

ETH-Zürich-September 19th 2007

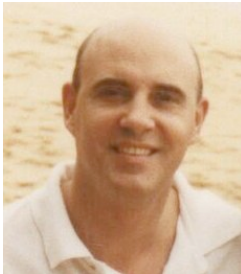
Constructive effects induced by heterogeneity: an application to a model for opinion formation

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Summary:

- Disordering role of heterogeneity and noise
- Ordering role of heterogeneity and noise
- Stochastic resonance induced by disorder
 - Bistable systems
 - Model for opinion formation

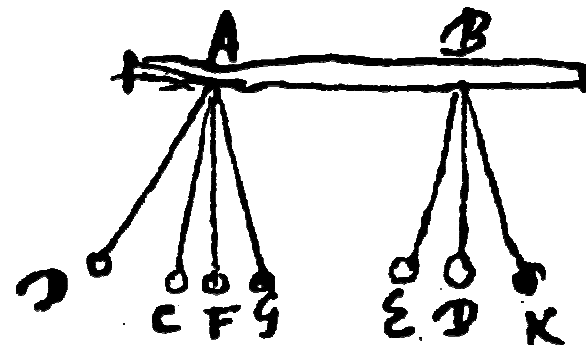
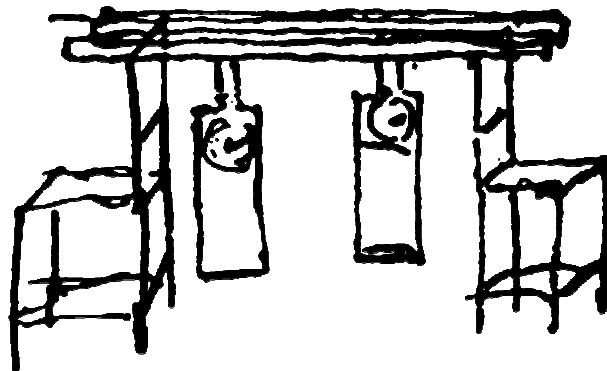
Is it possible to have perfect synchrony between two different systems?
 Two independent clocks that initially display the same time will, eventually, show different times.

That's why we "Synchronize our clocks before an action"

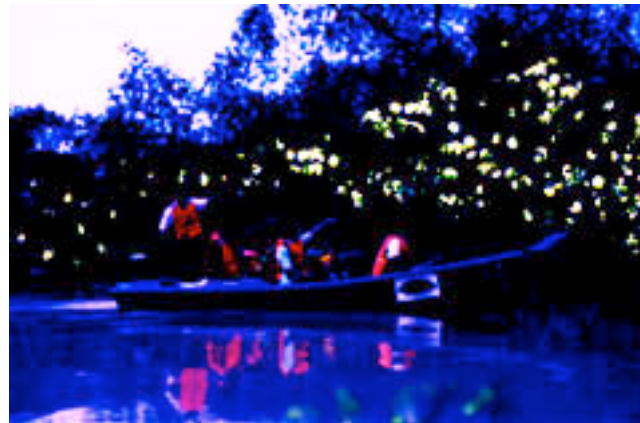
Can we keep two clocks permanently synchronized?

Christiaan Huygens showed that this is possible by coupling them in the appropriate manner

Journal des Sçavans, 16 march 1665



Fireflies in Malaysia, Thailand and New Guinea



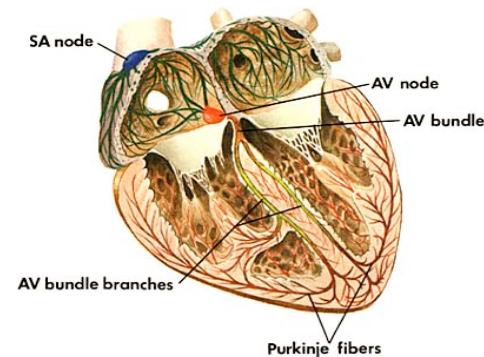
J. Buck & E. Buck, *Scientific American* May 1976.

S.H. Strogatz, I. Steward, *Scientific American*, Nov. 1993

The dark side of the moon



Sinoatrial node in heart: pacemaker of
~ 10000 synchronized cells.



Applause rate in a concert.

Z. Neda et. al. Nature **403**, 849 (2000).

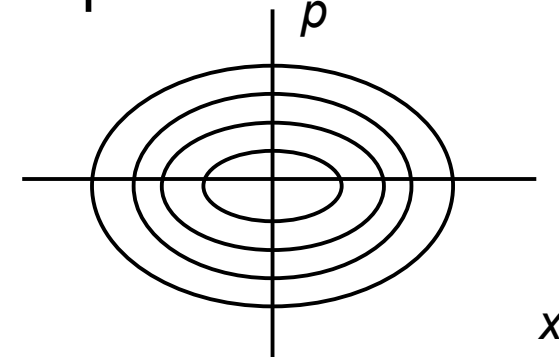


Basic ingredients [Winfree (1967)] : Coupled oscillators

Harmonic oscillator:

$$\dot{x} = \frac{p}{m}$$

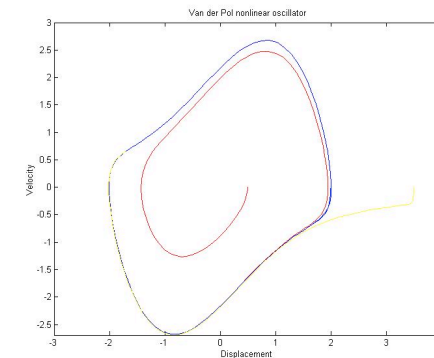
$$\dot{p} = -kx$$



Limit cycle attractor

(van der Pol):

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

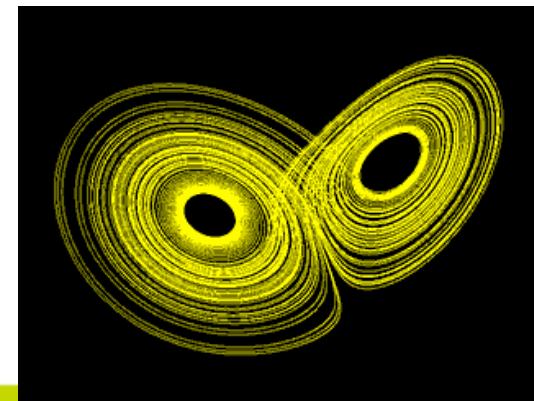


Chaotic attractor $\dot{x} = p(y - x)$

(Lorenz):

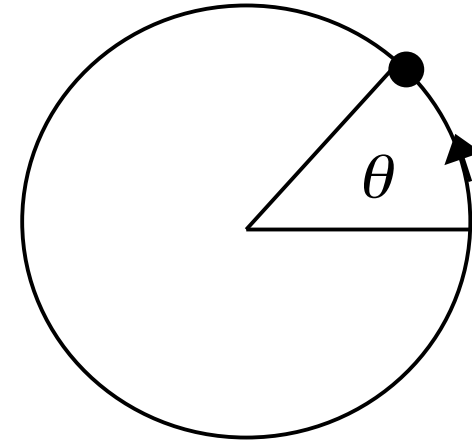
$$\dot{y} = -xz + rx - y$$

$$\dot{z} = xy - bz$$



Kuramoto model [1975]

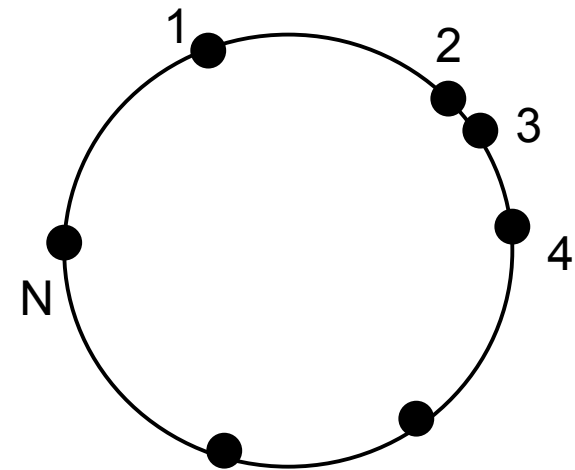
Simplest oscillator: $\dot{\theta} = \omega$



Many of them

$$\dot{\theta}_j = \omega + \eta_j \quad j = 1, \dots, N$$

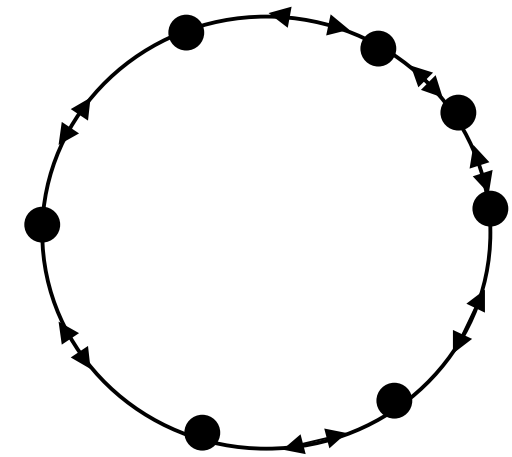
Rotate independently of each other unless.....



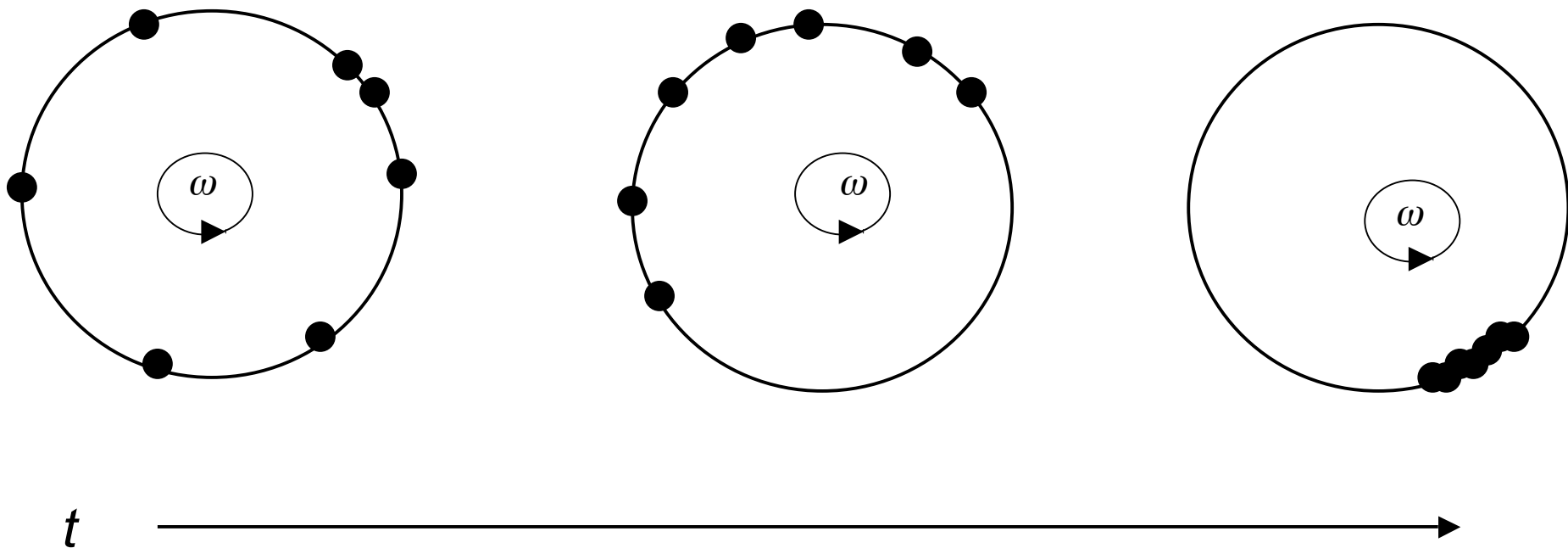
Coupling!!!

