Normal Mode Analysis with the Chesapeake Bay

Under Non-Optimal Conditions

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Normal Mode Analysis (NMA)

- Chesapeake Bay focused
- Calculate eigenmodes using a Helmholtz decomposition
- Use the eigenmodes as an orthonormal basis set
- Decompose a data set (velocity fields) into their modal components (amplitudes)
- Produce a time-dependent power spectrum
- Make predictions using eigenmodes over full domain

Previous Work

- Zel'Dovich - 1985 - vector decomposition
- Eremeev, Kirwan, et al. - 1992 - Black Sea
- Lipphardt, Kirwan, et al. - 2000 - Monterey Bay

NMA - Difficulties

- Computing eigenmodes over complex boundaries can be questionable (how close to ortho-normal).
- NMA requires full data sets (in space).
- Hardware sensors collect data sporadically (time irregular).
- Eigenmodes exhibit two behaviors (global and local).
- Differing methods exist to extract amplitudes (Galerkin, LSHA, other fitting).

NMA – Non-Optimal Conditions

- Chesapeake Bay – less than perfect
  - Large estuary – 180x50 miles
  - Significant sources – Atlantic Ocean, large watershed, five major rivers
  - Boundary – fractal
  - Geometric variation – many regions of varying sizes
  - Man-made sources – several major cities, nuclear reactor
  - Salinity variation – southern – salty, northern – fresh
  - Shallow water bathymetry – average depth 8.4m, max ~30m
  - Significant biological component – hypoxia
  - Community/Political pressure to act – Executive Order (2009)

NMA – Non-Optimal Conditions

- Few data collection stations – approx. 30 online
- At each location, collect time-series of many variables (water height, velocity vector, salinity, temperature,…)
- Given a limited number of extracted modes, can NMA meet its goals in the Chesapeake Bay?
- Previous systems (Black Sea, Monterey Bay)
  - Succeeded on simpler geometries.
  - Enjoyed richer data sets (Monterey has 70% coverage).
  - Found 85% of kinetic energy populated in low numbers of modes (Black Sea).
  - Nowcasts provide current information of estuary (full domain prediction).

NMA

- Problem Statement: Can information be taken at a few select locations (under-sampled) over long periods of time (full time series) such that the spatio-temporal data set is full enough to extract amplitudes well.

Eigenmodes – features

- Helmholtz Decomposition
  - Eigenmodes are the solution to Helmholtz equation.
  - Dirichlet (vorticity) / Neuman (divergence) boundary conditions used
  - Velocity vector fields calculated – $\mathbf{u}_{D,n}(\mathbf{x},t)$, $\mathbf{u}_{N,n}(\mathbf{x},t)$
  - Eigenvalues track with geometric length scales.
    - Leads to global modes and local modes.
- In order to provide 10+ eigenmodes over all regions of the Chesapeake Bay, 100+ eigenmodes need to be used.

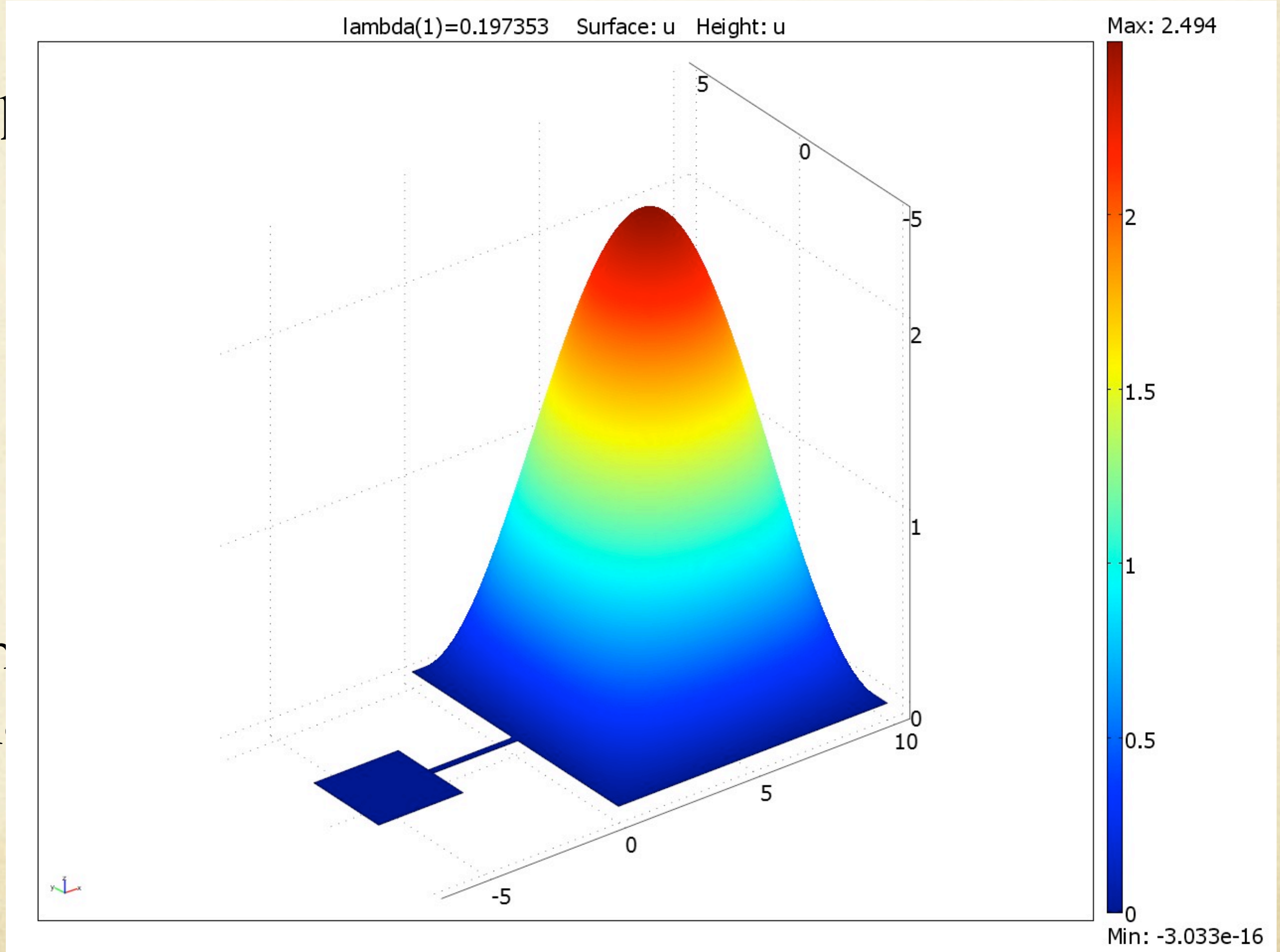
Eigenmodes - Interpretation

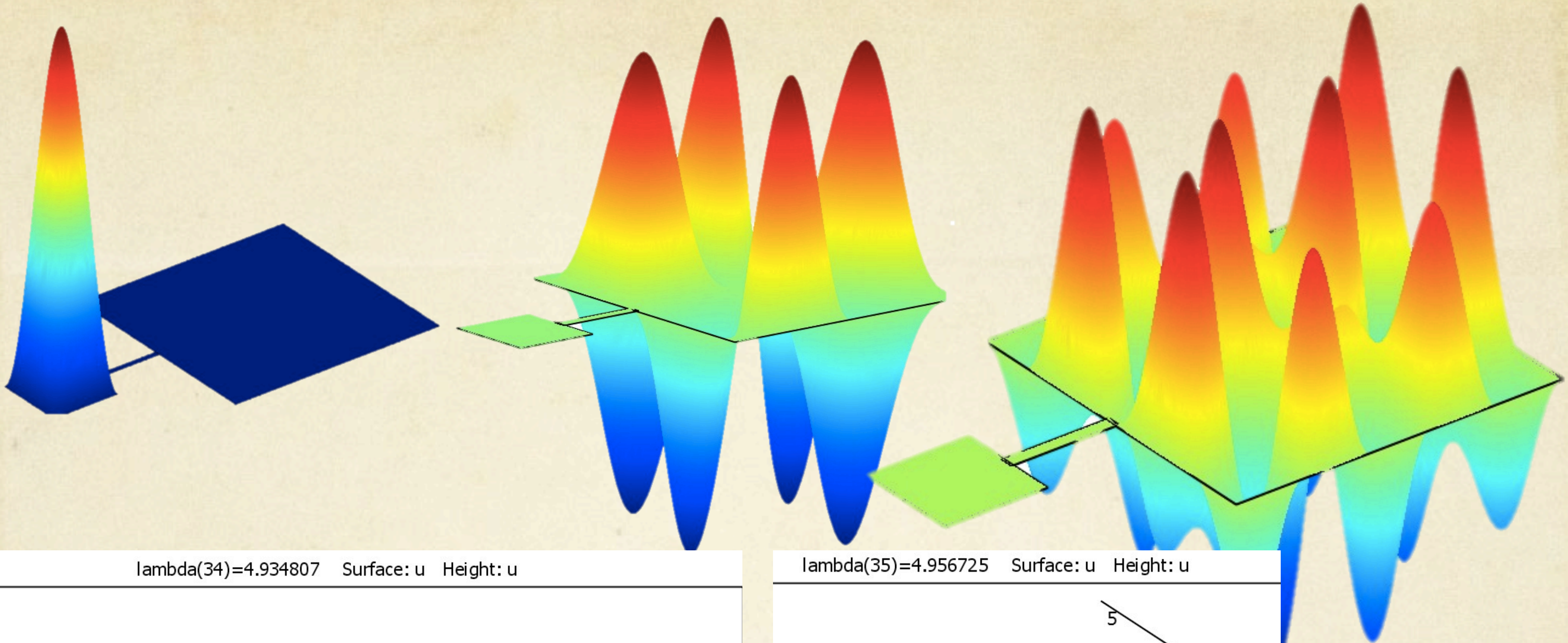
- A simple ortho-normal basis set (an alphabet).
- A physical mode related to dynamics of the system.
- Physical modes should exhibit differing time-scales (diffusion).
- Question: What are the appropriate time-scales associated with modal structure?
- How quickly can energy be transferred from one mode to (FPU):
  - Another mode?
  - Many modes?
  - All modes (thermalize)?

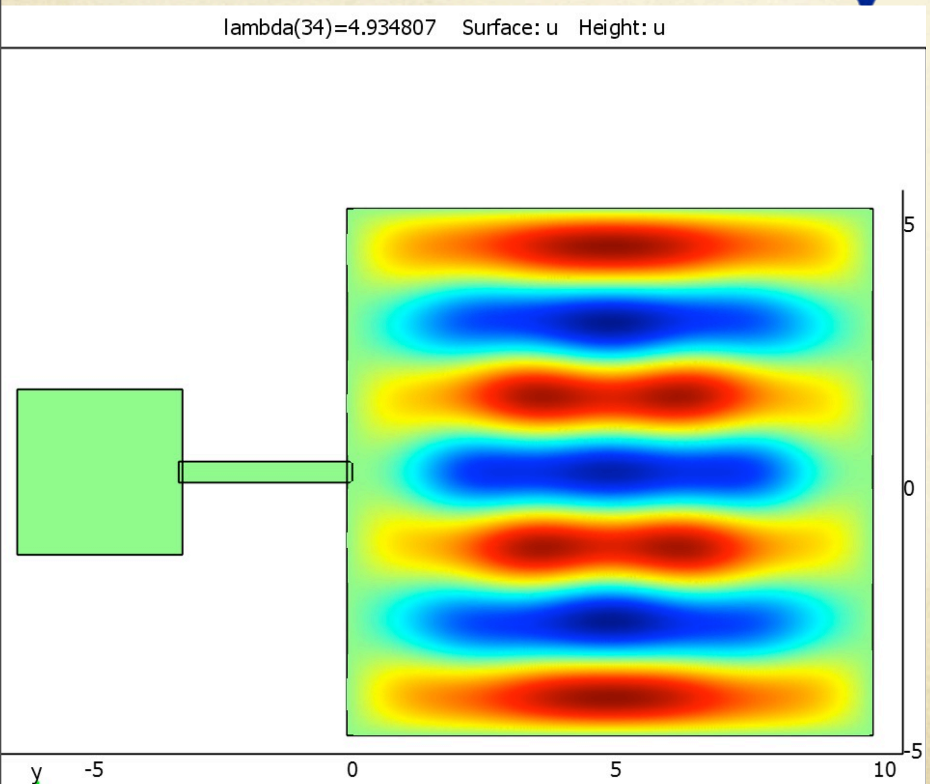
Eigenmodes – global/local

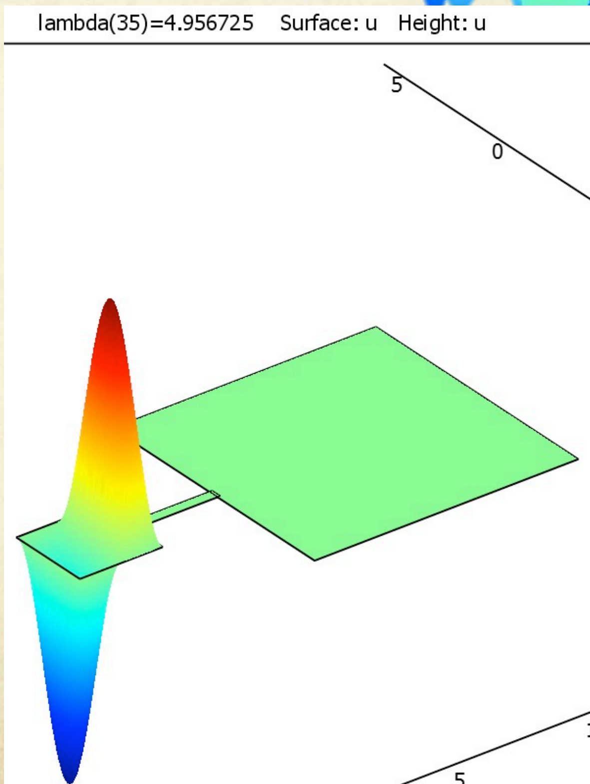
- Global modes – normal kind which cover the domain

