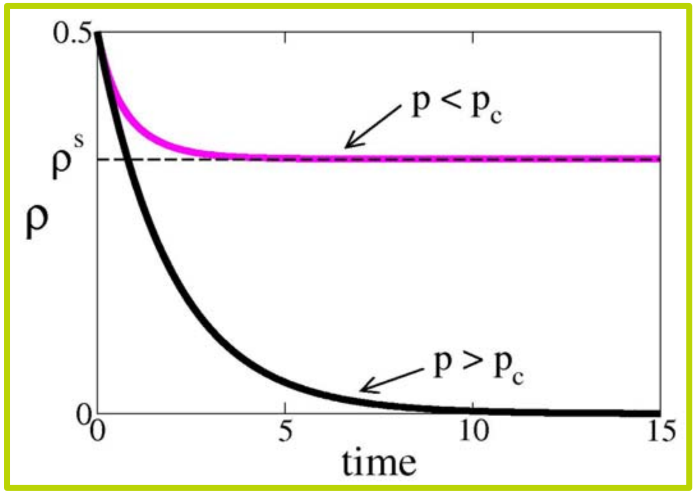
B(n,k) ≡ Prob. that *n* active links are connected to a node of degree k.

Mean-Field approximation: Probability that a given link is active ≈ ρ.

$$B_{n,k} \simeq \frac{k!}{n!(k-n)!}\rho^n(1-\rho)^{k-n}$$
$$\langle n\rangle_k = \sum_{n=0}^{k} n\, B_{n,k} \simeq \rho\, k$$
$$\langle n^2\rangle_k = \sum_{n=0}^{k} n^2 B_{n,k} \simeq \rho\, k + \rho^2 k(k-1)$$

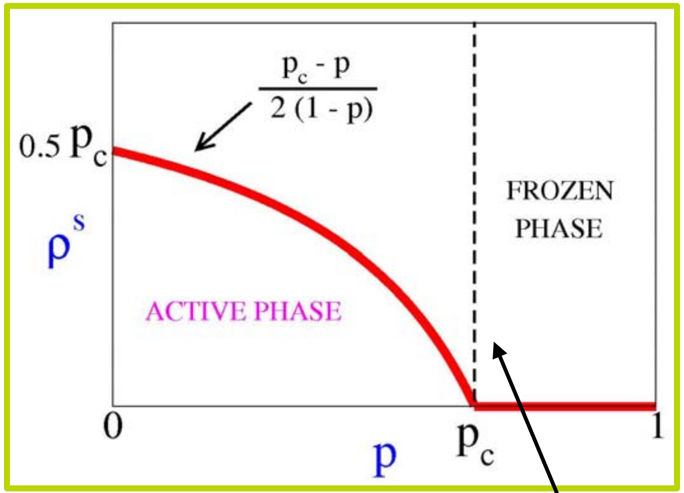
Master equation for the density of active links in the N → ∞ limit:

$$\frac{d\rho}{dt} = \frac{2\rho}{\mu}\left[(1-p)(\mu-1)(1-2\rho)-1\right]$$



Active - Frozen Transition at

$$p_c = \frac{\mu-2}{\mu-1}$$

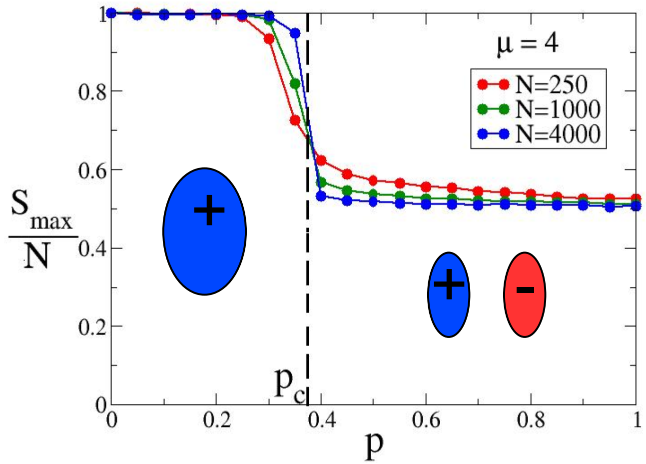


$\rho_{++} = \rho_{--} = 1/2$?