Kuramoto considers an **ensemble** of globally coupled oscillators:

$$\dot{\theta}_j = \omega + \eta_j + \frac{K}{N} \sum_{k=1}^N \sin(\theta_k - \theta_j) \quad j = 1, \dots, N$$



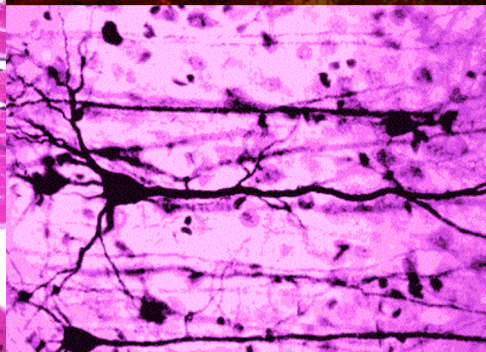
In the homogeneous case: $\eta_j = 0 \quad j = 1, \dots, N$



Stable synchronization (coherent behavior) occurs for any $K > 0$

Diversity exists

There are examples of systems with many different units strongly interacting. Despite not all units being identical, the system exhibits, in some situations a macroscopic coherent behavior.



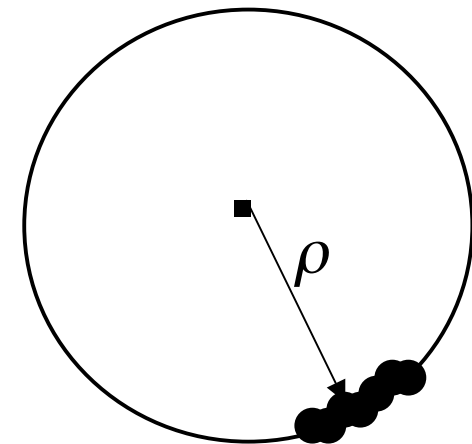
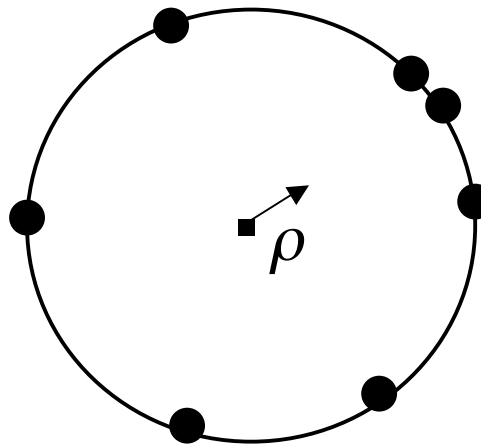
Kuramoto model:

$$\dot{\theta}_j = \omega_j + \eta_j + \frac{c}{N_j} \sum_j \sin(\theta_j - \theta_i)$$

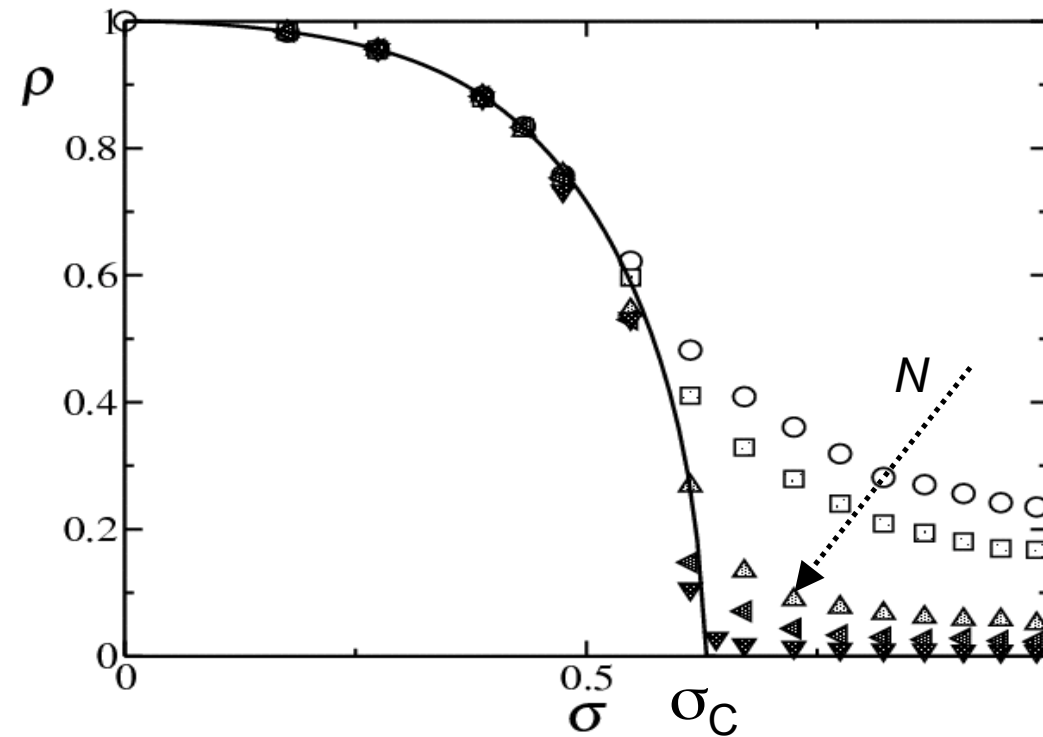
$$\langle \eta_i \rangle = 0, \langle \eta_i^2 \rangle = \sigma^2$$

$$\rho = \left| \frac{1}{N} \sum_i e^{i\theta_i} \right|$$

Radius of the center of mass of the oscillators



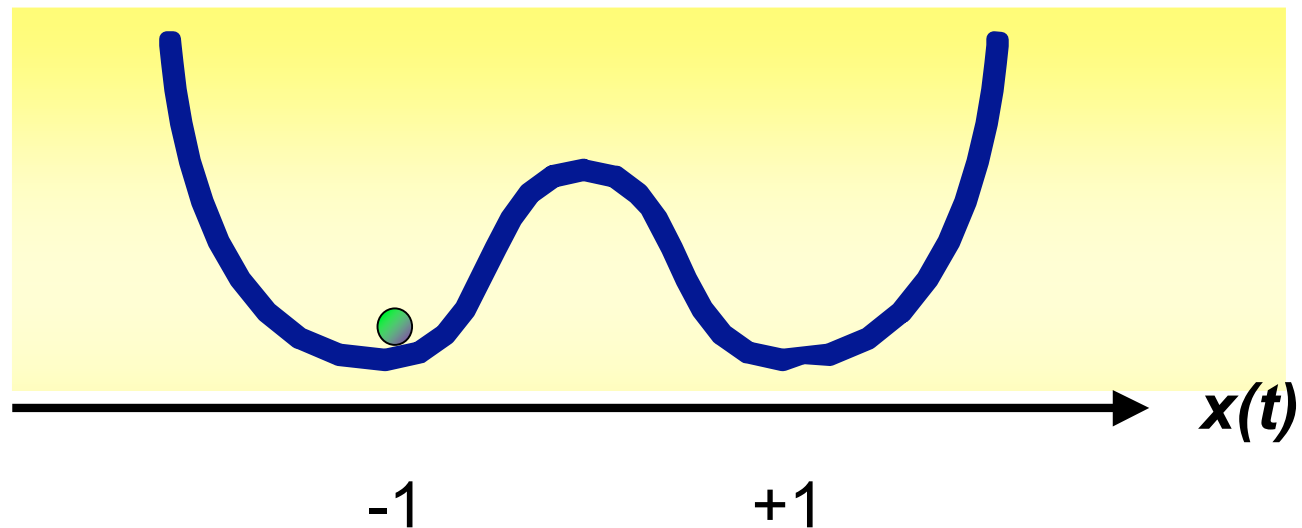
Diversity destroys coherent motion!!



This is a general result

Basic model for binary choice (Landau model):

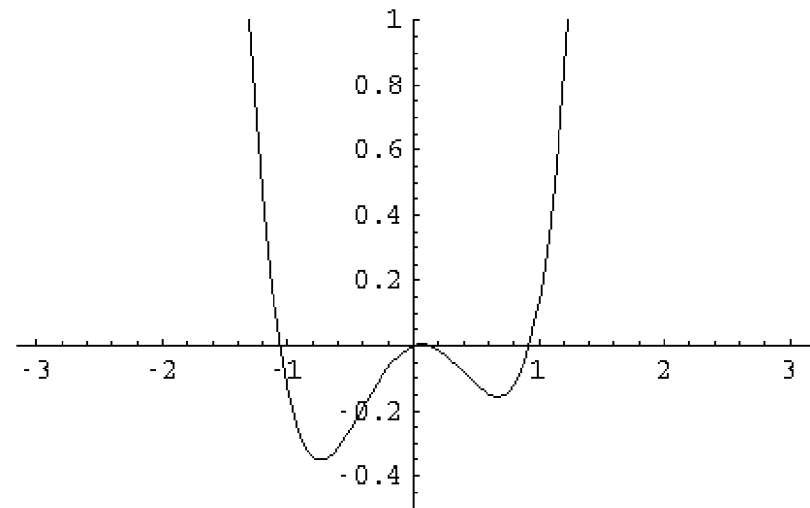
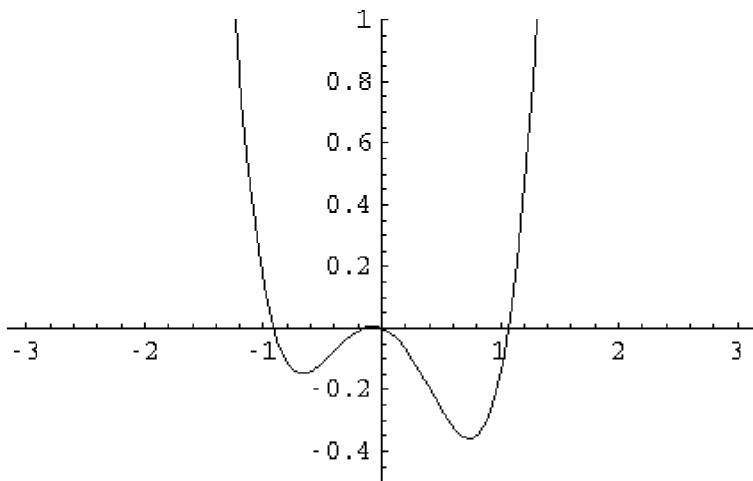
$$\dot{x}(t) = x(t) - x(t)^3$$



Relaxation in a symmetric potential

Diversity is introduced in the asymmetry of the potential:

$$\dot{x}(t) = x(t) - x(t)^3 + \eta$$



$\eta > 0$ favors the well on the right $\eta < 0$ favors the well on the left

Landau model:

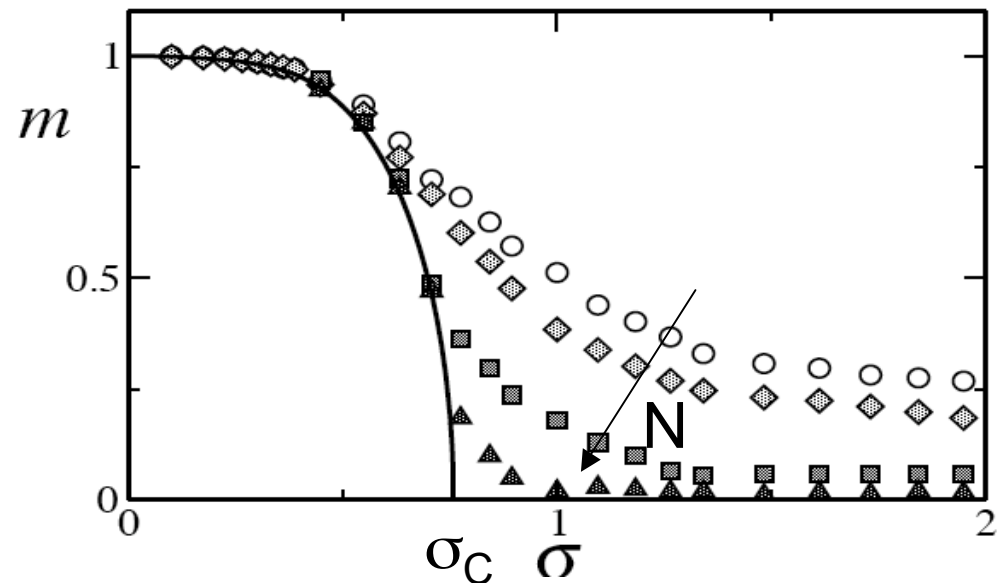
$$\frac{dx_i}{dt} = x_i - x_i^3 + \eta_i + \frac{C}{N_i} \sum_{j=1} (x_j - x_i)$$