- Local modes – only fit spatially in one location in the domain



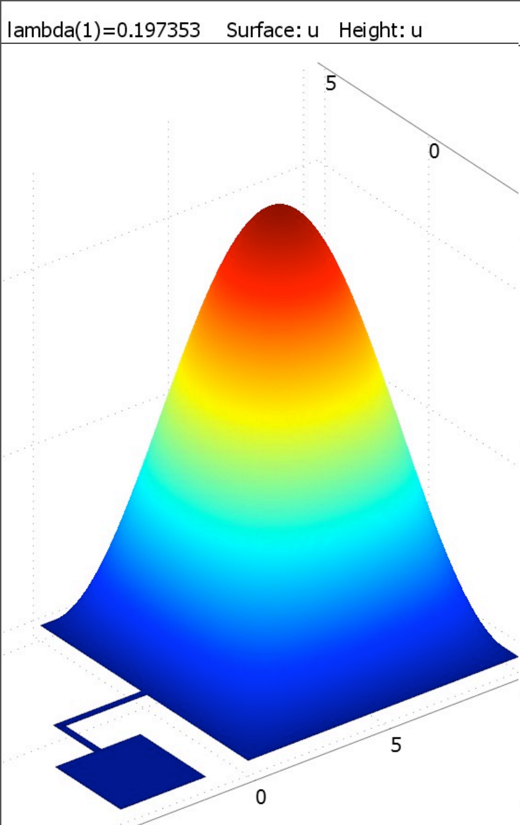


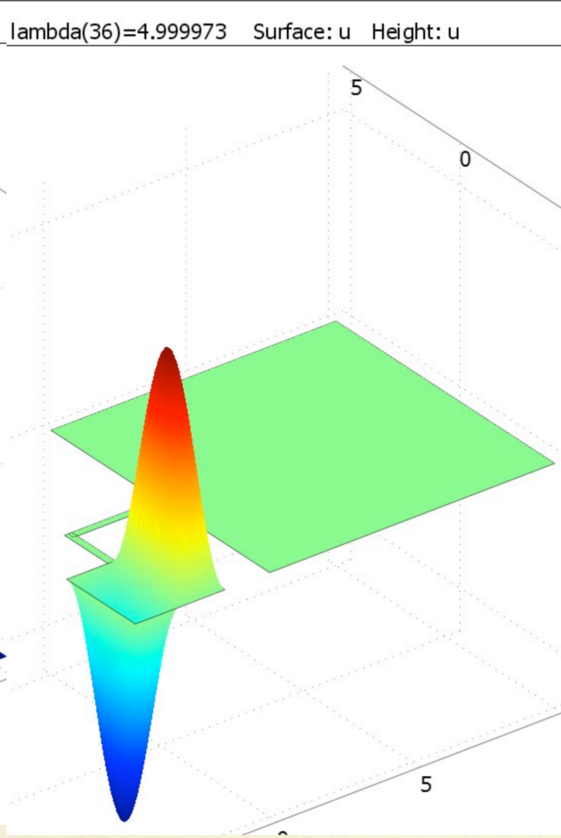


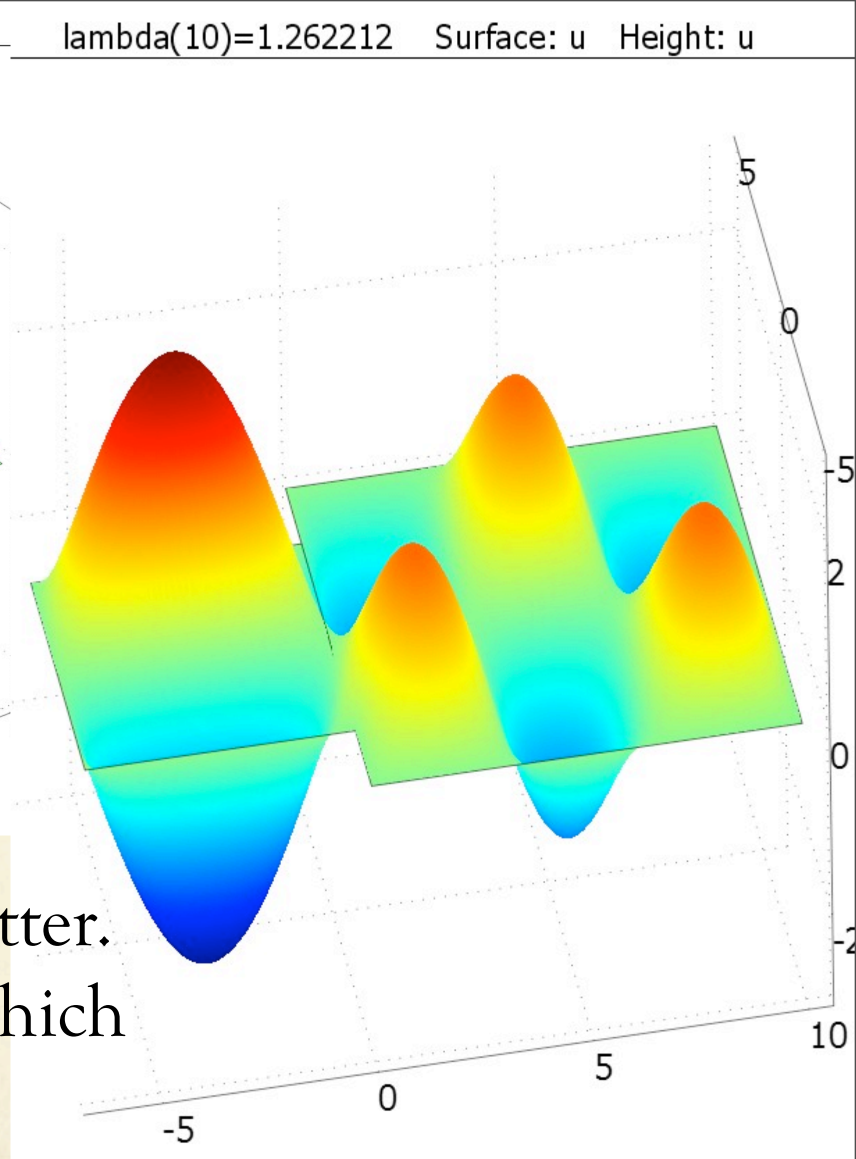


Very sensitive to eigenvales (lengths).

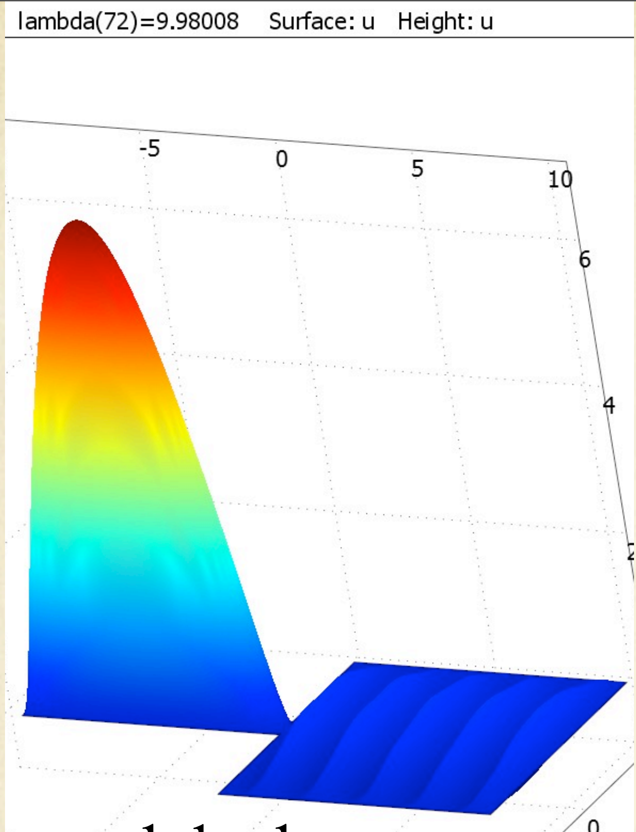
$$\lambda \quad = \pi^2 / L^2 * (n_x{}^2 + n_y{}^2)$$

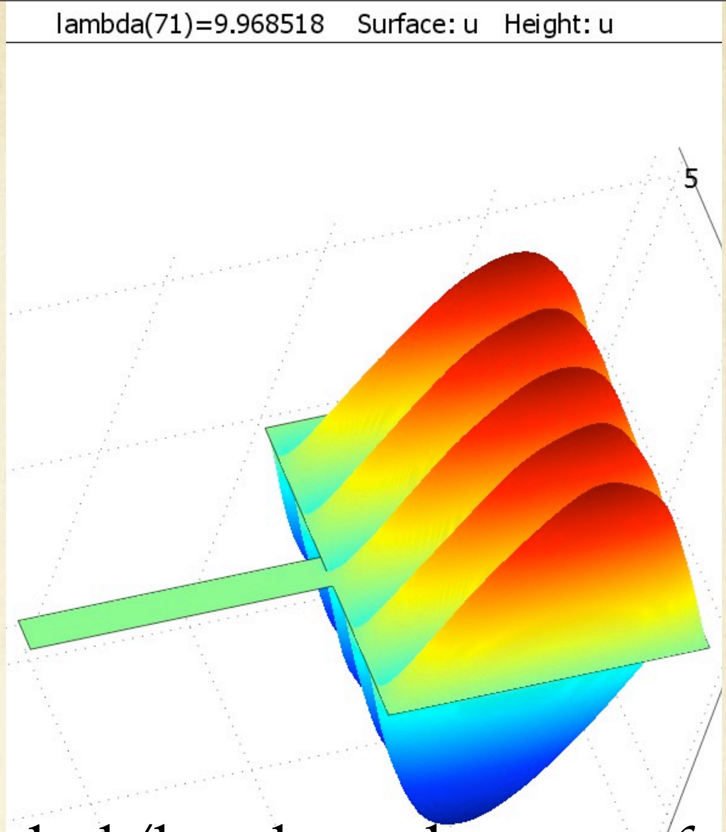






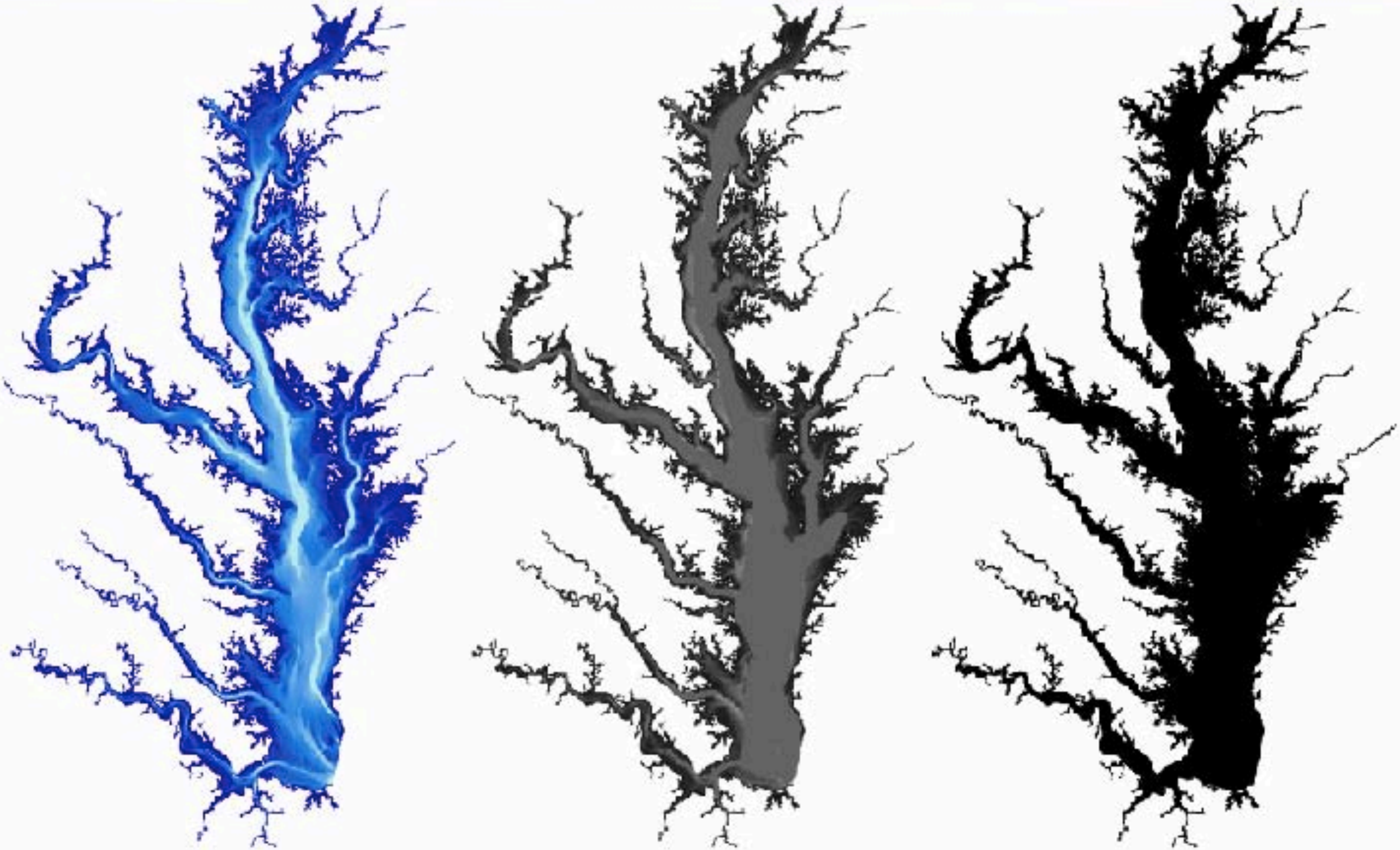
- The connecting path does not matter.
- There can be degenerate modes which are global/local.
- When allowed, modes can exist which stretch one wavelength and compress another to accommodate a compromised mode.