- **Active phase:** Links continuosly being rewired and nodes flipping states
- **Frozen phase:** Fixed network where connected nodes have the same state

Fragmentation Transition

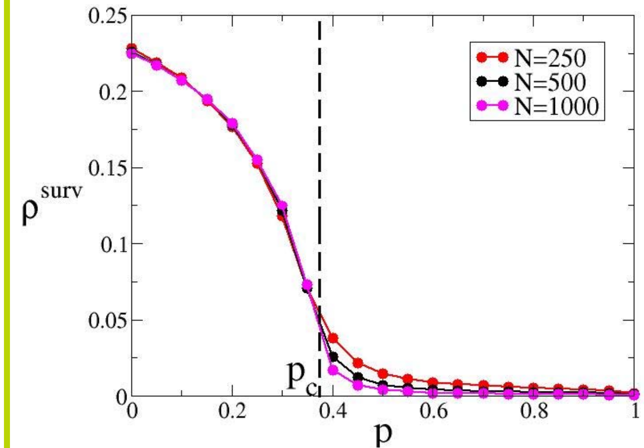
Size of largest network component.



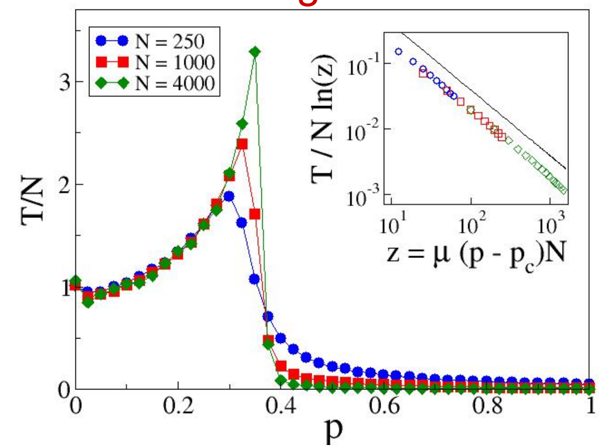
Active phase → Connected network ($S_{max}/N = 1$)
(N = ∞)
Frozen phase → Fragmented network ($S_{max}/N \approx 0.5$)

- $p<p_c$: **slow rewiring keeps network connected until system fully orders and freezes in a single component.**
- $p>p_c$: **fast rewiring leads to fragmentation of network into two components before system reaches full order.**

Active links in surviving runs.



Convergence times



$$T \sim \begin{cases} N\,(p_c - p)^{-1} & \text{for } p \lesssim p_c \\ (p-p_c)^{-1}\ \ln\left[\mu(p-p_c)N\right] & \text{for } p \gtrsim p_c \end{cases}$$

Why does the network break above p_c ?: a first-passage problem

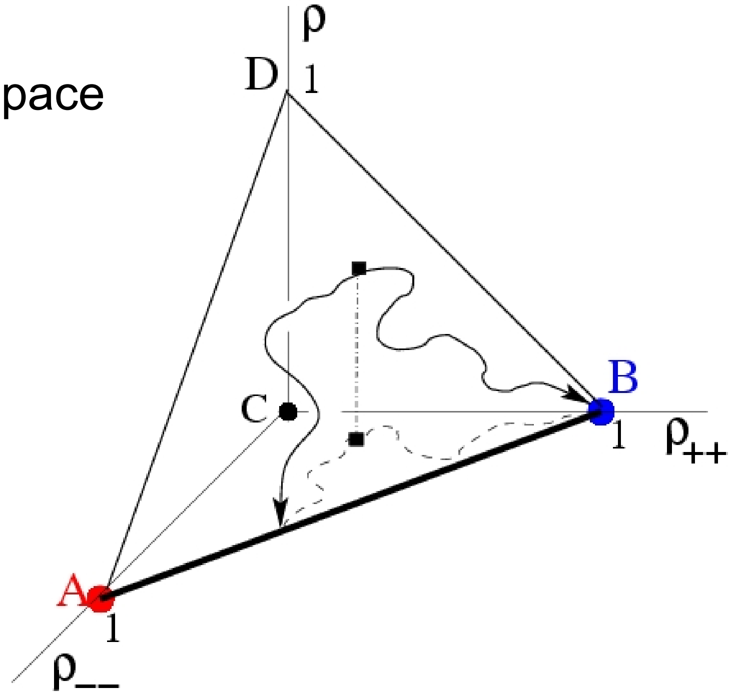
One + → - interaction:

rewire: $$(N_{--}, N_{++}, N_{+-}) \rightarrow (N_{--}, N_{++}+1, N_{+-}-1)$$

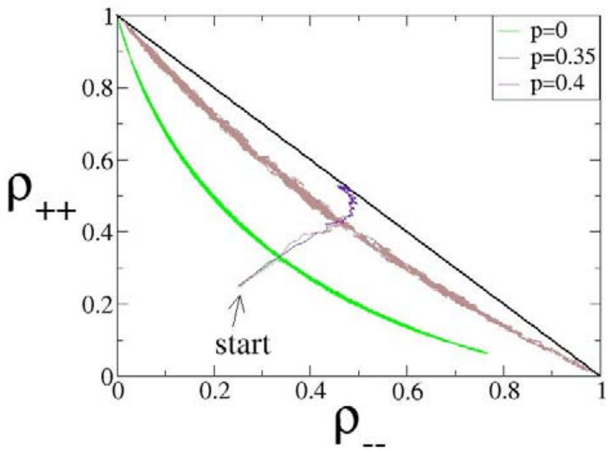
copy: $$(N_{--}, N_{++}, N_{+-}) \rightarrow (N_{--}+n, N_{++}+n-k, N_{+-}+k-2n)$$

no update: $$(N_{--}, N_{++}, N_{+-}) \rightarrow (N_{--}, N_{++}, N_{+-})$$

System performs a random walk on the densities' space

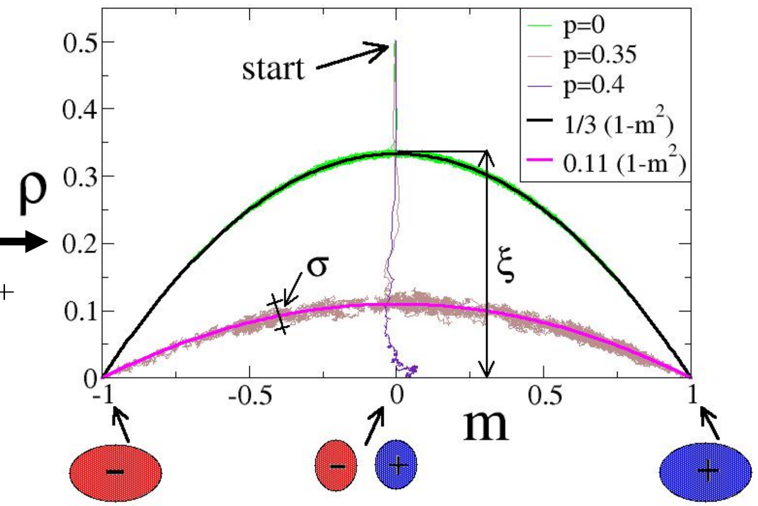


SYSTEM	SPACE OF DENSITIES
State	Point (□$_{--}$, □, □$_{++}$)
Evolution	Motion of RW in ABD □
Frozen state	Absorbing boundaries



$m = \rho_{++} - \rho_{--}$

$\rho = 1 - \rho_{--} - \rho_{++}$



absorbing points

$$p < p_c: \quad \rho \sim \xi(1 - m^2) \quad \rightarrow \quad m = -1, 1$$ (one component)

$$p > p_c: \quad \rho \sim e^{-t/\tau} \quad \rightarrow \quad m = 0$$ (two components)

$p<p_c$: slow rewiring keeps network connected until system fully orders and freezes in a single component.

$p>p_c$: fast rewiring leads to fragmentation of network into two components before system reaches full order.

Conservation laws for the voter model in complex networks

Suchecki, Krzysztof; Eguiluz, Victor M.; San Miguel, Maxi
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Physical Review E **72**, 036132(1-8) (2005)

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Analytical Solution of the Voter Model on Uncorrelated Networks

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