Order parameter:

$$m = \left| \frac{1}{N} \sum_{i=1}^N x_i \right|$$

$$\langle \eta_i \rangle = 0$$

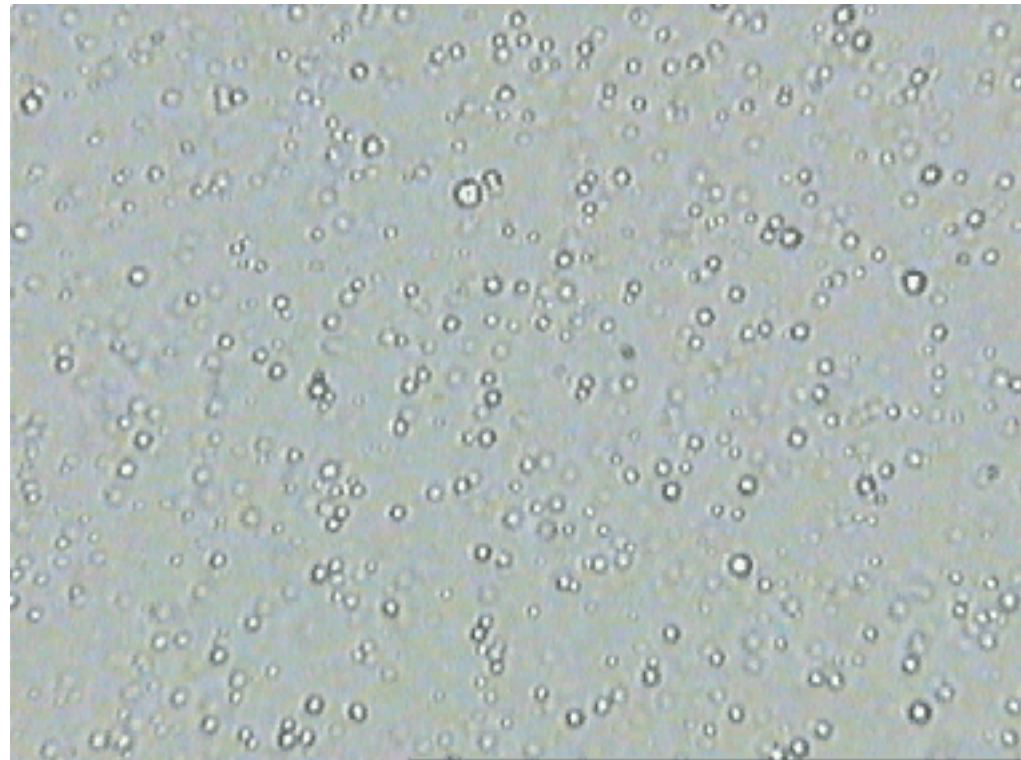
$$\langle \eta_i^2 \rangle = \sigma^2$$



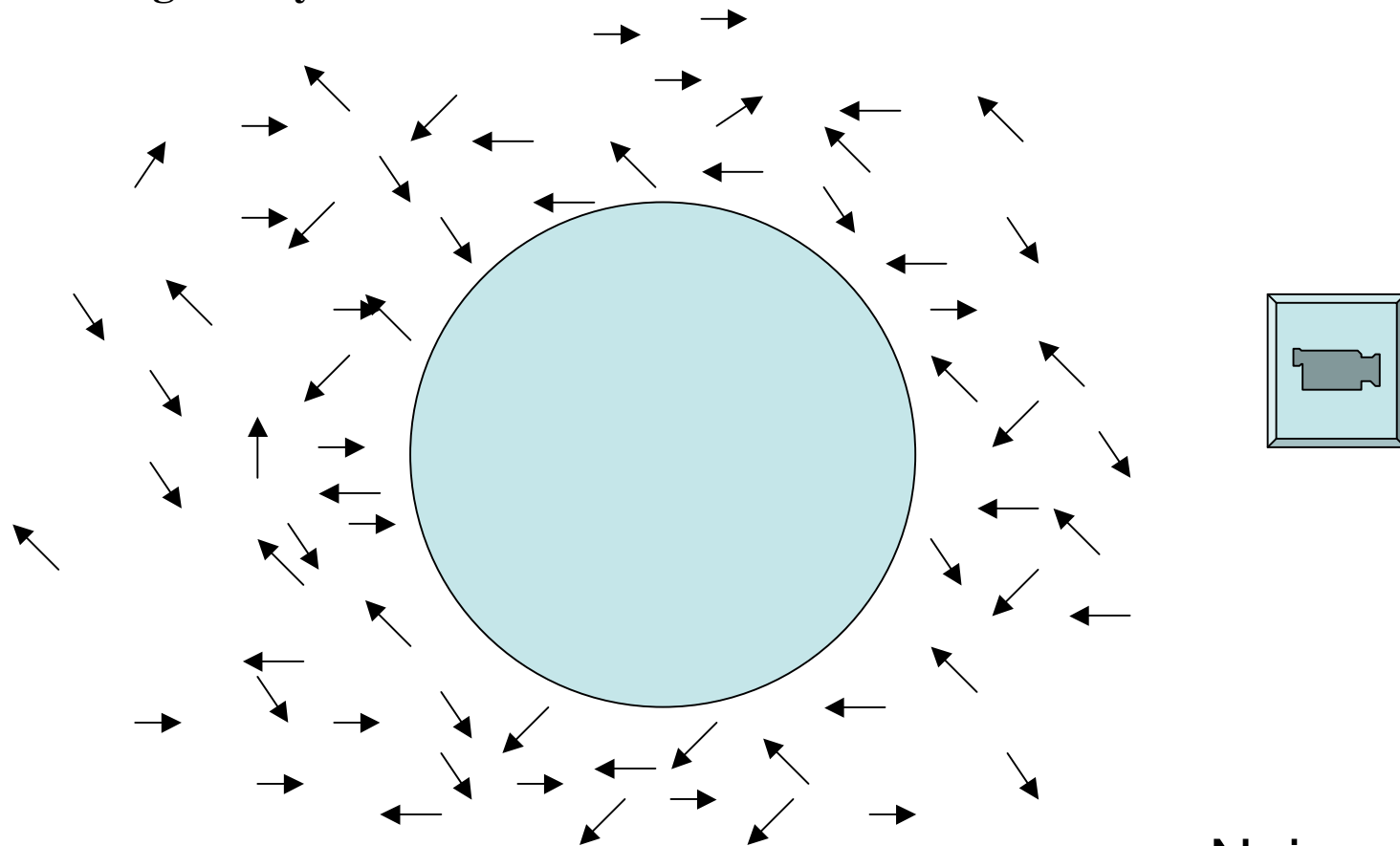
Diversity is not the only source of disorder
“Noise” (fluctuations of microscopic origin) is another

It all started in 1828 with a Scottish botanist

Robert Brown (1828) was looking at pollen in water through a microscope.

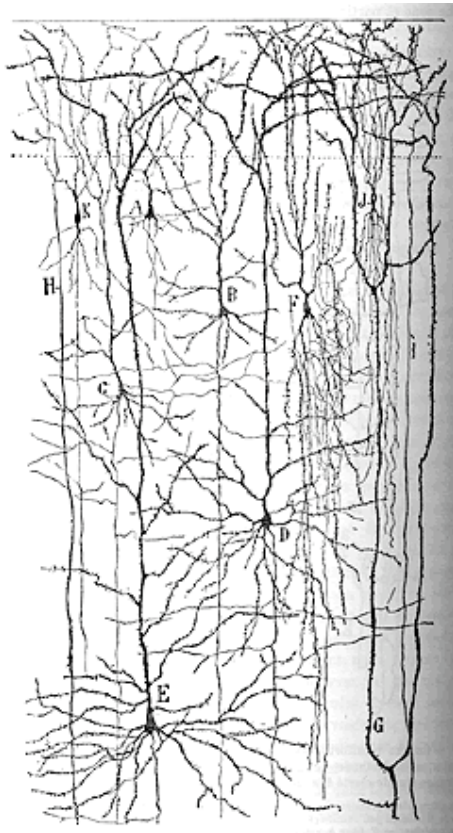


Explanation was given by Einstein in 1905



$$\vec{F} = \vec{F}_{ext} - \gamma \vec{v} + \xi(t)$$

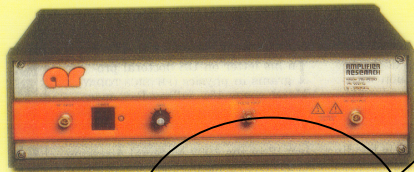
Noise everywhere!!!!



Electrical currents with many connections and noise

$$\begin{aligned}\tau_v \dot{v} &= v\left(v - \frac{1}{2}\right)(1 - v) - w, \\ \tau_w \dot{w} &= v - w - \beta + \epsilon \cos(\omega t) + \xi(t)\end{aligned}$$

Silence Is Golden.



High-speed blanking circuitry in AR Quiet Amps keeps RF waveforms square, and reduces noise to near thermal. You detect even the smallest transients from your sample—no small matter in the world of NMR/MRI where received signals are notoriously prone to decay. Application and recovery stay fast and noise-free. Other features in Quiet Amps keep the effects of applied RF true to form—gated, rapid pulse rise and fall time, no-droop with long pulse width due to Class A operation, wide bandwidth. The 75AP250 and 250AP250 headline this group, which includes instruments with a variety of features to meet a variety of NMR/MRI applications. Don't keep questions to yourself. Drop us a call. Or visit our Website at www.ar-amps.com.

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Circle number 13 on Reader Service Card

Reduces noise to
near thermal.

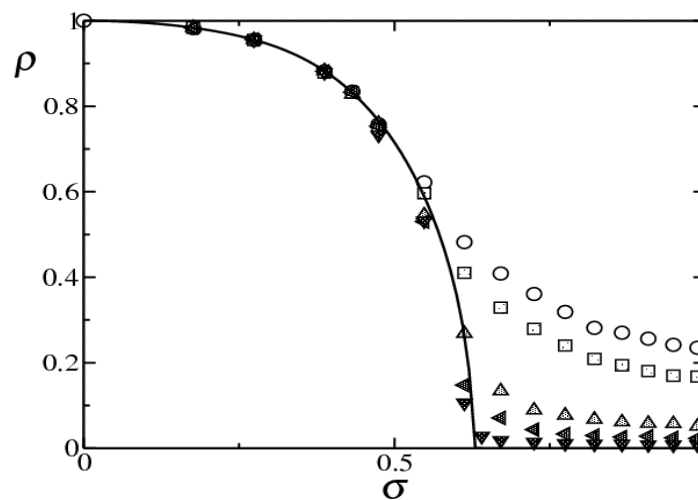
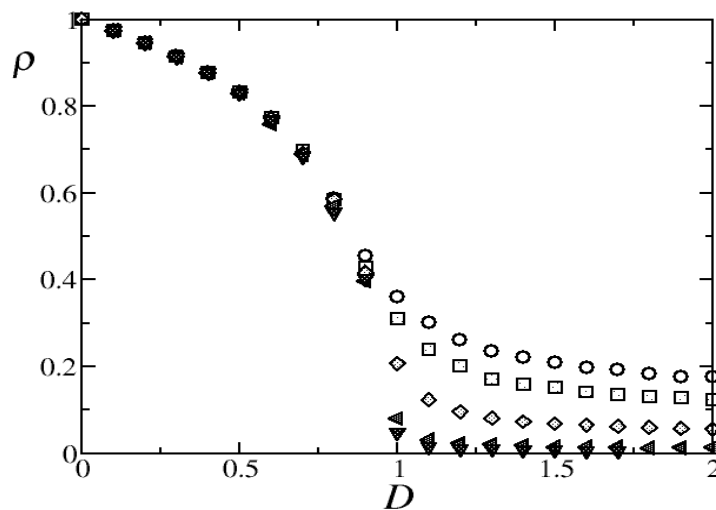
Internal noise:
proportional to
temperature T

Kuramoto model with noise:

$$\frac{d\theta_i}{dt} = \omega + \xi_i(t) + \frac{C}{N_j} \sum_j \sin(\theta_j - \theta_i)$$

$$\langle \xi_i(t) \rangle = 0, \langle \xi_i^2(t) \rangle = D$$

$$\langle \eta_i \rangle = 0, \langle \eta_i^2 \rangle = \sigma^2$$



$$\rho = \left| \frac{1}{N} \sum_i e^{i\theta_i} \right|$$

Landau model with noise:

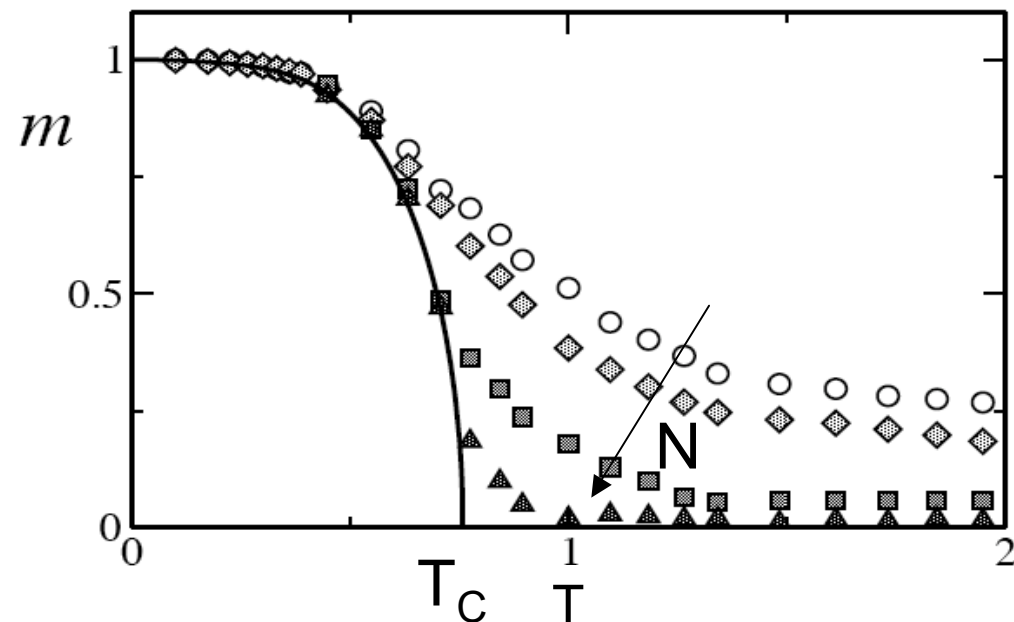
$$\frac{dx_i}{dt} = x_i - x_i^3 + \frac{C}{N_i} \sum_{j=1} (x_j - x_i) + \xi_i(t)$$

$\xi_i(t)$ is thermal noise

$$\langle \xi_i(t) \rangle = 0$$

$$\langle \xi_i^2(t) \rangle \propto T$$

$$m = \frac{1}{N} \sum_{i=1}^N x_i$$



No Noise Is Good Noise

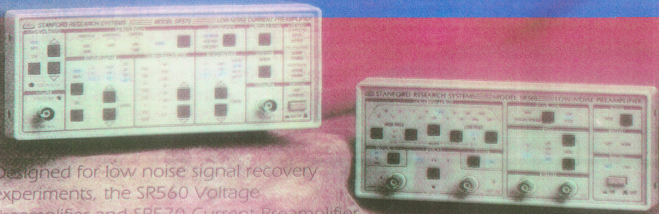
SR570 Current Preamp

- 5 fA/√Hz input noise
- 1 MHz bandwidth
- 1 pA/V max. sensitivity
- Adjustable bias voltage and input offset current

SR560 Voltage Preamp

- 4 nV/√Hz input noise
- 1 MHz bandwidth
- Gain from 1 to 50,000
- True differential or single-ended input

Low Noise Preamplifiers.....\$2195 (U.S. list)



Designed for low noise signal recovery experiments, the SR560 Voltage Preamplifier and SR570 Current Preamplifier are the industry's standards. These general purpose instruments are ideal for amplifying and conditioning very small signals and offer solutions for a variety of photonic and low temperature applications. Both preamplifiers feature a 1 MHz bandwidth, configurable filters, line or battery operation and an RS-232 computer interface.



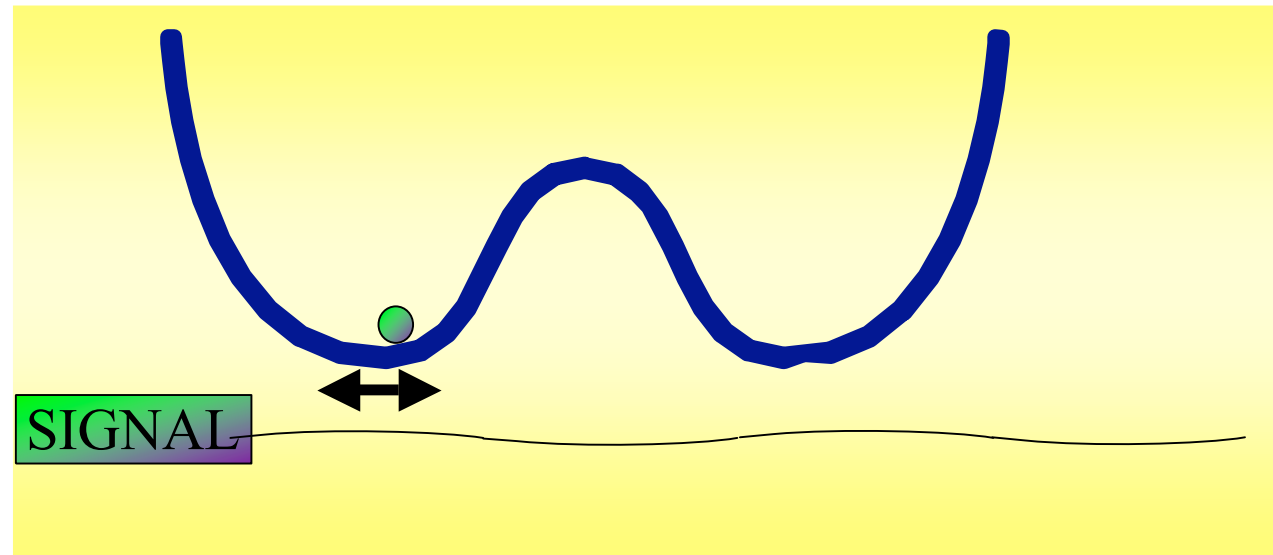
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Noise disorders

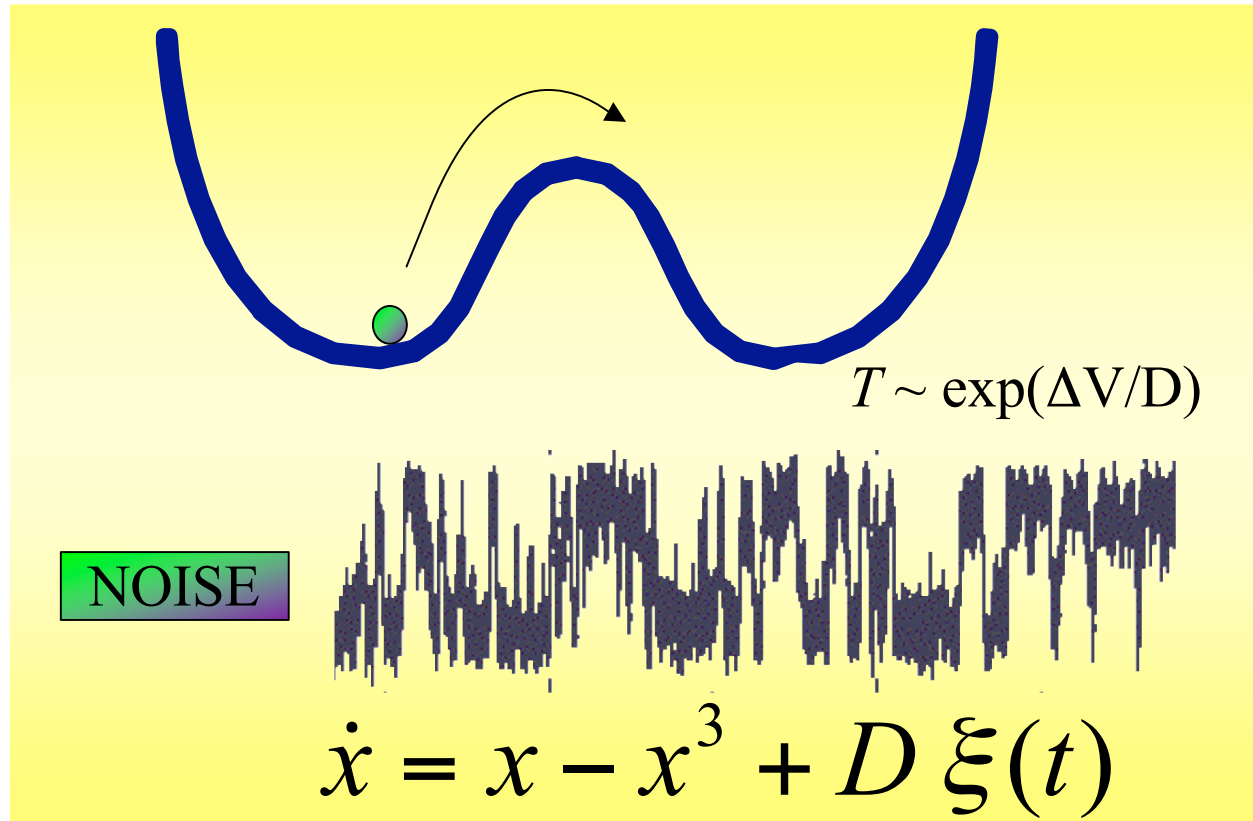
....but some noises are better than others.

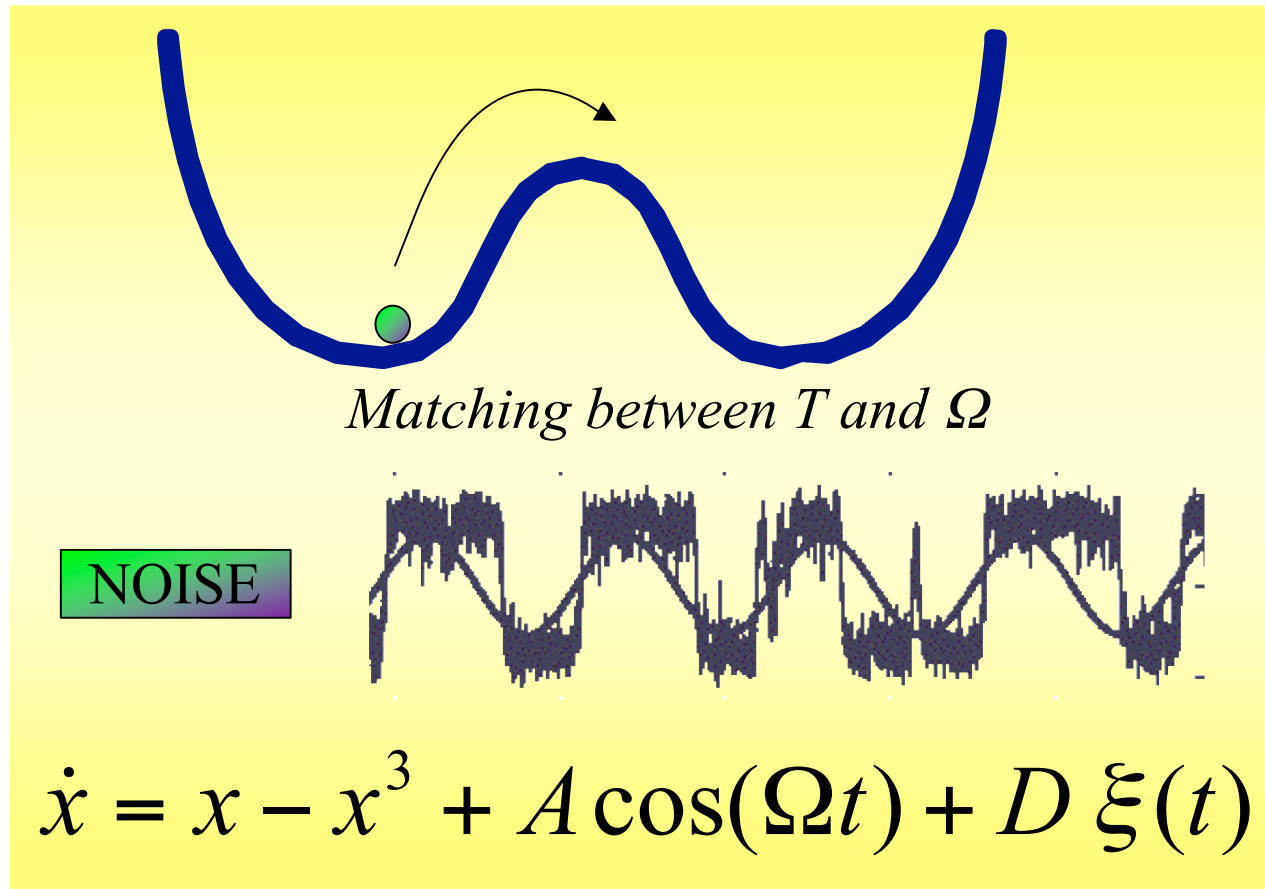
Stochastic Resonance (1981)