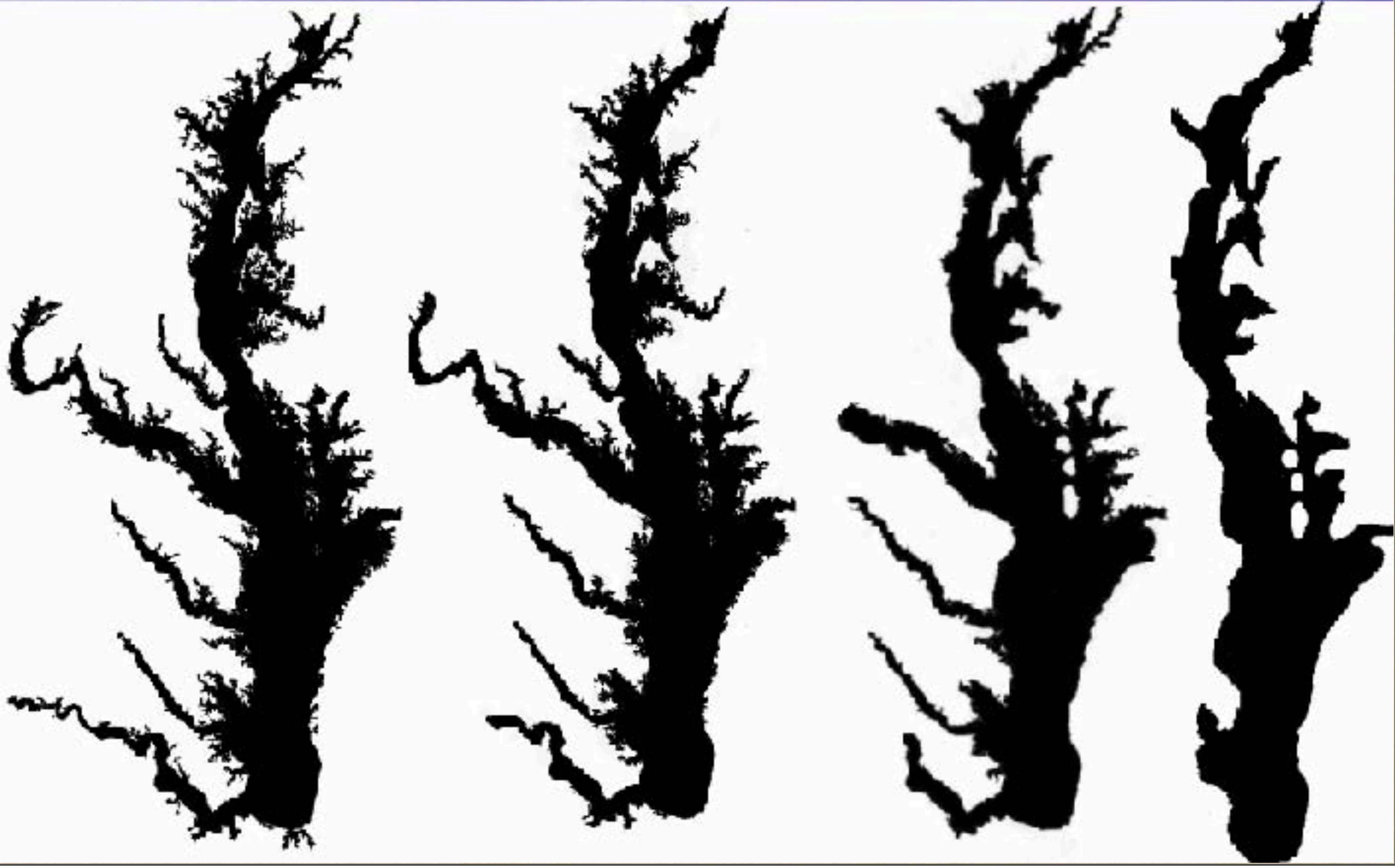


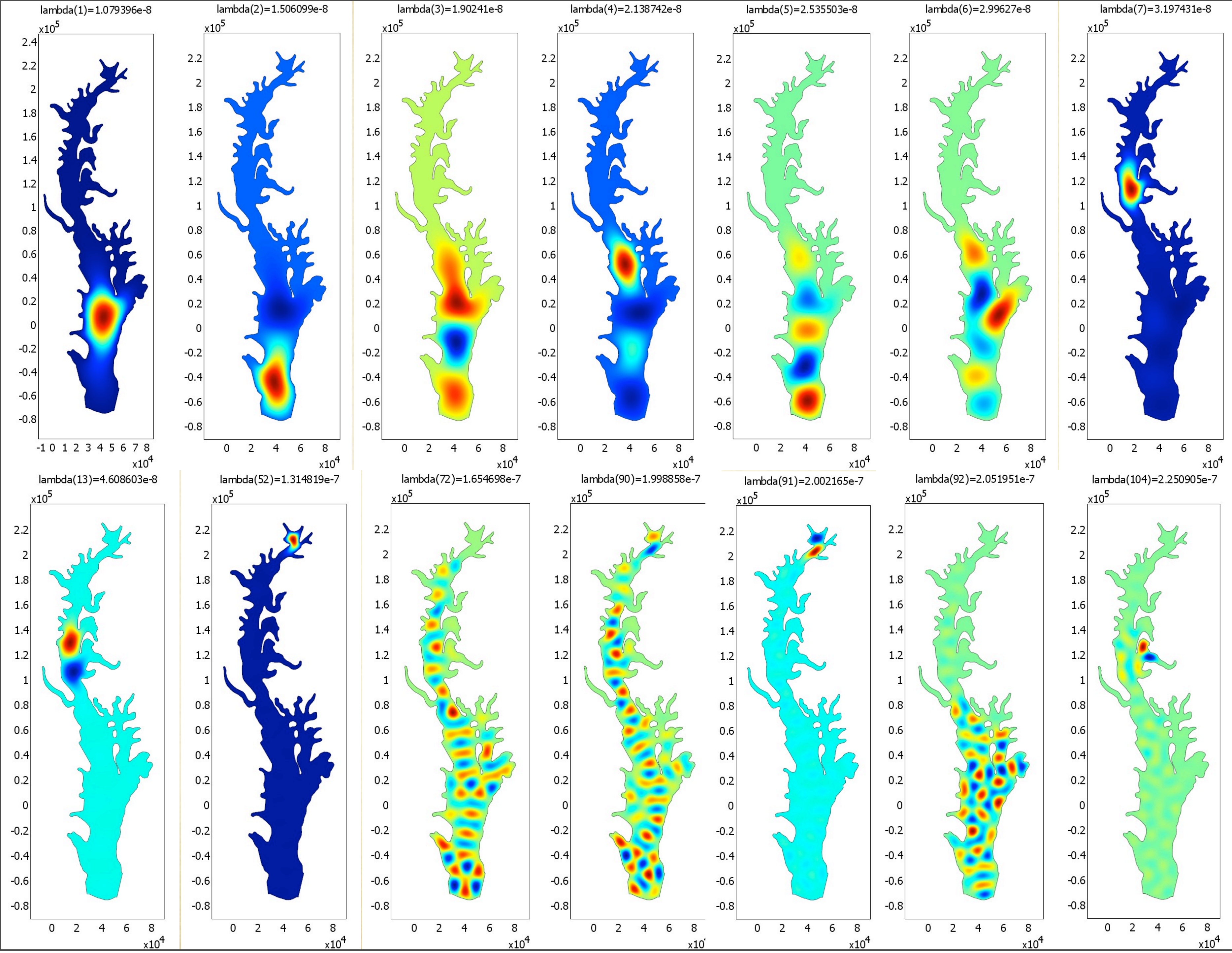
- Toy models demonstrate the global/local modes exist for many simple geometries. Such as:
    - Two geometries "connected" by waterway.
    - Bay/Ocean.
    - River/Bay.
- Global/local mode structures should be expected whenever lengths cover multi-scales.

Image Processing of the Chesapeake



Approximated Boundaries

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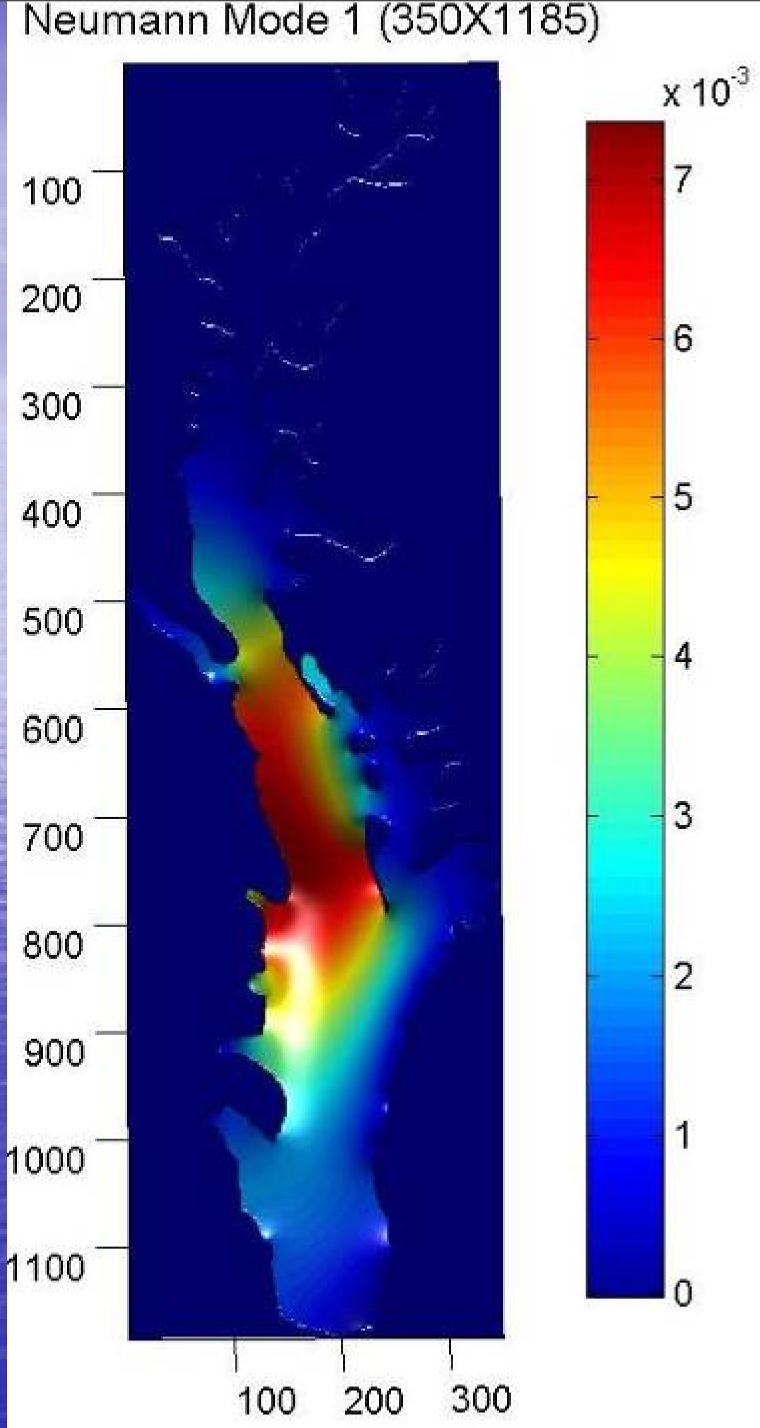
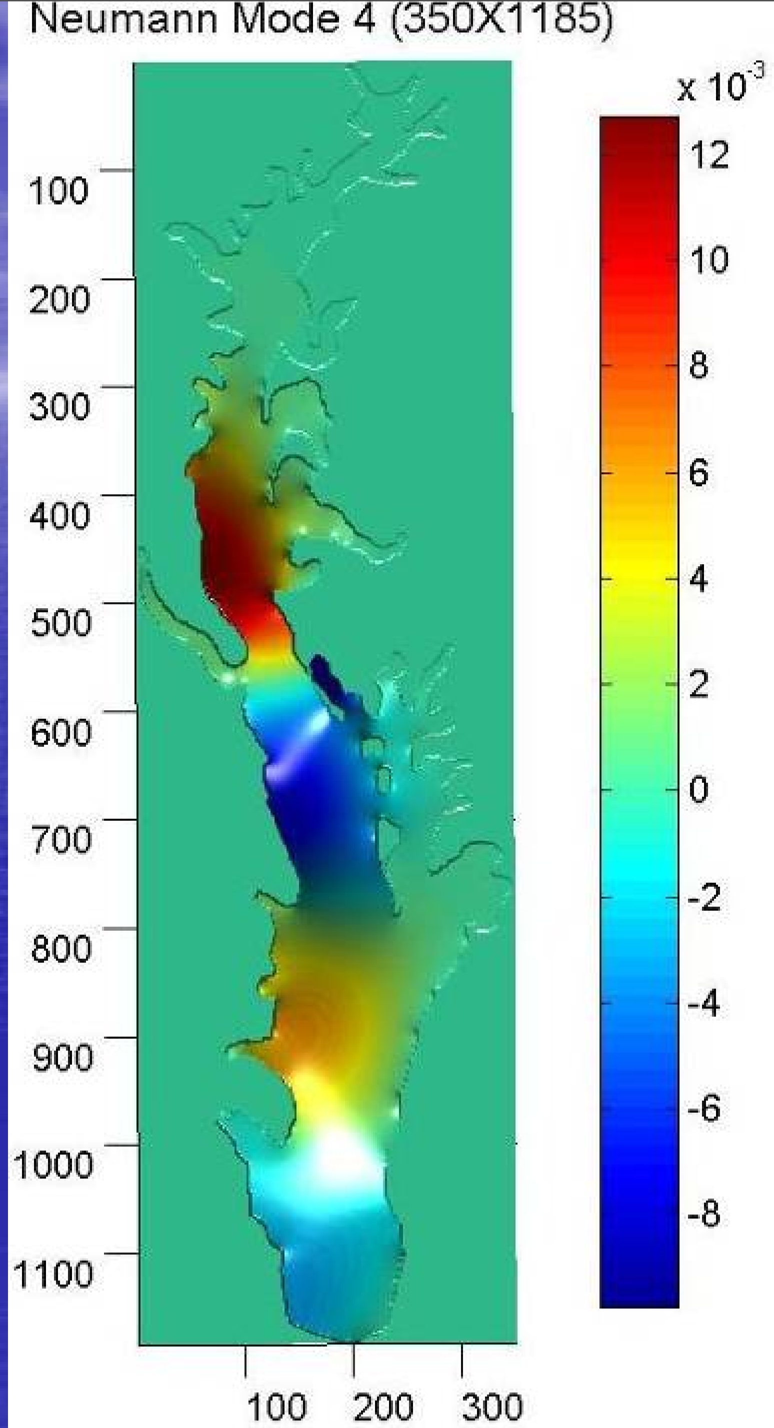
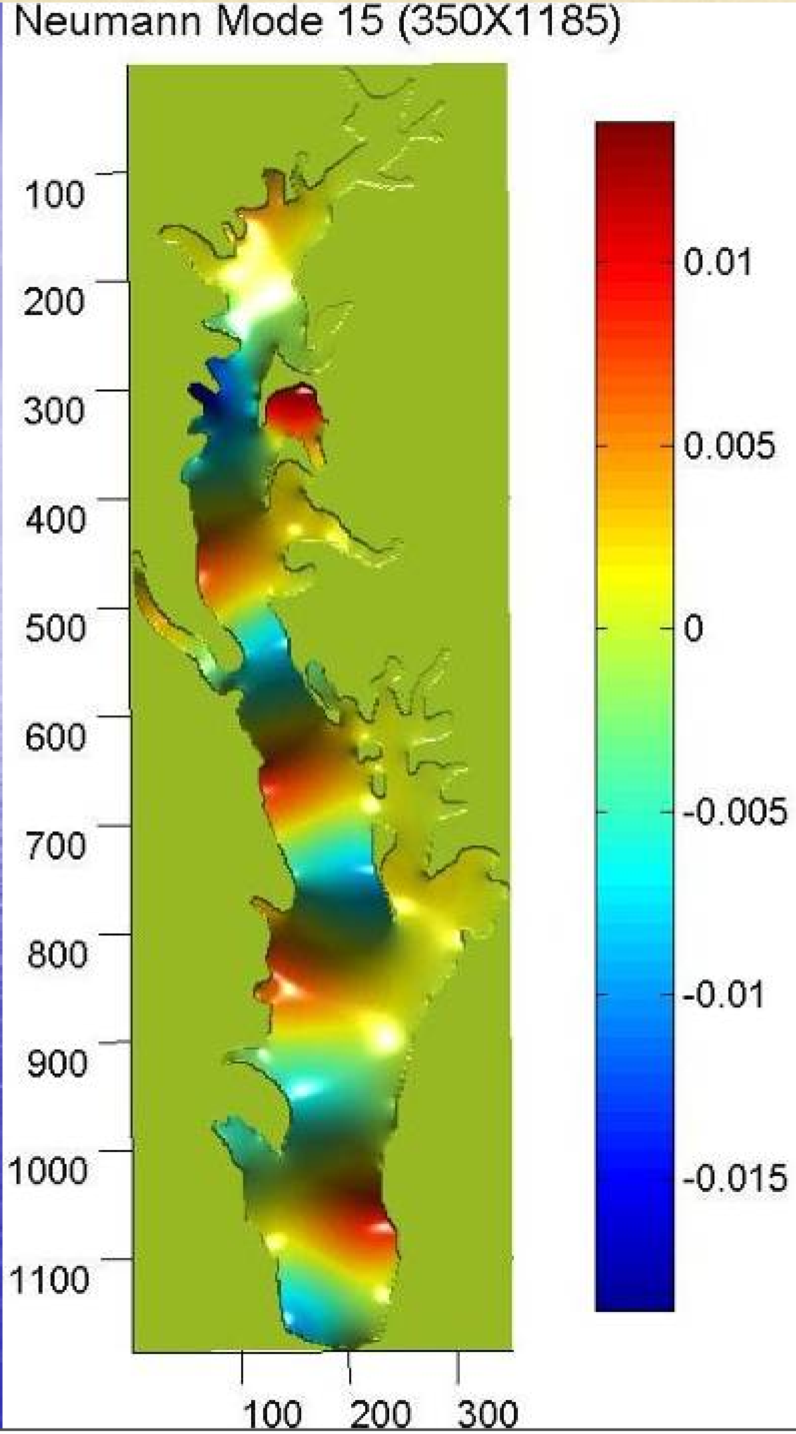
lambda(2)=1.506099e-8
lambda(3)=1.90241e-8
lambda(4)=2.138742e-8
lambda(5)=2.535503e-8
lambda(6)=2.99627e-8
lambda(7)=3.197431e-8
lambda(13)=4.608603e-8
lambda(52)=1.314819e-7
lambda(72)=1.654698e-7
lambda(90)=1.998858e-7
lambda(91)=2.002165e-7
lambda(92)=2.051951e-7
lambda(104)=2.250905e-7
x10^5
x10^4