-Amplification of sub-threshold signals helped by noise

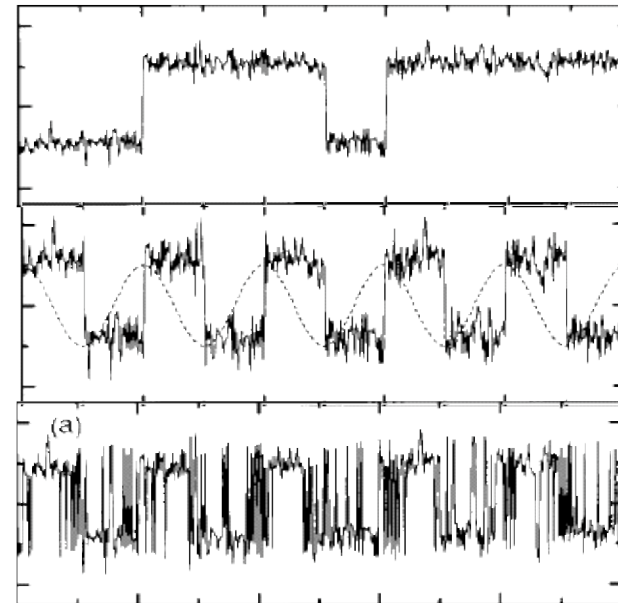
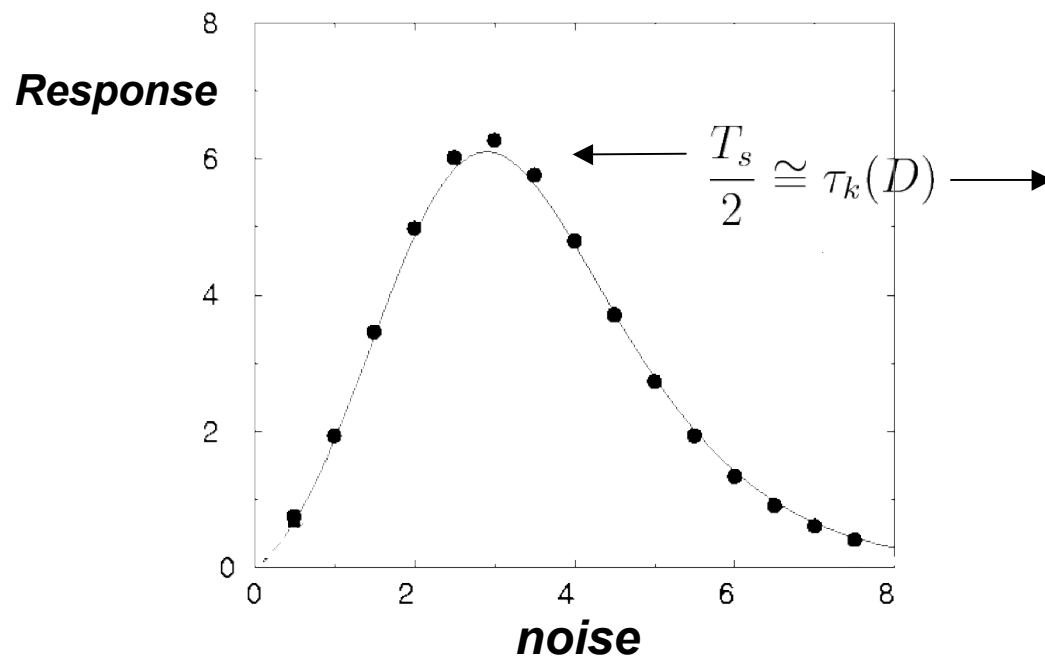


$$\dot{x} = x - x^3 + A \cos(\Omega t)$$

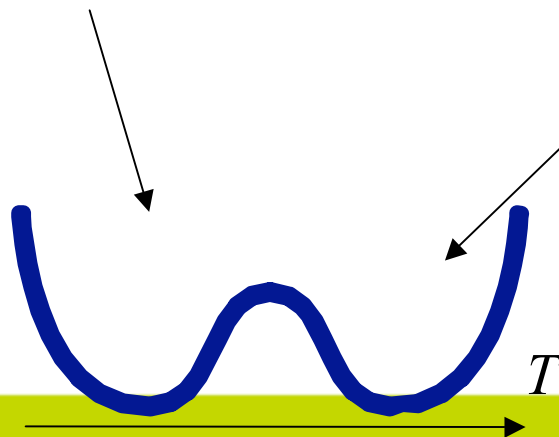




$$\frac{d}{dt}x(t) = -x(t)^3 + bx(t) + A \sin\left(\frac{2\pi}{T_s}t\right) + \xi(t)$$



Earth climate



Weak signal: earth eccentricity ($\sim 3 \cdot 10^4$ y)
Noise: Changes in Sun luminosity due to solar stains, clouds, CO_2 , etc...

Global temperature changes
Bi-stable climate potential



New results begin here

The role of diversity in a forced system

What is the role of diversity when a weak external periodic forcing acts upon a system composed by many, diverse, bistable units ?

Diversity can make the response easier

If a population makes only umbrellas, they are O.K. only in rainy years.

If they make only swim suits, they are O.K. in sunny years.

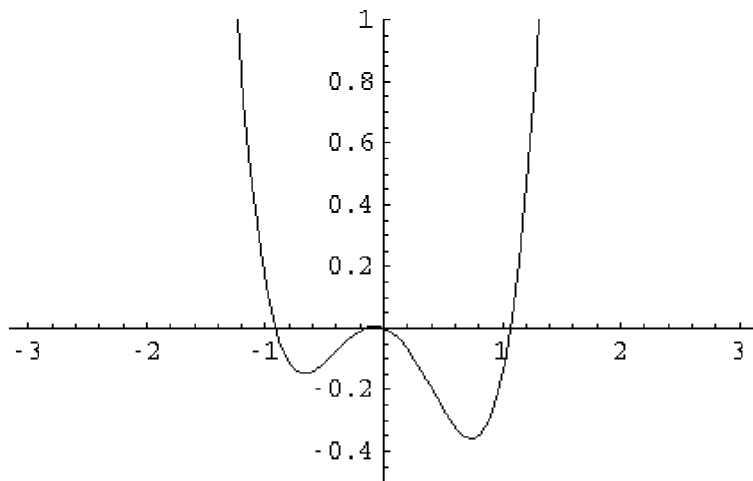
But if a fraction of the population makes umbrellas and another fraction swim suits, they can be O.K. always

...provided they is some coupling between people.

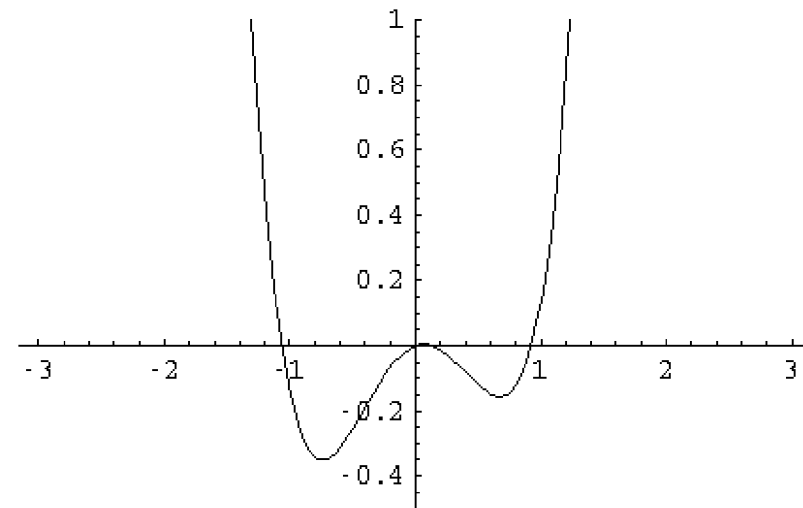
Never put all the eggs in the same basket!!

Diversity is introduced in the asymmetry of the potential:

$$\dot{x} = x - x^3 + a$$



$a > 0$ favors the well on the right
(positive forces)



$a < 0$ favors the well on the left
(negative forces)

An ensemble of coupled bistable units

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + \underbrace{a_i}_{\text{diversity parameter}} + \underbrace{\frac{C}{N} \sum_j (x_j(t) - x_i(t))}_{\text{global coupling}} + \underbrace{A \sin(\omega t)}_{\text{periodic signal} \dots \text{weak} \dots}$$

- a_i is a parameter distributed according to a **Gaussian distribution** function $g(a)$, of mean 0 and variance σ^2

... **no noise** ...

applet

Some theory:

$$\dot{x}_i(t) = x_i(t) - x_i(t)^3 + a_i + \frac{C}{N} \sum_j (x_j(t) - x_i(t)) + A \sin(\omega t)$$

The macroscopic variable

$$X(t) = \frac{1}{N} \sum_{i=1}^N x_i(t)$$

$$\dot{X} = X(1 - 3M) - X^3 + A \sin(\omega t)$$

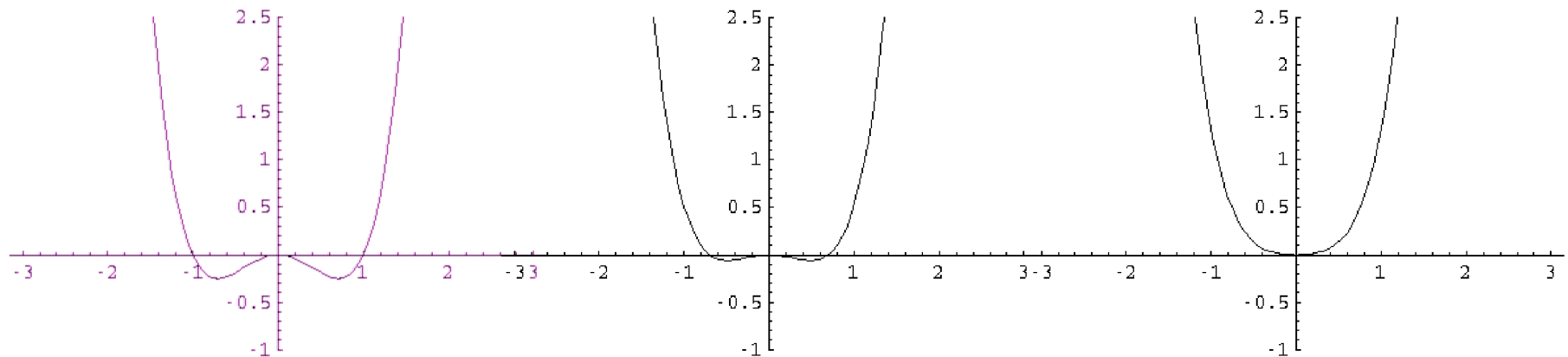
$$\frac{dX}{dt} = -\frac{\partial V(X, M)}{\partial X} + A \sin(\omega t)$$

$$M = \frac{1}{N} \sum_j (X - x_j)^2 = \frac{1}{N} \sum_j \delta_j^2$$

Effective global potential

$$V(X, M) = \frac{X^4}{4} - \frac{X^2(1 - 3M)}{2} \quad \frac{1}{N} \sum_i \delta_i^2 = M$$

- No diversity implies $M=0$ because of the coupling
- Increasing diversity, M increases and the barrier between minima lowers. Eventually, the system undergoes a transition from a bistable to a monostable potential



Linear regime ----->non-linear regime----->linear regime

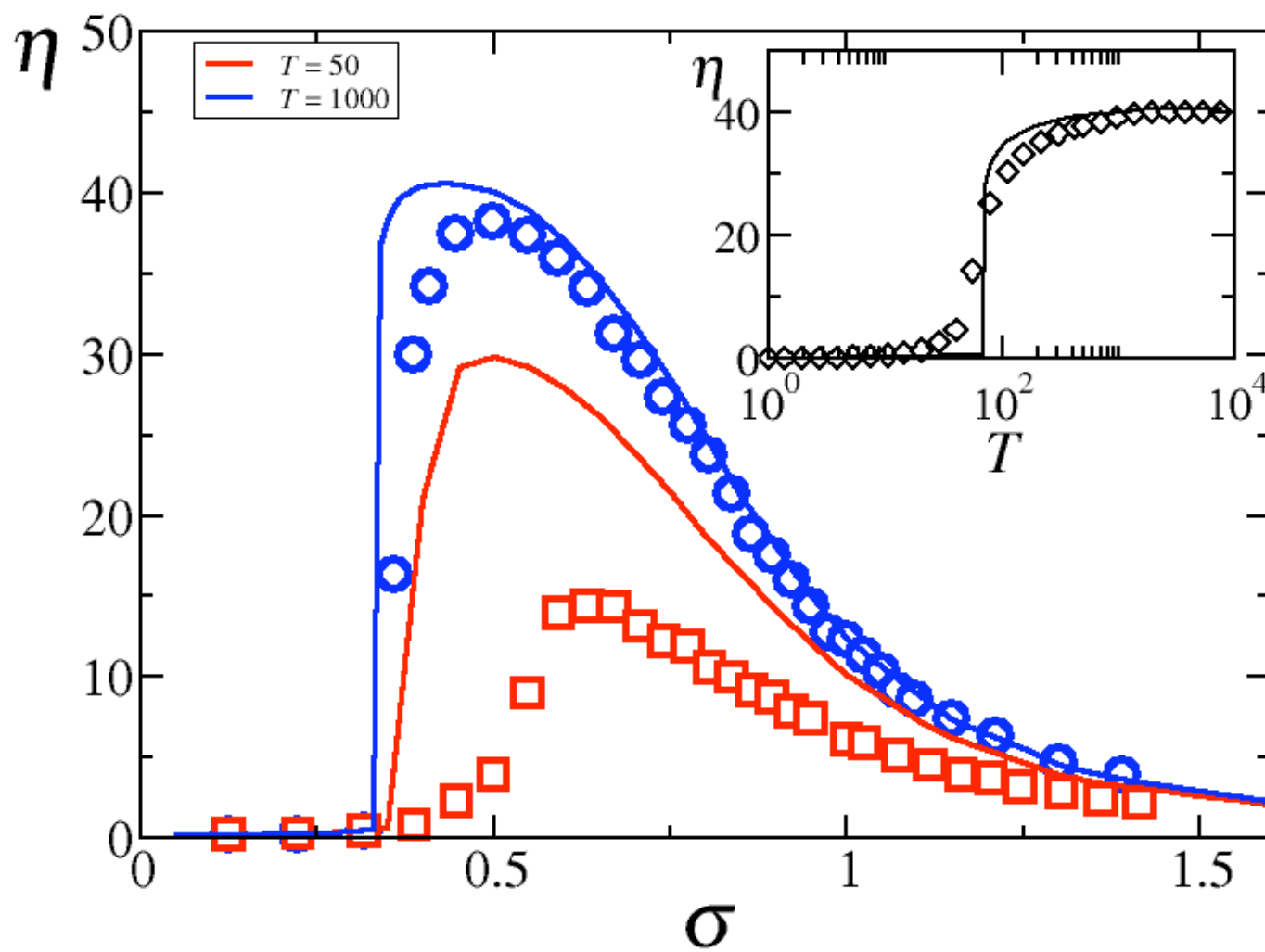
▶ M increases.

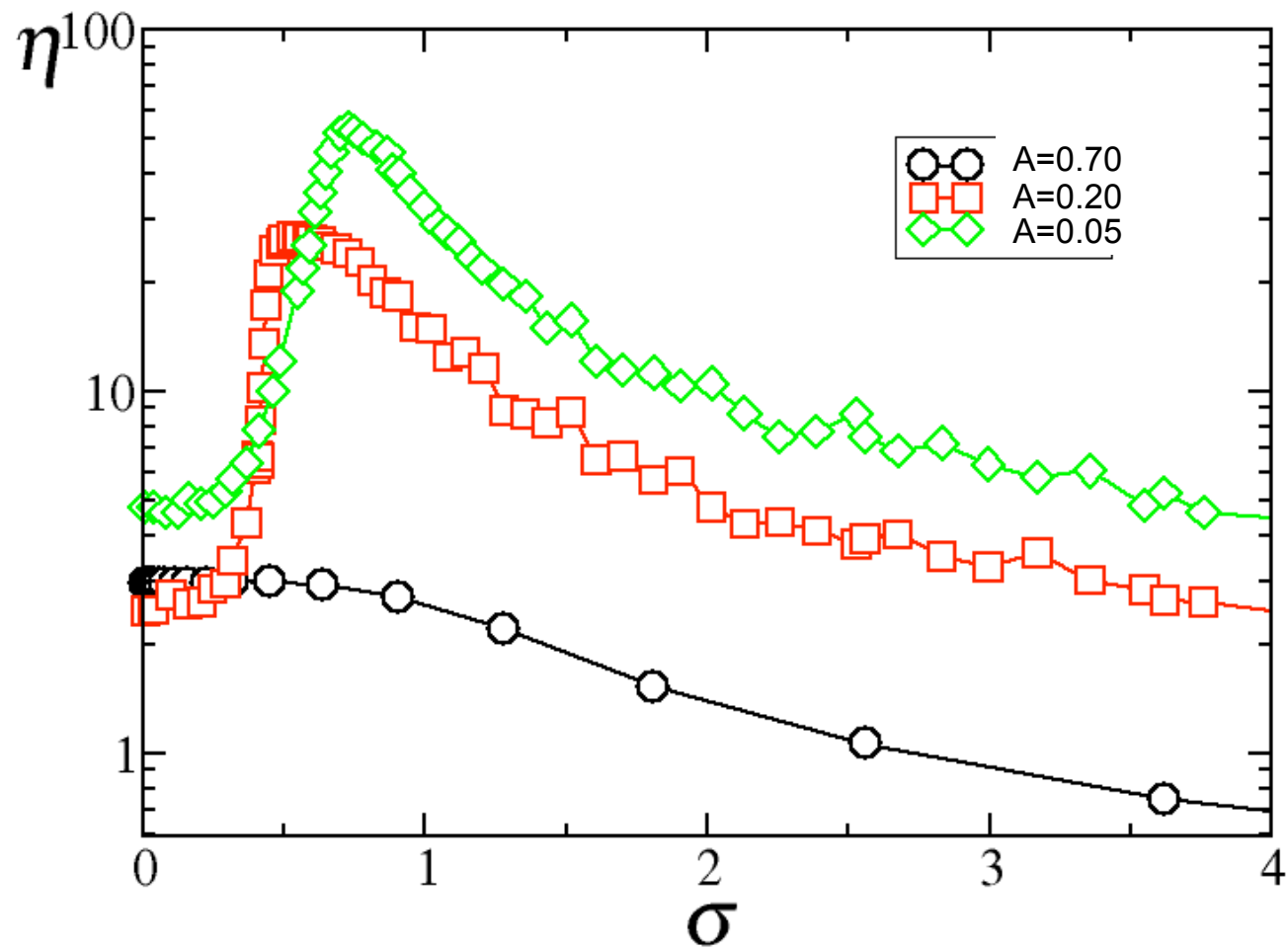
Effective global potential

$$V(X, M) = \frac{X^4}{4} - \frac{X^2(1 - 3M)}{2} \quad \frac{1}{N} \sum_i \delta_i^2 = M$$

- No diversity implies $M=0$ because of the coupling
- Increasing diversity, M increases and the barrier between minima lowers. Eventually, the system undergoes a transition from a bistable to a monostable potential

This is independent on the mechanism that originates the increase in M





Bistable system: *summary*

We have demonstrated the existence of a mechanism by which a weak (sub-threshold on the average) forcing is optimally followed by a large coupled bistable system in the presence of diversity

How general is this phenomenon?

Excitable system

As a paradigmatic example of excitable system, we studied the FitzHugh-Nagumo model

$$\begin{aligned}\epsilon \dot{x} &= x - \frac{1}{3}x^3 - y \\ \dot{y} &= x + a + D\xi(t)\end{aligned}$$

For a single unit, for $|a| < 1$ The system is oscillatory; otherwise, it is excitable (there is a Hopf bifurcation on this parameter).

Let us couple many, diverse, FitzHugh-Nagumo units

$$\epsilon \dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{k}{N} \sum_{j=1}^N (x_j - x_i)$$

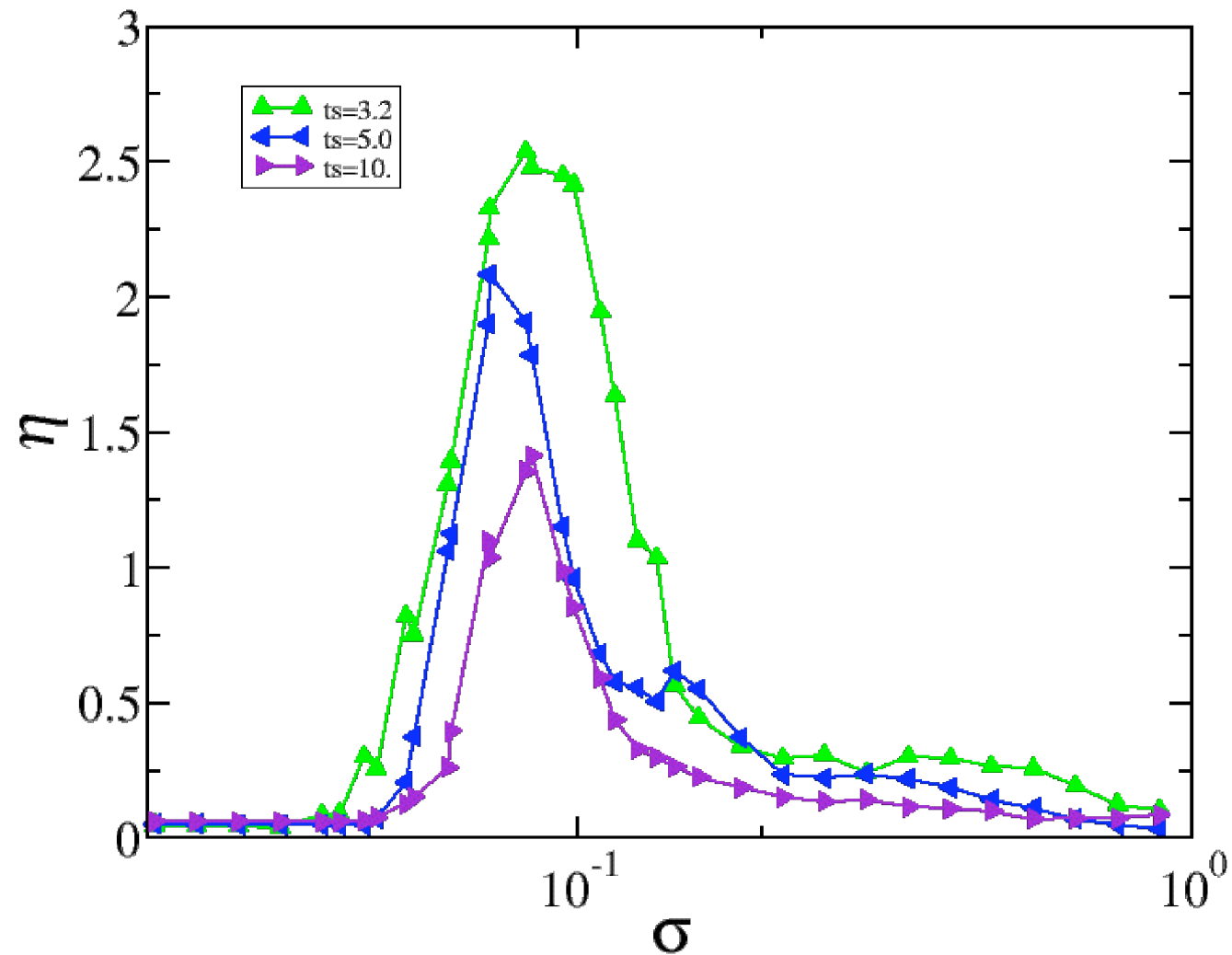
$$\dot{y}_i = x_i + a_i + \cancel{D\xi_i(t)} + A \sin\left(\frac{2\pi}{T_s} t\right)$$