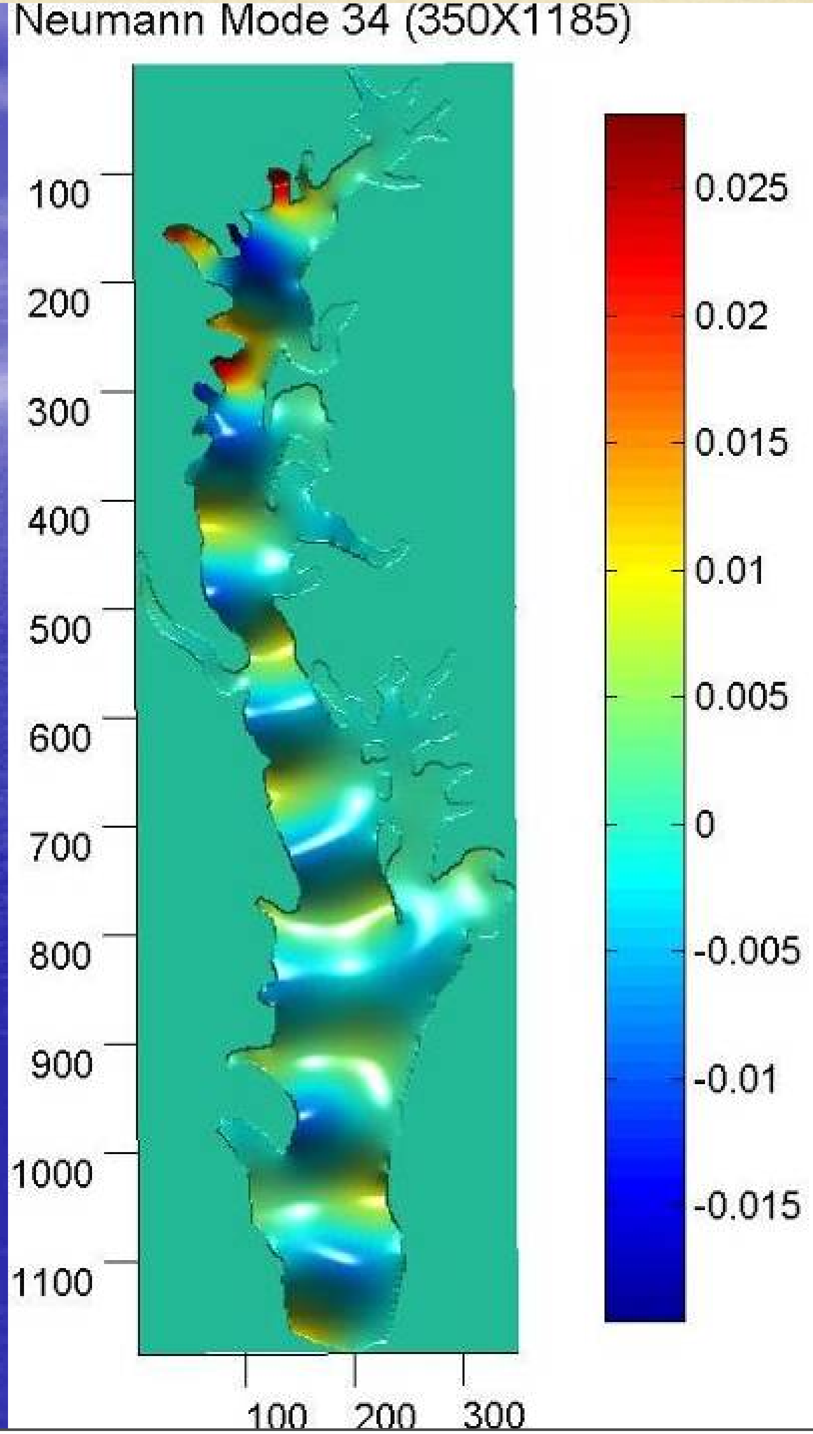


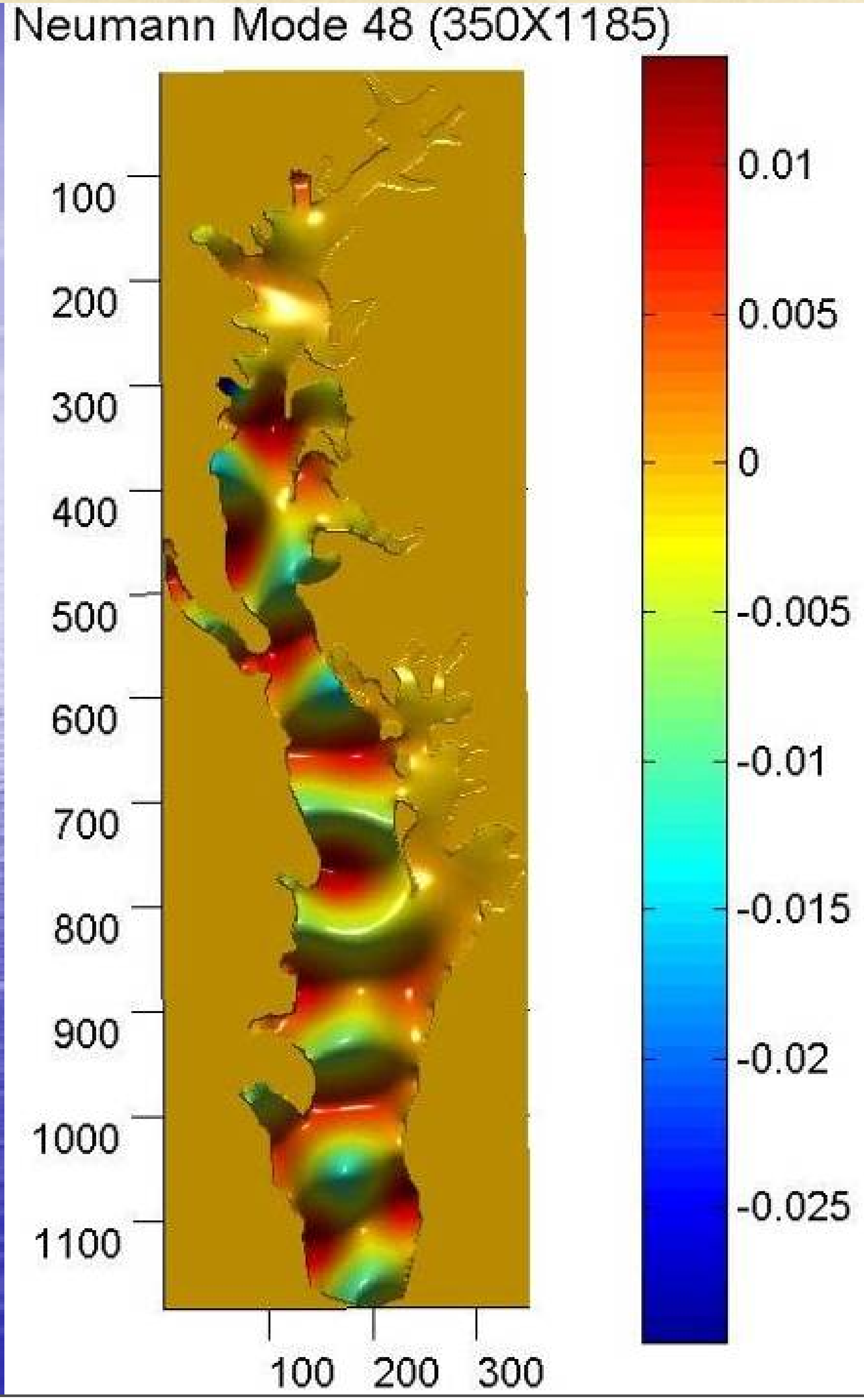
Neumann Mode 1 (350X1185)
x 10^-3
0
1
2
3
4
5
6
7
100
200
300
400
500
600
700
800
900
1000
1100
100
200
300