$D=0$

The diversity is put in this parameter, the one that controls the bifurcation, distributed according to a Gaussian distribution of mean a and variance σ^2

applet

Excitable system: *numerical results*



Model for opinion formation

[M. Kuperman, D. Zanette, *Eur. Phys. Jour. B*, 26 387 (2002)]

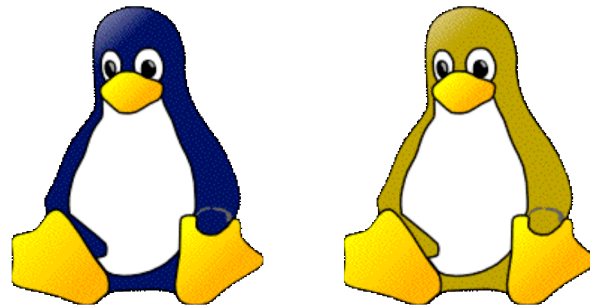
Opinion is a binary variable:

Individuals have an opinion: $\mu_i = +1, -1$

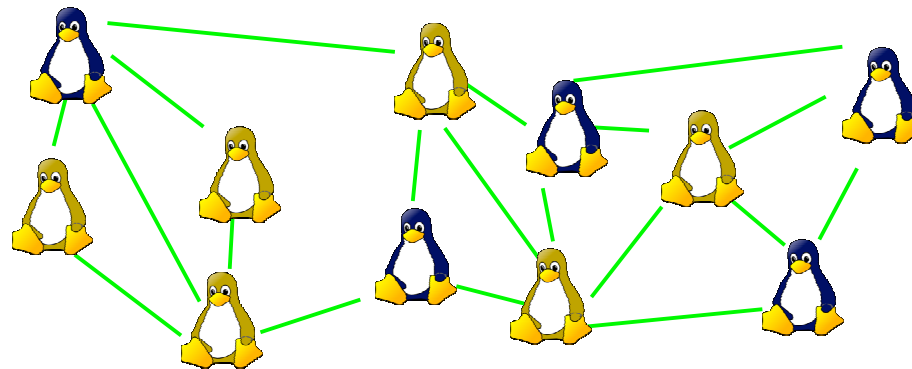
Opinion $\mu_i(\mathbf{t})$ changes by 3 effects:

- **Social pressure**
- **Advertising**
- **Random effects**

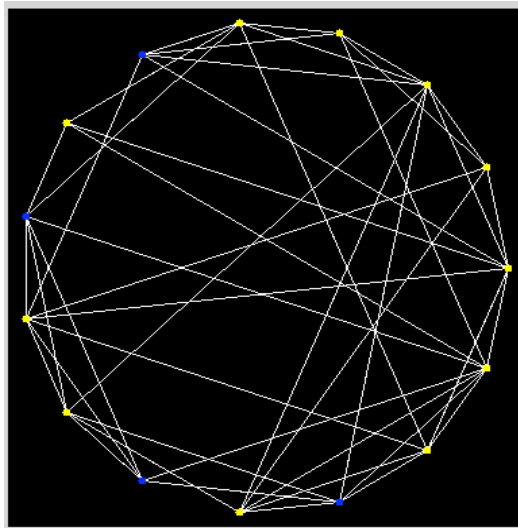
-System formed by N individuals which have one of two opinions



-Each individual has a set of neighbors



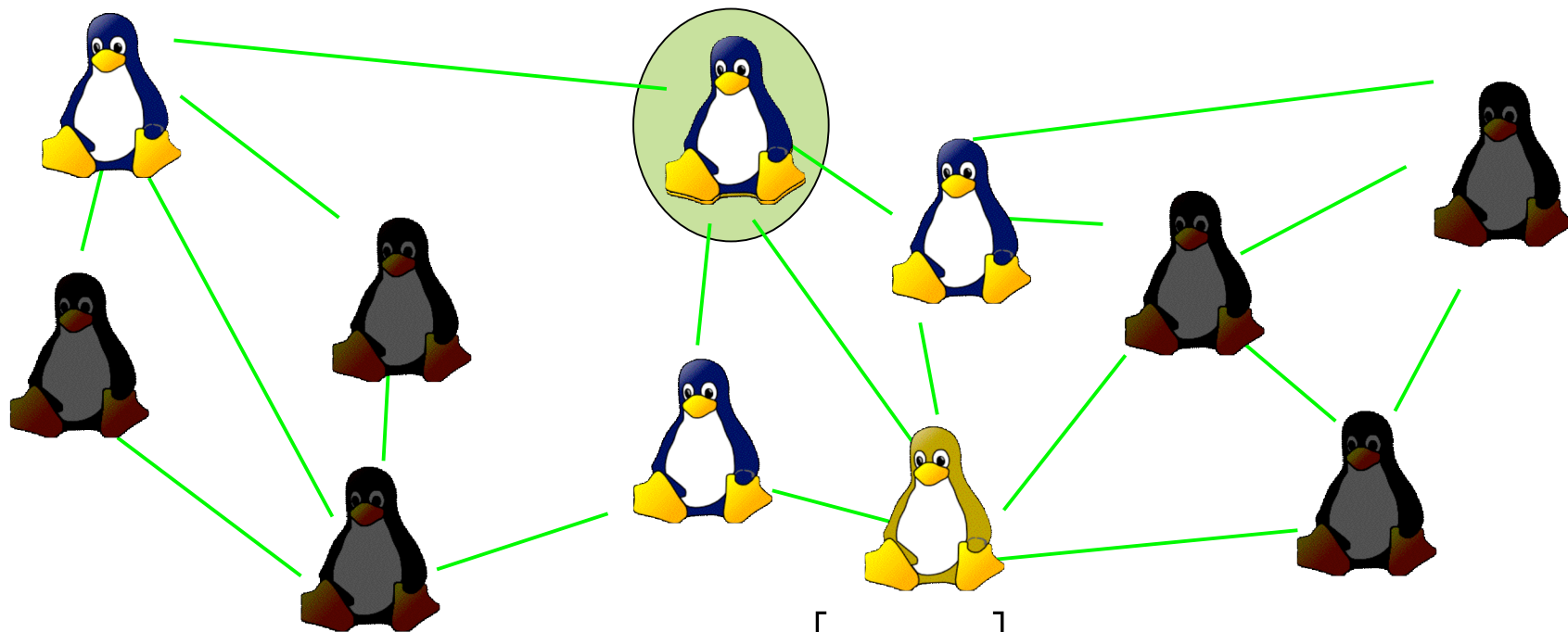
The network of neighbors is a small-world one:



- ❖ A regular 1D network built up to k neighbors
- ❖ with probability p each link is rewired, i.e. another destination node is selected

- Opinion Update: 1 social pressure

An individual is randomly chosen and takes the majority opinion of his neighbors



$$\mu_i(t) = \text{sign} \left[\sum_{j \in n(i)} \mu_j(t) \right]$$

- Opinion Update: 2 advertising

The preference for one of each opinions is assumed to change periodically in the form

$$\cos(\omega t) \begin{cases} < 0 \\ > 0 \end{cases} \begin{cases} \longleftrightarrow \text{Blue Penguin} \\ \longleftrightarrow \text{Yellow Penguin} \end{cases}$$

With probability $|\varepsilon \cos(\omega t)|$ the favored opinion is taken

- Opinion Update: 3 *random choice*

With probability η a random opinion is taken



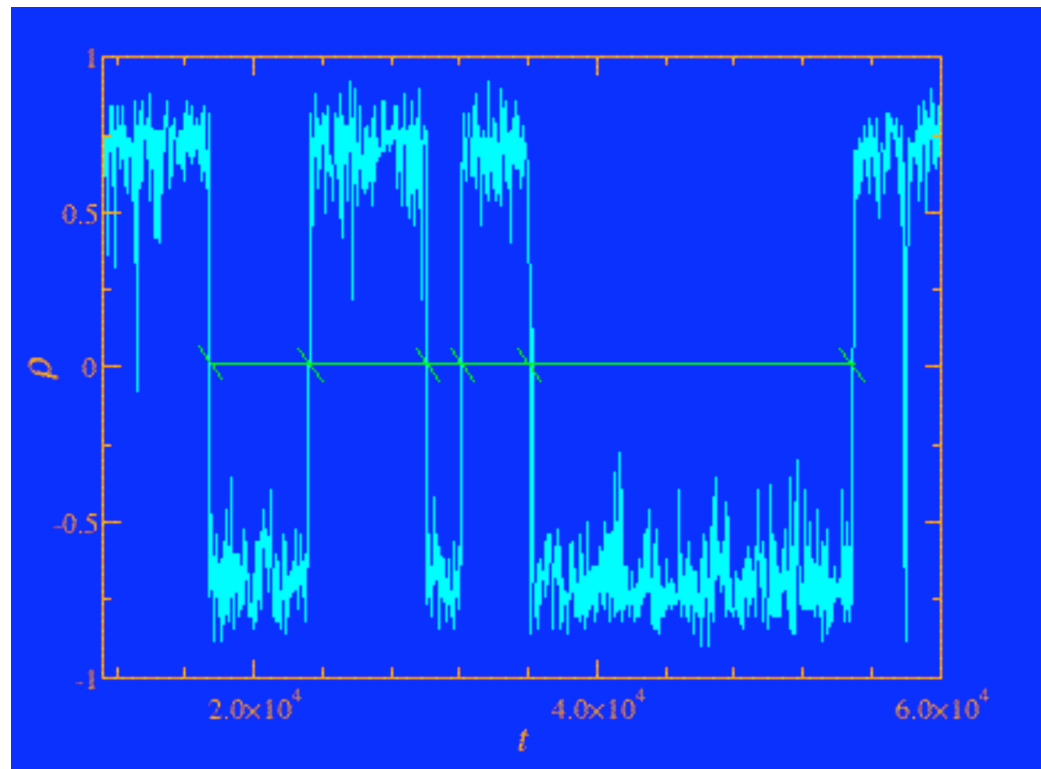
-Steps **1,2,3** are applied ***CONSEQUENTLY***

-After each repetition, t increases by $1/N$

Results: Dynamical Evolution

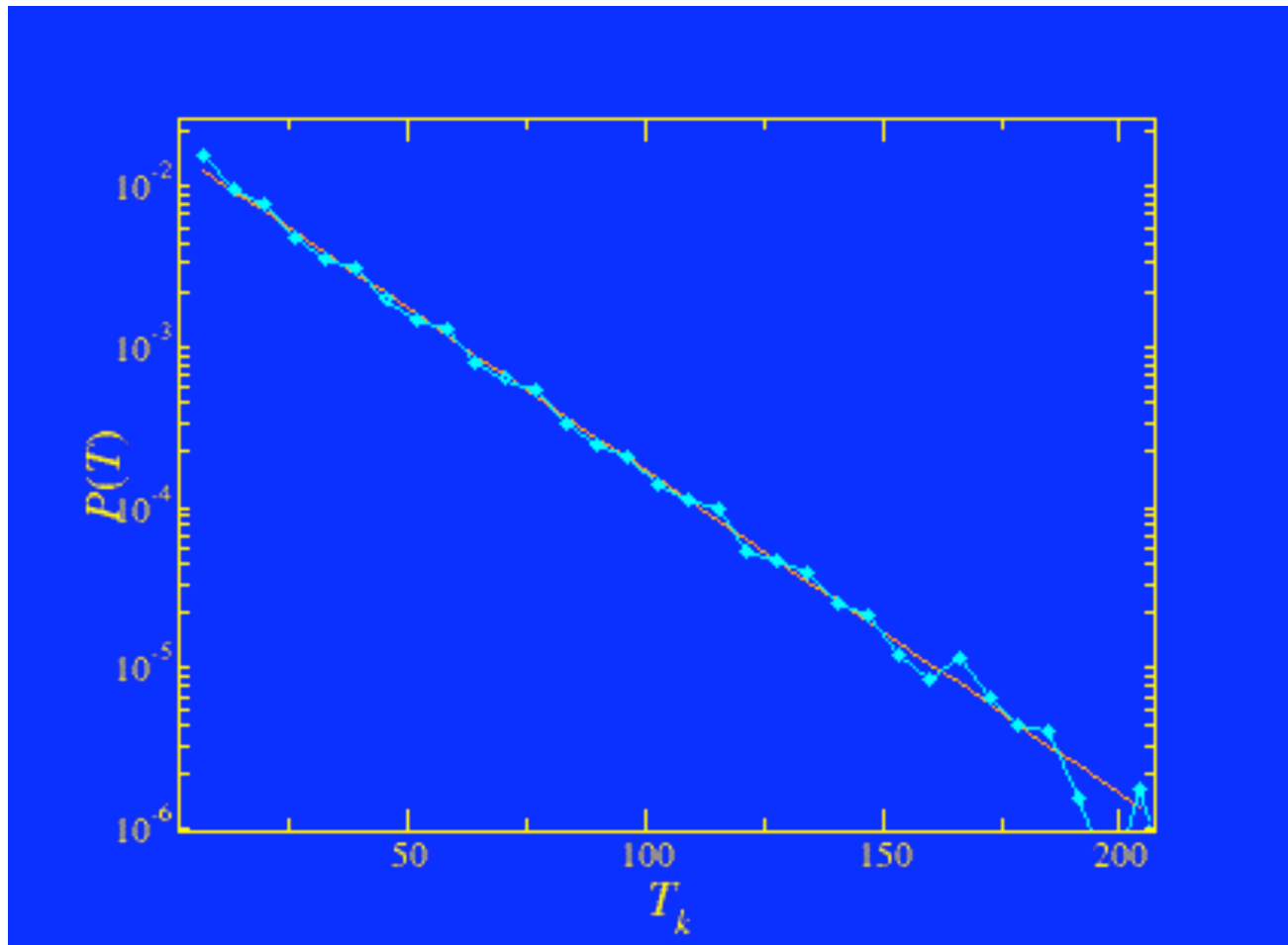


i) In absence of external forcing, behaves as a bistable system

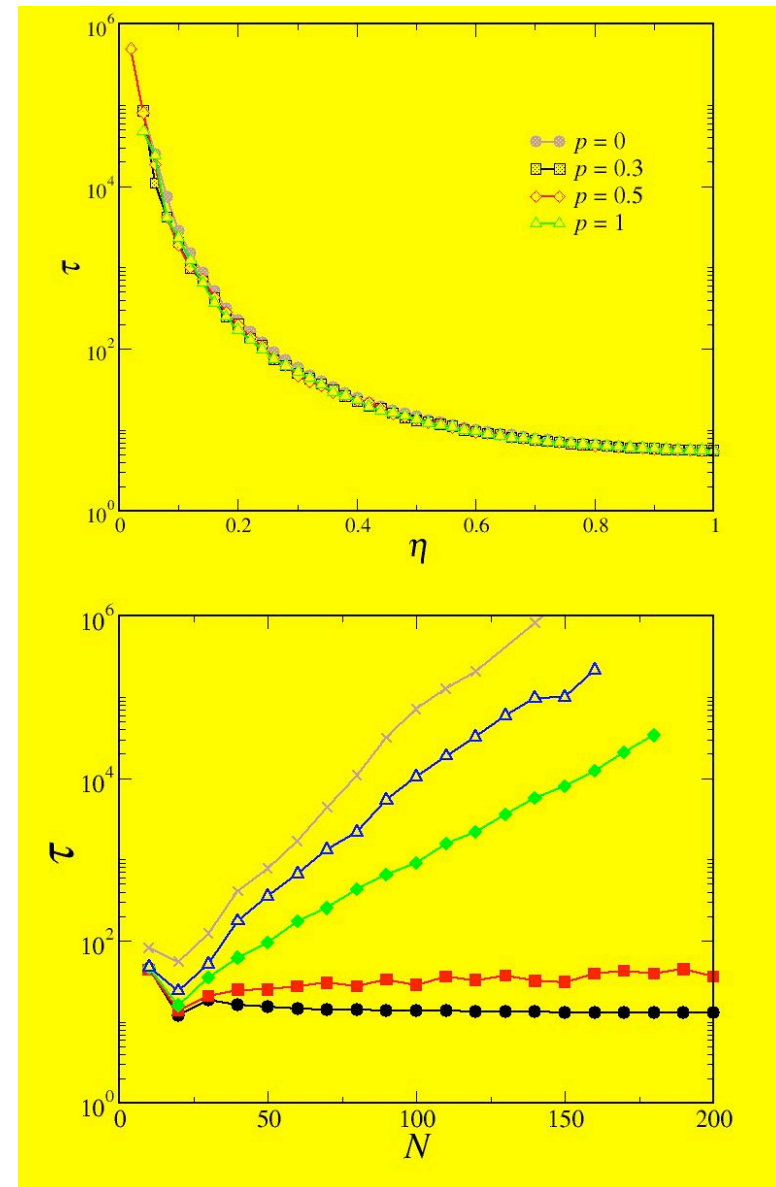


ii) the residence time distribution follows Kramer's law

$$p(T) = \tau \exp(-T/\tau)$$



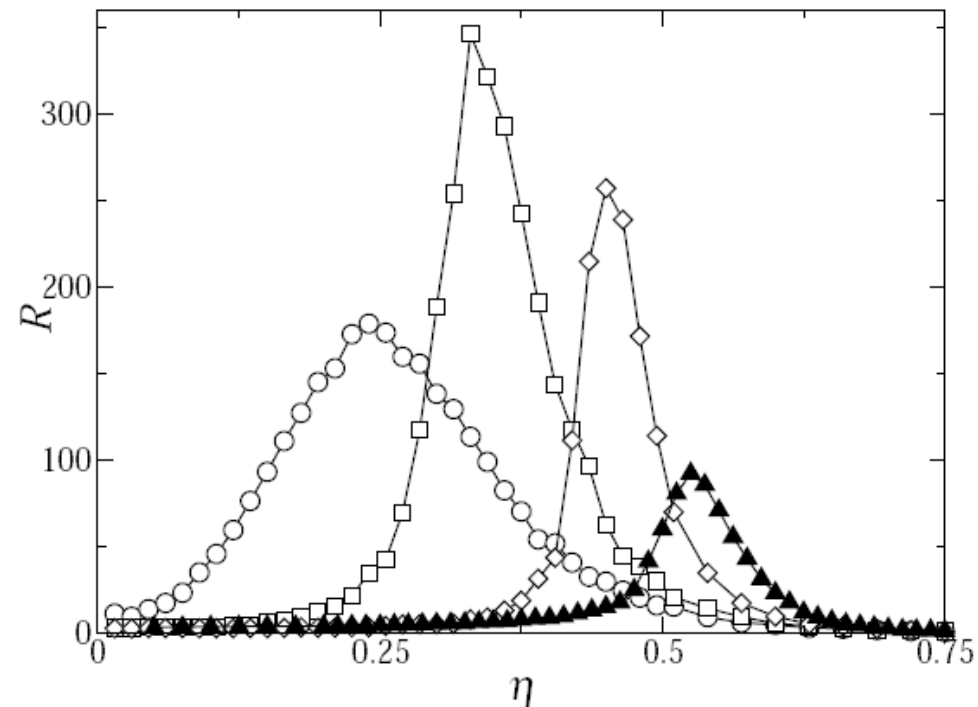
τ decreases with noise rate



τ increases with systems size

-We have all the ingredients for stochastic resonance:

- Bistable system (two opinions)
- Coupling (social pressure)
- Noise (random choice “free will”)
- External forcing



Diversity introduced through *preferences* θ_i :

$$\langle \theta_i \rangle = 0, \quad \langle \theta_i^2 \rangle = \sigma^2$$

Step 1: social pressure

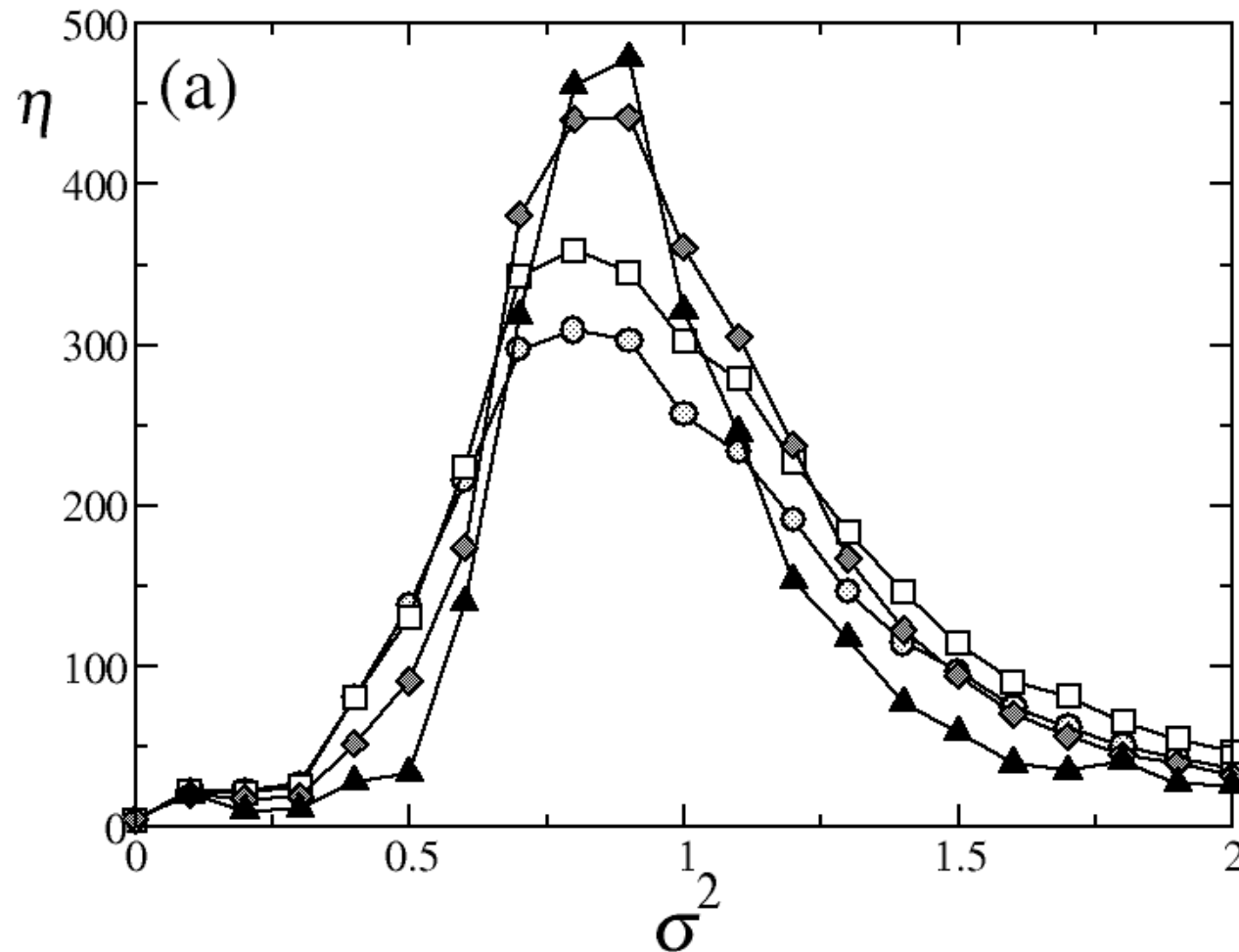
$$\mu_i(t) = \text{sign} \left[\sum_{j \in n(i)} \mu_j(t) + \theta_i \right]$$

Step 2: advertising

With probability $|\varepsilon \sin(\omega t) + \theta_i|$

set $\mu_i(t) = \text{sign}[\sin(\omega t) + \alpha \theta_i]$

Diversity-induced Resonance



Conclusions

- In bistable coupled systems, there is an optimal amount of the dispersion that optimizes the response to an external forcing.
- This is a rather general phenomenon similar to that of Stochastic Resonance. It also happens in excitable systems.
- In a majority opinion formation model, an external influence works optimally under the right amount of diversity in individual preferences.
- Any source of disorder leads to similar results
- Real systems could, or could not, be taking advantage of these mechanisms. Experimental verification in some candidate systems is most welcome to check whether these diversity-induced effects are taking place.

Thank you... for being different.

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