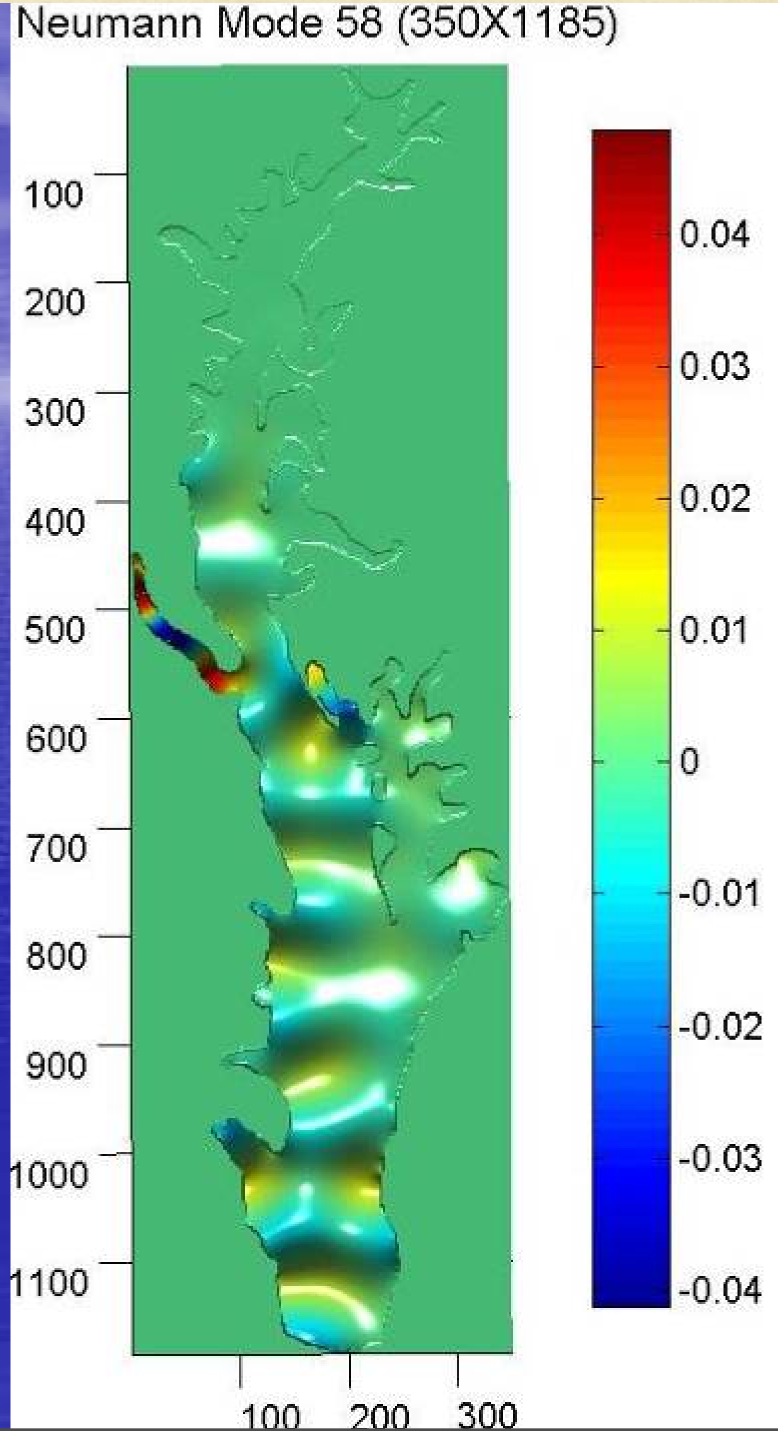


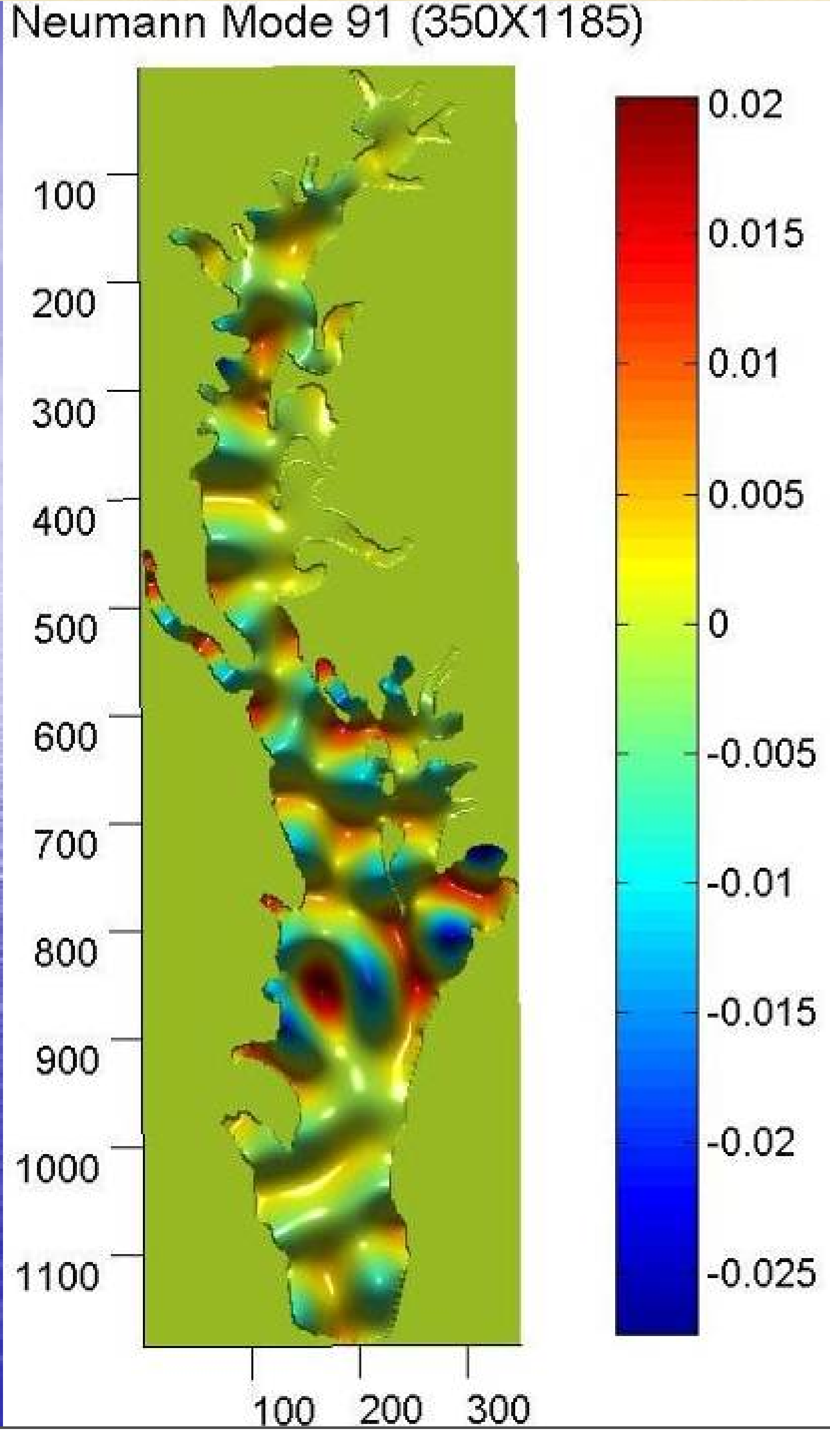
Neumann Mode 4 (350X1185)
x 10^-3
12
10
8
6
4
2
0
-2
-4
-6
-8
100
200
300
400
500
600
700
800
900
1000
1100
100
200
300

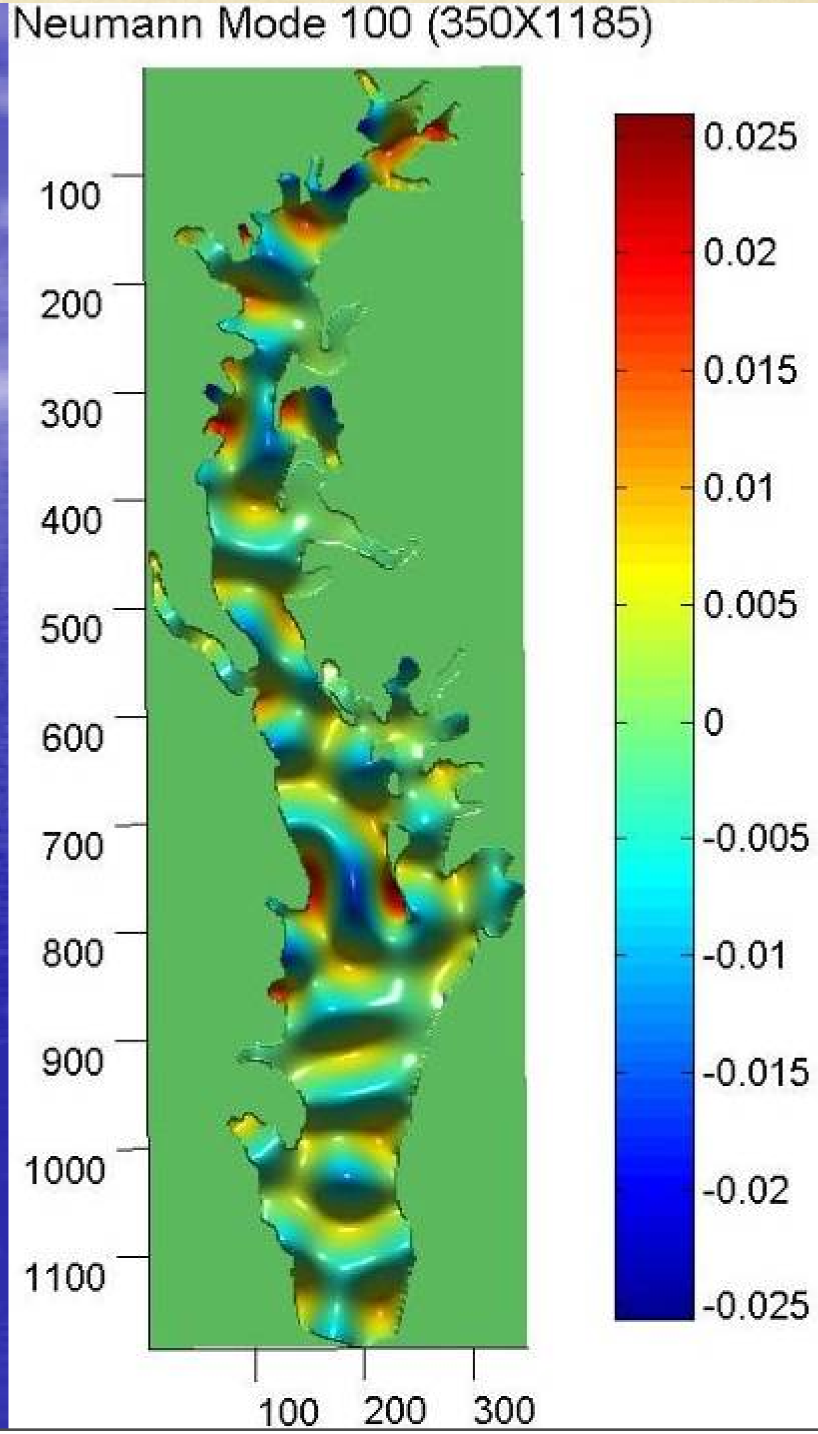

Neumann Mode 15 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.01
0.005
0
-0.005
-0.01
-0.015


Neumann Mode 34 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015


Neumann Mode 48 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025


Neumann Mode 58 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04


Neumann Mode 91 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025

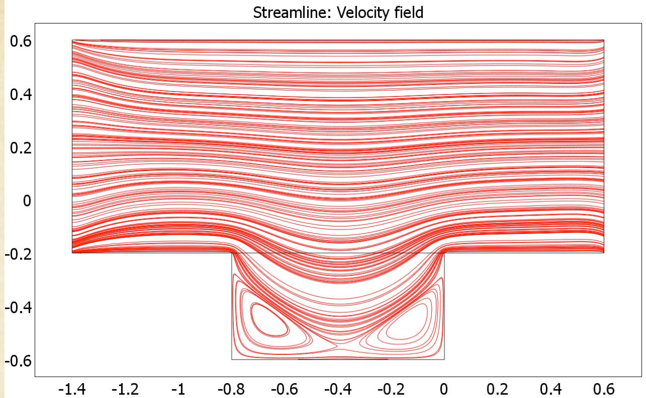

Neumann Mode 100 (350X1185)
100
200
300
400
500
600
700
800
900
1000
1100
100 200 300
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025

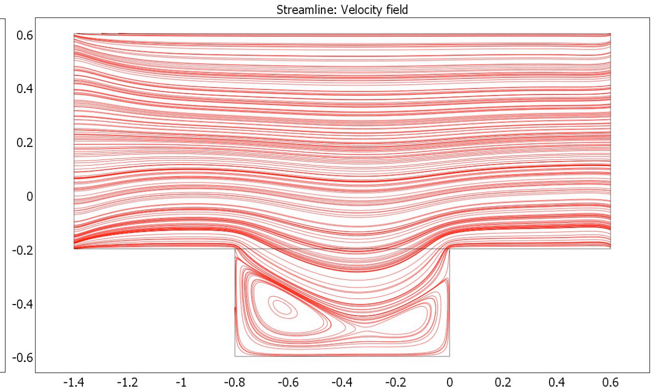
Cavity Driven Flow

- By increasing the forcing along the main channel, do any higher modes ever occur other than lowest?
- How fast can one mode “relax” into other modes?
- In order for physical gyres to be found, DHTs are required to *guide* the BVP.

INLET:

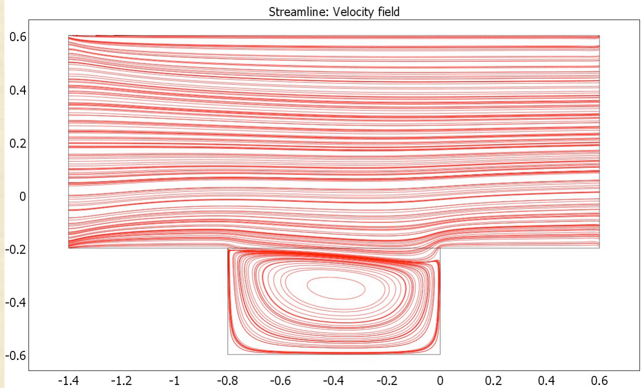
V=1 m/s

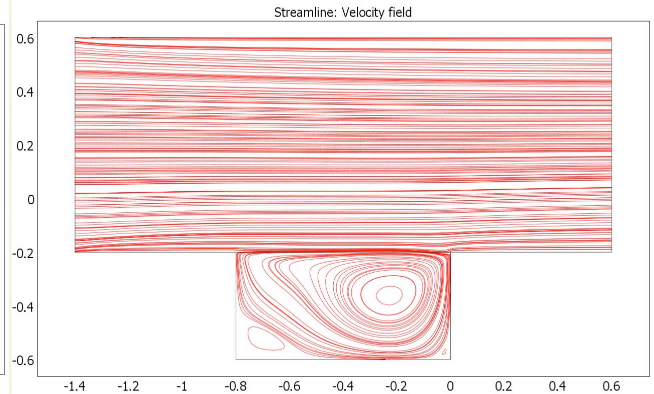




V = 10 m/s

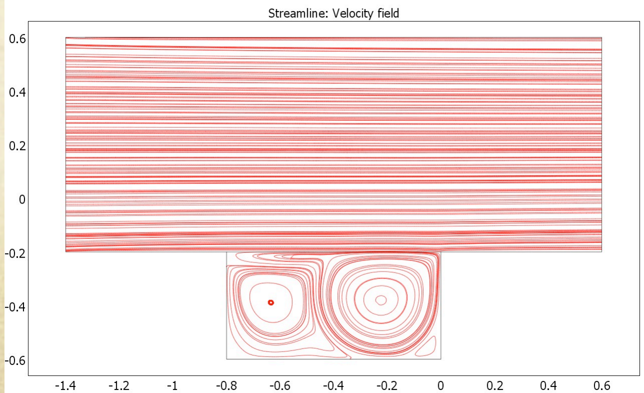
V= 100 m/s

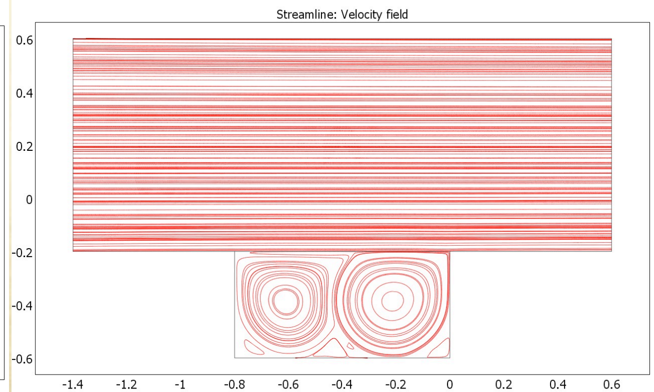




V = 1000 m/s

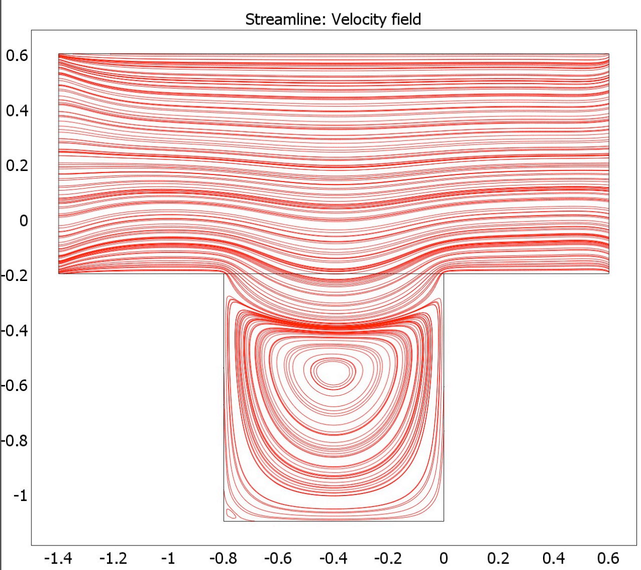
V=10000 m/s

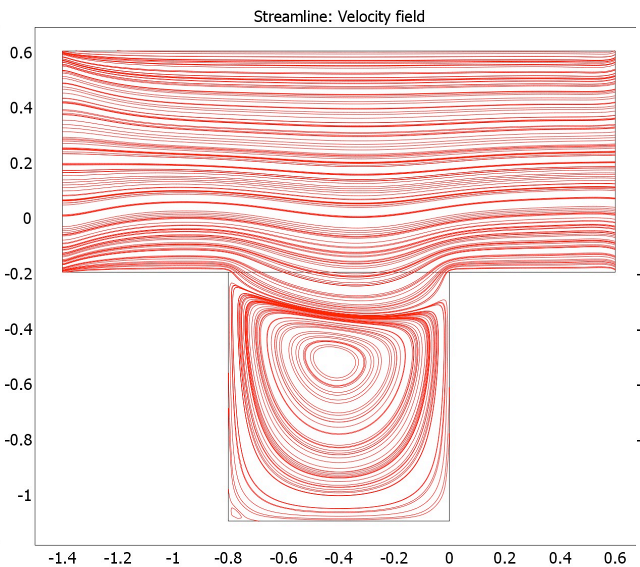


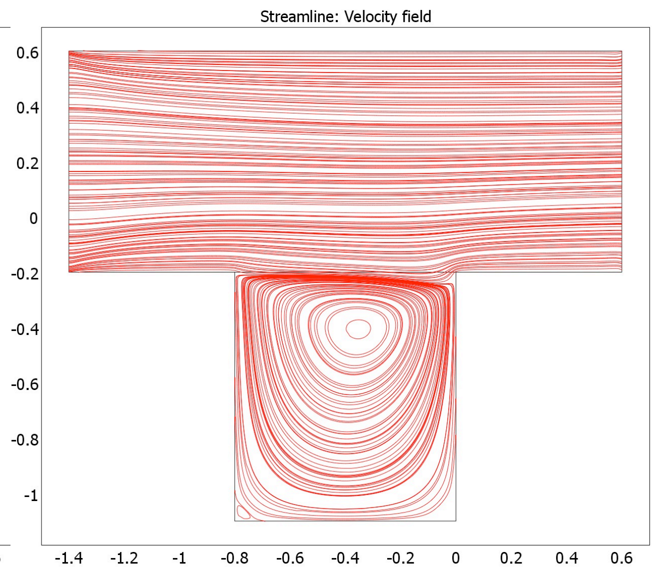


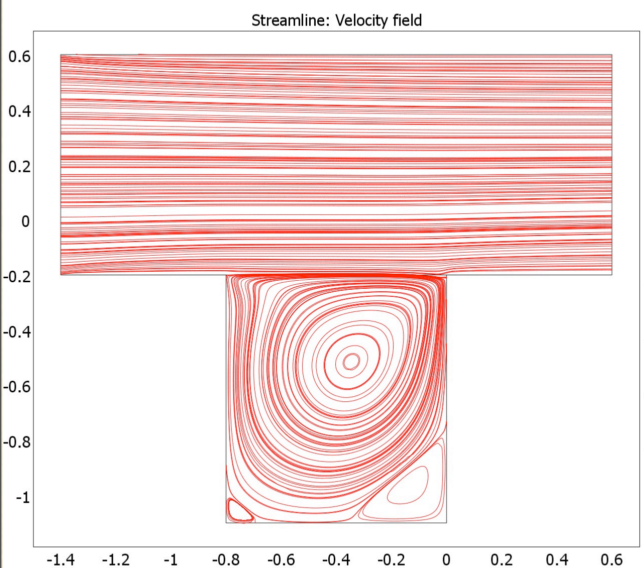
V=10^6 m/s

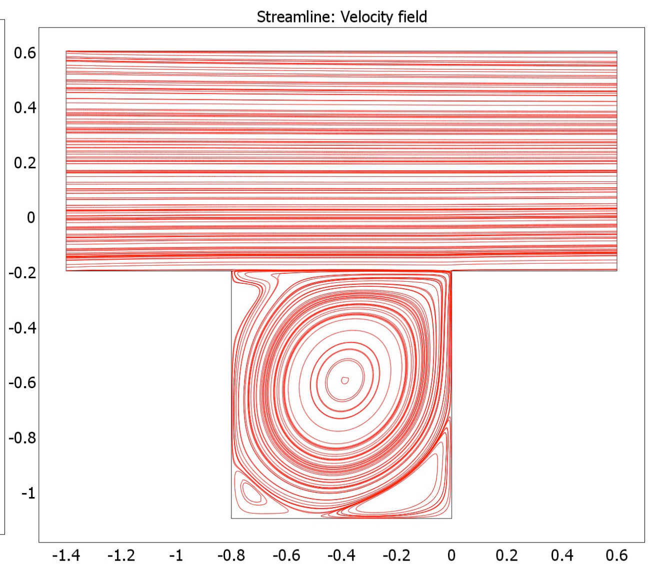
INLET: V=1 m/s V=10 m/s V=100 m/s

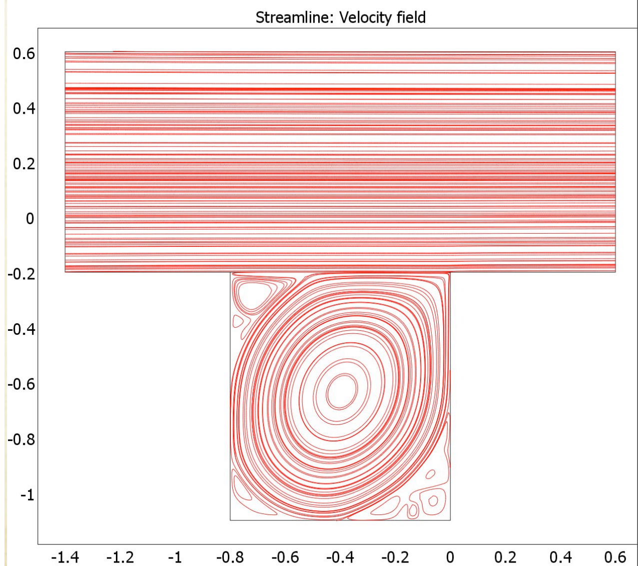








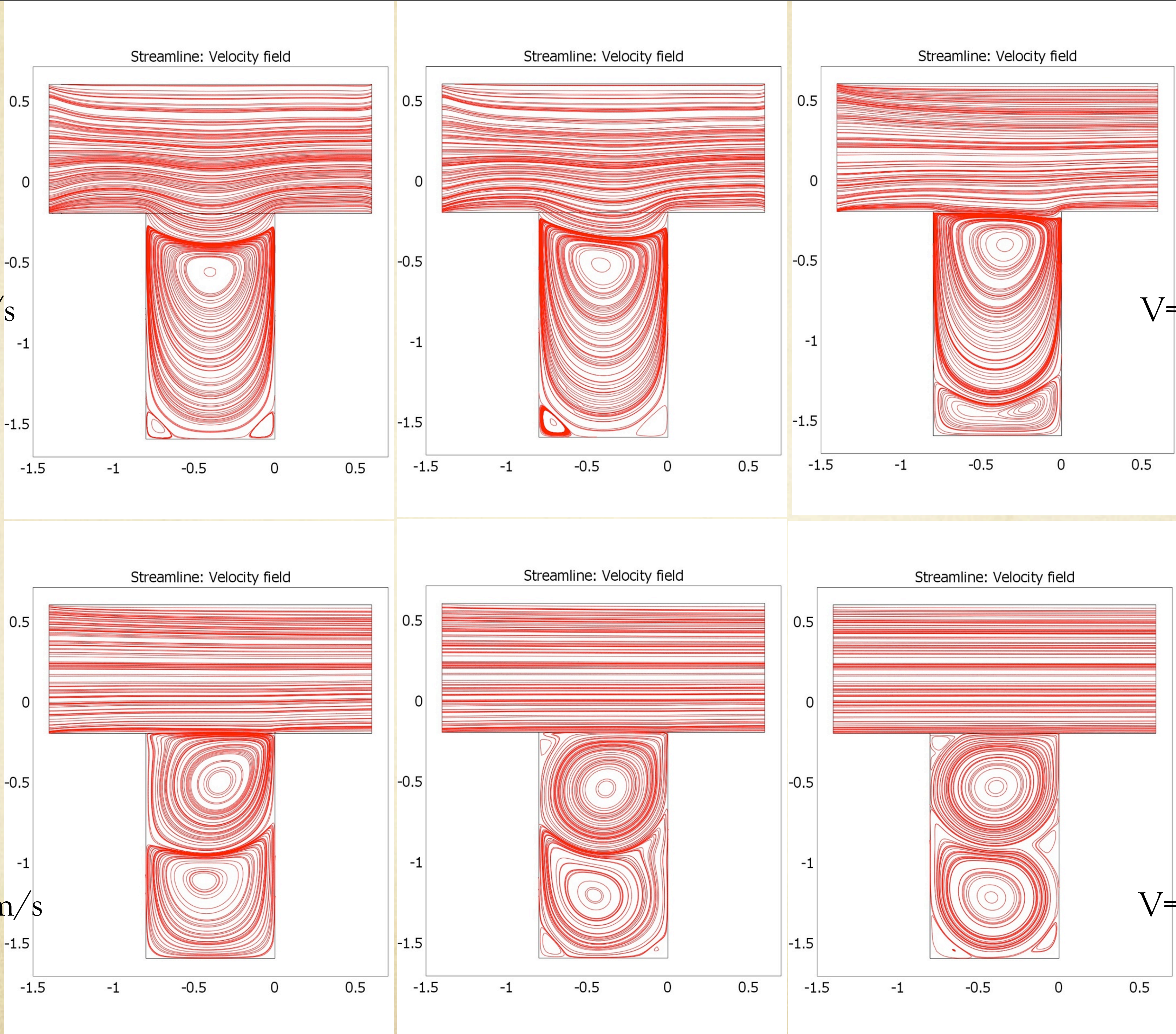




INLET: V=1000 m/s V=10000 m/s V=10^6 m/s

V=1m/s

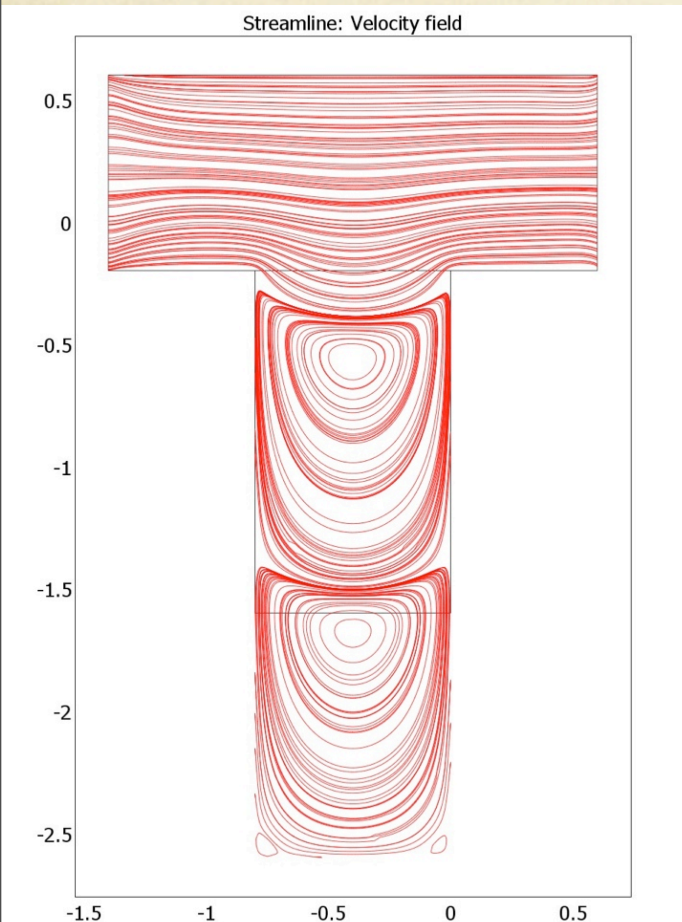
V=10^2m/s



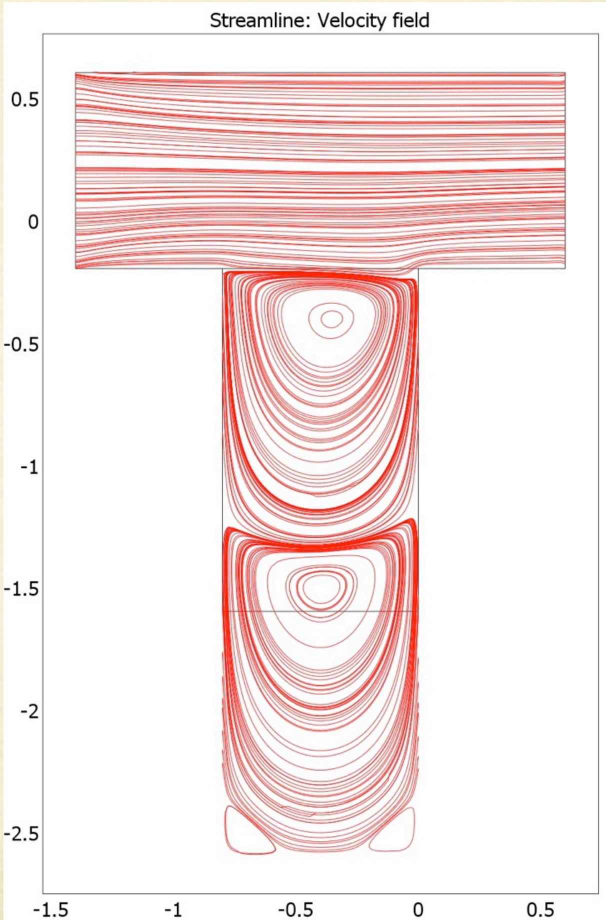
V=10^3m/s

V=10^5m/s

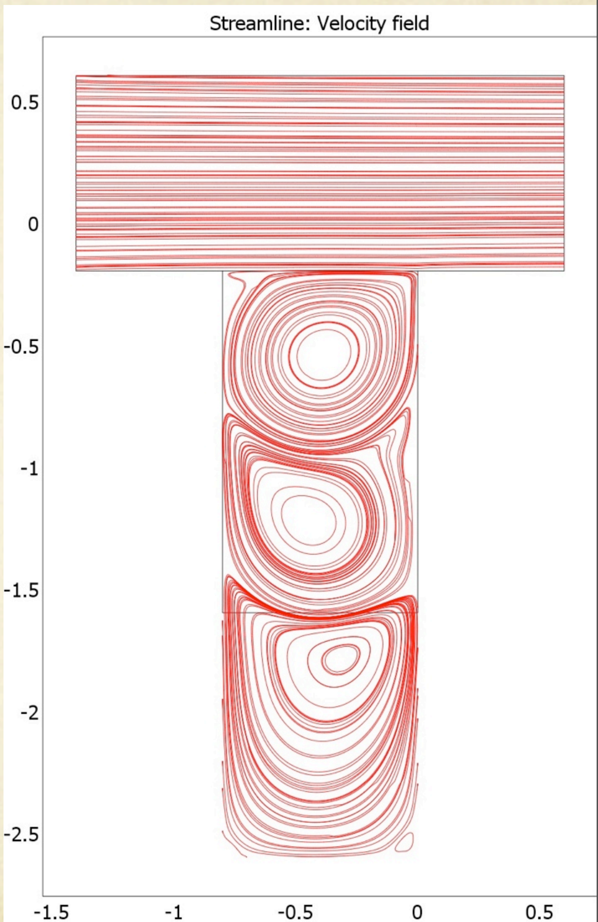
3 Modes Deep



V=1m/s



V=100m/s



V=10000m/s

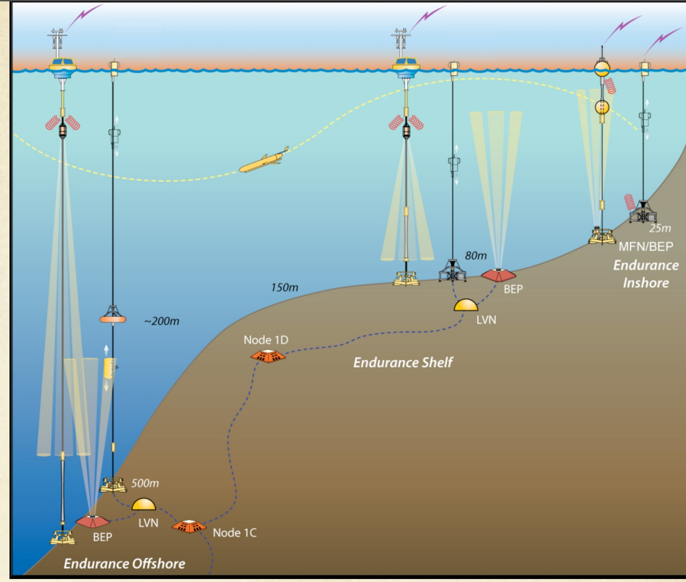
Data Collection for 3D modes

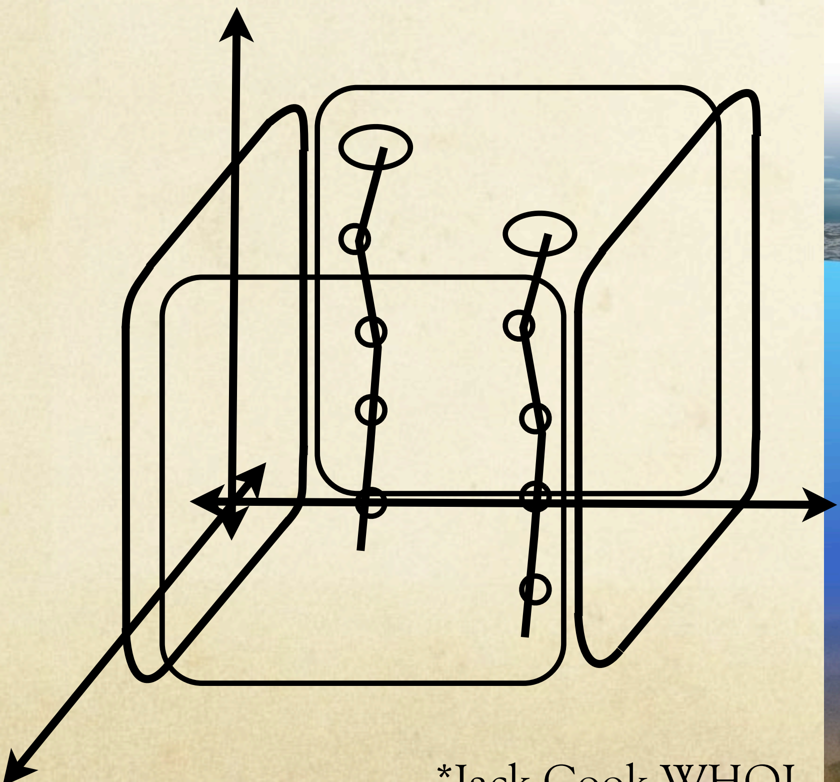
- Assume most of data is collected regularly on the surface of the water.
- At a limited set of locations, data is collected at depth down to a limited depth and for a finite number of depths.
- Lagrangian drifter/glider data exists along transects of the domain (specific to one location at one time).
- Satellite feature extraction only works when coverage permits.

Most of this data will be taken on the top layer (surface).

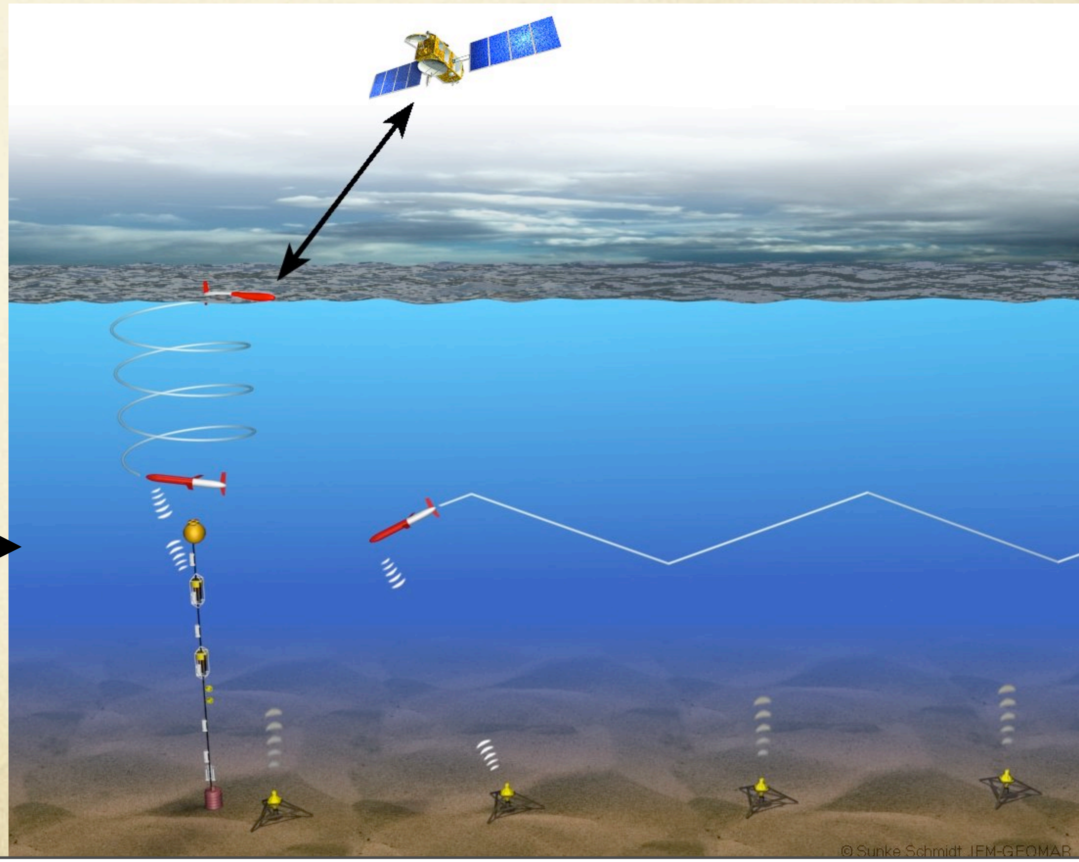
Some data at depth is taken.

Glider data taken along a transect.





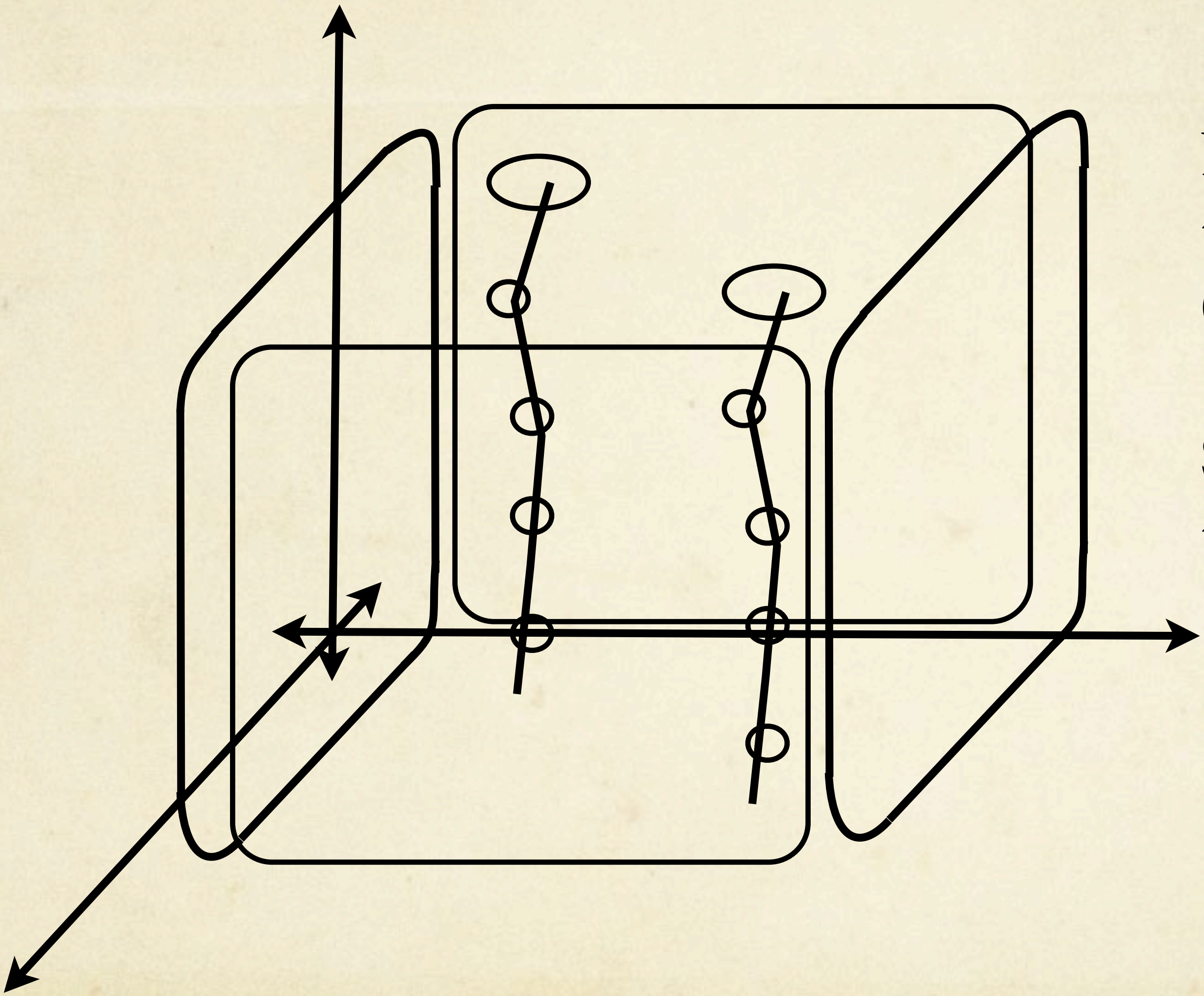
*Jack Cook WHOI



3D Modes - limitations

- Most of the data is collected on one face of a 3D mode.
- Coverage of the face will be limited (less than 100%).
- A few lines along at depth (less than 10% coverage).
- Drifter/glider data force issue of how to incorporate spatio-temporal information in a meaningful way to extract amplitudes of modes.
- Shannon sampling theorem – need as many spatial points as modes to extract (more or less).
- Compressive sensing addresses this partially.
- Can times-series data be used as additional data points – effectively increasing the size of the data set spatially – providing enough coverage to be useful?

3D Mode Data Collection



Most of this data will be taken on the top layer (surface).

Some data at depth is taken.

Possible Solution

- Collect data at a limited number of locations/times

$$f(\mathbf{r_i}, t_j) \quad \mathbf{u}(\mathbf{r_i}, t_j)$$

- Select a set of times for a window (N_t in window)

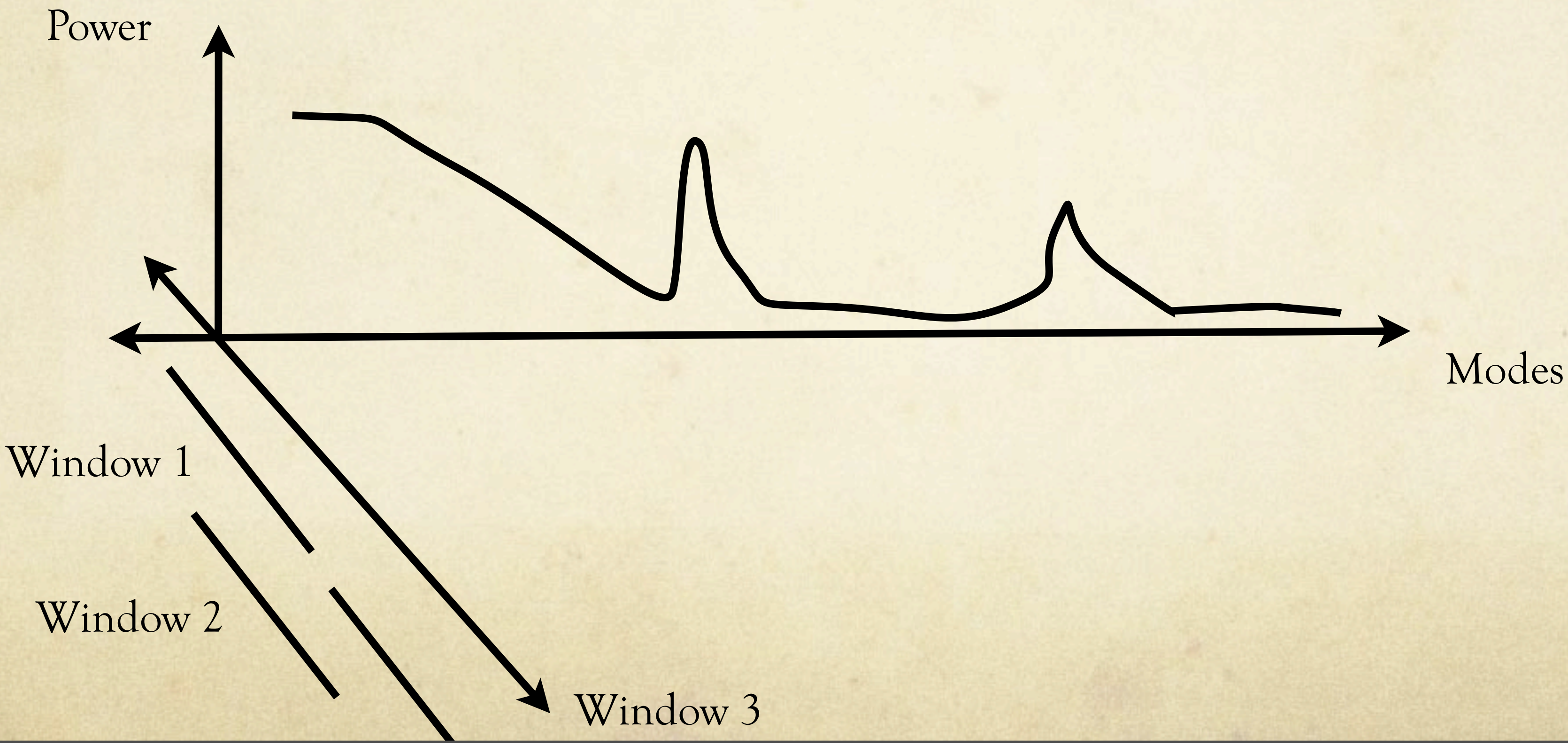
$$t_1 \ldots t_j \ldots . t_{N_t}$$

- Treat all data within the time window as if constant wrt. amplitudes of the modes.

- Fit the amplitudes (A_n, B_n) to the extended data set.

Power Spectrum over Time

- Assume spectrum is constant over a time window
- Calculate the spectra over multiple overlapping windows.



Fitting the Data

$$
\begin{pmatrix}
u(\mathbf{r_1},t_1)\\
u(\mathbf{r_1},t_2)\\
u(\mathbf{r_1},t_3)\\
\dots\\
u(\mathbf{r_1},t_{N_t})\\
v(\mathbf{r_1},t_1)\\
v(\mathbf{r_1},t_2)\\
v(\mathbf{r_1},t_3)\\
\dots\\
v(\mathbf{r_1},t_{N_t})\\
u(\mathbf{r_2},t_1)\\
\dots\\
u(\mathbf{r_2},t_{N_t})\\
v(\mathbf{r_2},t_1)\\
\dots\\
v(\mathbf{r_2},t_{N_t})\\
\dots\\
u(\mathbf{r_{N_p}},t_1)\\
\dots\\
u(\mathbf{r_{N_p}},t_{N_t})\\
v(\mathbf{r_{N_p}},t_1)\\
\dots\\
v(\mathbf{r_{N_p}},t_{N_t})
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial}{\partial y}\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \frac{\partial}{\partial x}\phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
\dots\\
\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
-\frac{\partial}{\partial x}\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \frac{\partial}{\partial y}\phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
\dots\\
\dots\\
\dots\\
\psi_1(\mathbf{r_1}) \;\; \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \;\; \phi_1(\mathbf{r_1}) \;\; \phi_2(\mathbf{r_1})\dots\phi_N(\mathbf{r_1})\\
\psi_1(\mathbf{r_2}) \;\; \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \;\; \phi_1(\mathbf{r_2}) \;\; \phi_2(\mathbf{r_2})\dots\phi_N(\mathbf{r_2})\\
\dots\\
\dots\\
\dots\\
\psi_1(\mathbf{r_2}) \;\; \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \;\; \phi_1(\mathbf{r_2}) \;\; \phi_2(\mathbf{r_2})\dots\phi_N(\mathbf{r_2})\\
\psi_1(\mathbf{r_{N_p}}) \;\; \psi_2(\mathbf{r_{N_p}})\dots\psi_N(\mathbf{r_{N_p}}) \;\; \phi_1(\mathbf{r_{N_p}}) \;\; \phi_2(\mathbf{r_{N_p}})\dots\phi_N(\mathbf{r_{N_p}})\\
\dots\\
\dots\\
\dots\\
\psi_1(\mathbf{r_{N_p}}) \;\; \psi_2(\mathbf{r_{N_p}})\dots\psi_N(\mathbf{r_{N_p}}) \;\; \phi_1(\mathbf{r_{N_p}}) \;\; \phi_2(\mathbf{r_{N_p}})\dots\phi_N(\mathbf{r_{N_p}})
\end{pmatrix}
\begin{pmatrix}
A_1\\ A_2\\ A_3\\ \dots\\ A_N\\ B_1\\ B_2\\ B_3\\ \dots\\ B_N
\end{pmatrix}
$$

Simplified...

$$f(\mathbf{r},t) = \sum_{i=1..N} A_i(t) * \Psi_i(\mathbf{r})$$

$$\begin{pmatrix} f(\mathbf{r_1},t_1) \\ f(\mathbf{r_1},t_2) \\ f(\mathbf{r_1},t_3) \\ \dots \\ f(\mathbf{r_1},t_{N_t}) \\ f(\mathbf{r_2},t_1) \\ f(\mathbf{r_2},t_2) \\ f(\mathbf{r_2},t_3) \\ \dots \\ f(\mathbf{r_2},t_{N_t}) \\ \dots \\ \dots \\ \dots \\ f(\mathbf{r_p},t_1) \\ f(\mathbf{r_p},t_2) \\ f(\mathbf{r_p},t_3) \\ \dots \\ f(\mathbf{r_p},t_{N_t}) \end{pmatrix} = \begin{pmatrix} \psi_1(\mathbf{r_1}) & \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \\ \psi_1(\mathbf{r_1}) & \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \\ \psi_1(\mathbf{r_1}) & \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \\ & \dots \\ \psi_1(\mathbf{r_1}) & \psi_2(\mathbf{r_1})\dots\psi_N(\mathbf{r_1}) \\ \psi_1(\mathbf{r_2}) & \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \\ \psi_1(\mathbf{r_2}) & \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \\ \psi_1(\mathbf{r_2}) & \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \\ & \dots \\ & \dots \\ & \dots \\ \psi_1(\mathbf{r_2}) & \psi_2(\mathbf{r_2})\dots\psi_N(\mathbf{r_2}) \\ \psi_1(\mathbf{r_p}) & \psi_2(\mathbf{r_p})\dots\psi_N(\mathbf{r_p}) \\ \psi_1(\mathbf{r_p}) & \psi_2(\mathbf{r_p})\dots\psi_N(\mathbf{r_p}) \\ \psi_1(\mathbf{r_p}) & \psi_2(\mathbf{r_p})\dots\psi_N(\mathbf{r_p}) \\ & \dots \\ \psi_1(\mathbf{r_p}) & \psi_2(\mathbf{r_p})\dots\psi_N(\mathbf{r_p}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \dots \\ A_N \end{pmatrix}$$

Where:

p = number of locations

N_t = number of time samples

N = number of modes

Fitting and the Pseudo-inverse

- Classic fitting problem for a data set: (x_i, y_i)

$$\vec{y} = A\ \vec{x}$$

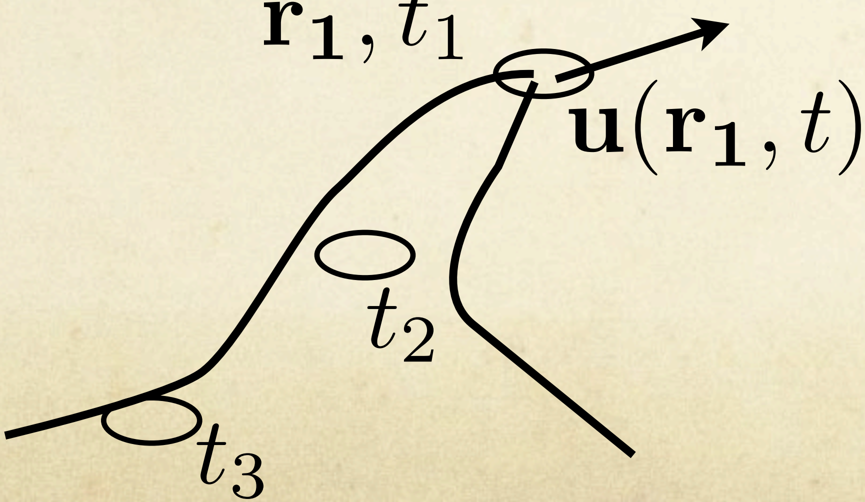
$$\vec{x} = A^{-1}\ \vec{y}$$

- $\vec{x} = \left(A^T\ A\right)^{-1}\ A^T\ \vec{y}$ if A is rectangular

- Rows and columns cannot have zero or repeated entries
- Poor condition number for A leads to unstable solution, $\vec{x}$
- $\delta\vec{x} = A^{-1}\ \delta\vec{y}$ where small variations in y lead to large variations in x

Backing out the Time Dependence

- For currents at $\mathbf{r_1}$, where is the current at t_2?
- Define a new position for each time by successively "backing out" the current.
- Prevents rows from repeating.
- Lowers the condition number of A.



Monte Carlo

- By adding a slight variation to either $\vec{y}$ or A, force errors in $\vec{x}$.

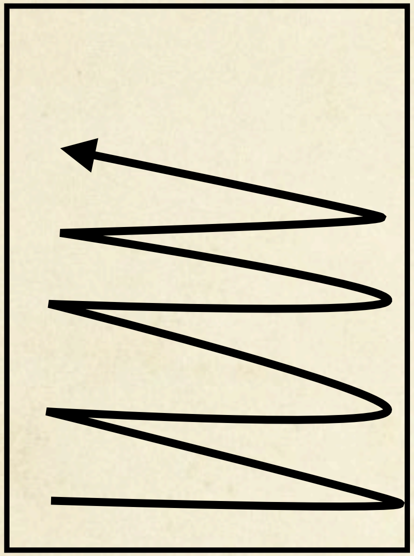
- Repeat the process 1000+ times and average the answer.

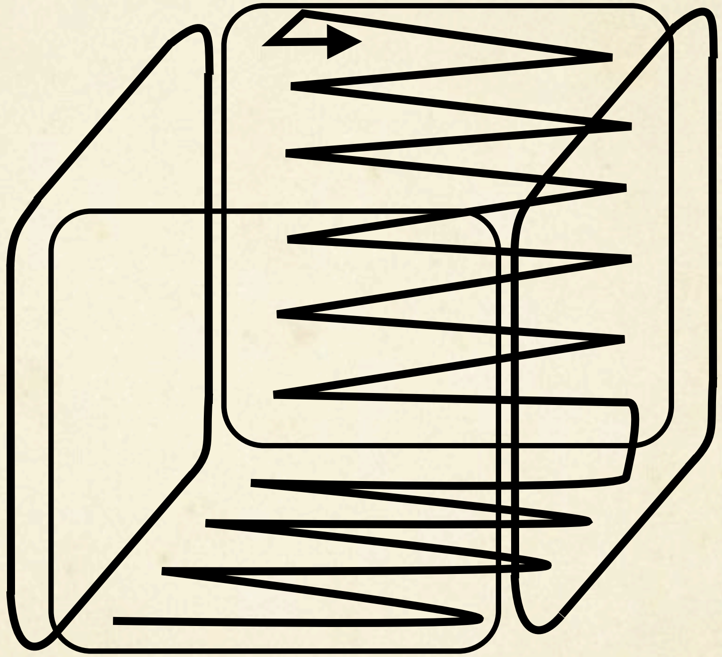
Power Spectra over Time

- Given a set of constant amplitudes for each time window, over all windows, assign a time dependence to each mode, $A_1(t)$...
- Re-adjust the modal matrix at each position based on the extracted time dependence for each mode.
- Repeat entire process until the time dependence for each mode becomes stable.

Data State Vector

- Whether 2D or 3D, state vector of the data is re-formatted as a column vector (rasterized).





- Number of populated terms in the state vector is always very sparse (typically 1%).

Conclusions

- Realistic Normal Mode Analysis must utilize sparse data over space.
- NMA provides an alphabet for how to discuss the power spectrum for a system - not a physical gyre representation.
- NMA - when successful provides full domain coverage given a limited number of spatially sampled locations.