Nonlinear Processes in Oceanic and Atmospheric Flows 3-6 July 2012

Out of Flatland: 3D Transport Barriers in Oceanography

Mohamed H. M. Sulman
Helga S. Huntley
B. L. Lipphardt, Jr.
A. D. Kirwan, Jr.
SMSP University of Delaware

P. Hogan
G. Jacobs
Naval Research Laboratory
SSC

Thanks to:

CONSORTIUM for ADVANCED RESEARCH and TRANSPORT of HYDROCARBONS in the ENVIRONMENT

Support from Office of Naval Research through MURI *OCEAN 3D+1 grant N00014-11-1-0087*

Ana Mancho and NLOA2012 Organizing Committee

These results are preliminary and have not been peer-reviewed.

No information contained in this presentation should be used or cited without permission of the authors.

Lagrangian Coherent Structures

- Defined by Haller (2000) in terms of FTLE (default metric used here)
- Other LCS characterizations
  - Joseph & Legras (2002) - FSLE
  - Mancho & Mendoza (2010) - minimal trajectories
  - Rypina - complexity
  - Mezic et al (2010) - mesohyperbolicity
  - Haller (2011) - geodesic material surfaces
- Most studies in GFD confined to 2D velocities
- Yet theory applies to $\mathbf{R}^n$

Are LCS important in GFD?

- MODE/POLYMODE (circa 1975) - Mesoscale eddies transport heat, salinity, and momentum
  But
  - How do eddies form?
  - How many eddies are there?
  - How do eddies exchange heat, etc with environment?
- Since MODE/POLYMODE
  - Growing Lagrangian user community
  - Dramatic oil spills
- Circa 1990 – Little Compton meeting. DST methods applied to 2D mesoscale and submesoscale transport

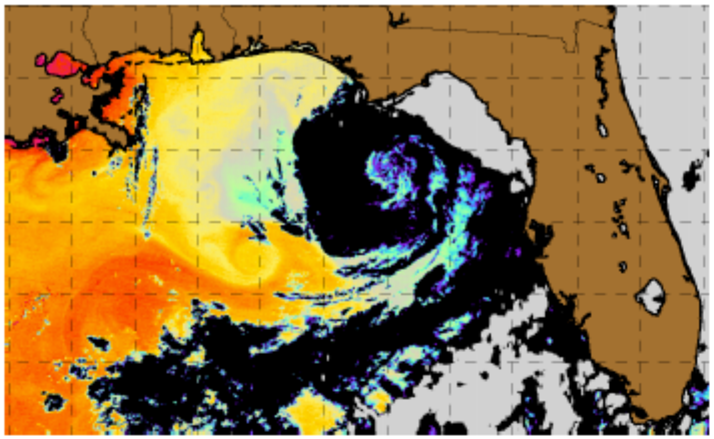
But Transport is *Volume* per time

- 3D computations too costly (Garth et al 2007)
- 3D velocities not always available in GFD
  (Branicki & Kirwan 2010, Bettencourt et al 2012)

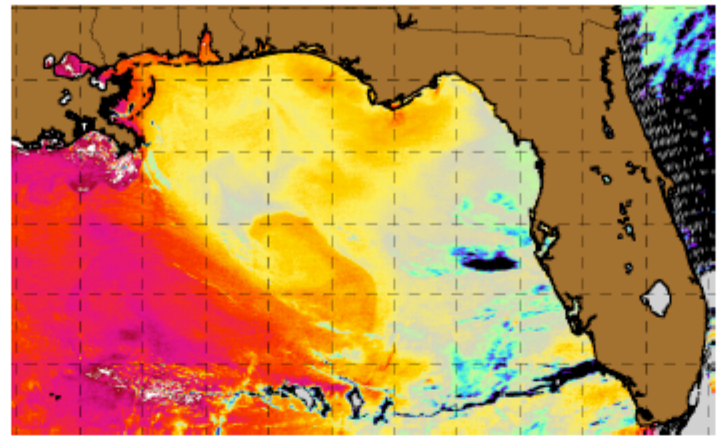
- *Question*:
  Can 3D LCS be estimated from 2D velocities?

Are LCS Important in Oceanography?

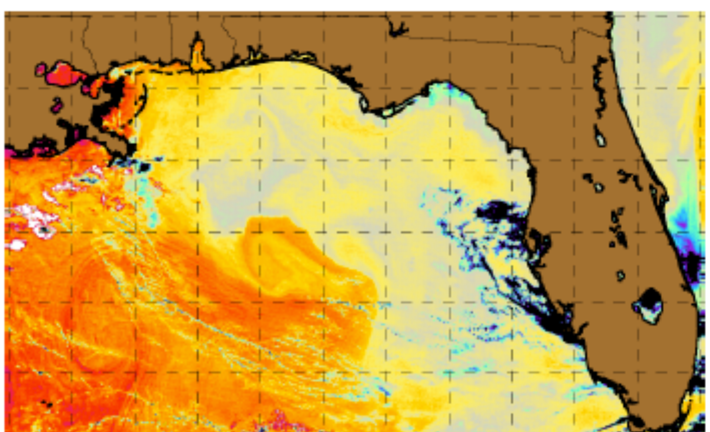
June 26



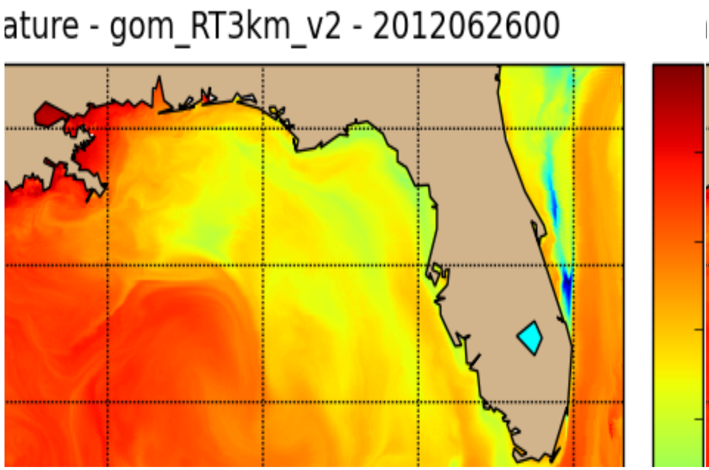
June 27

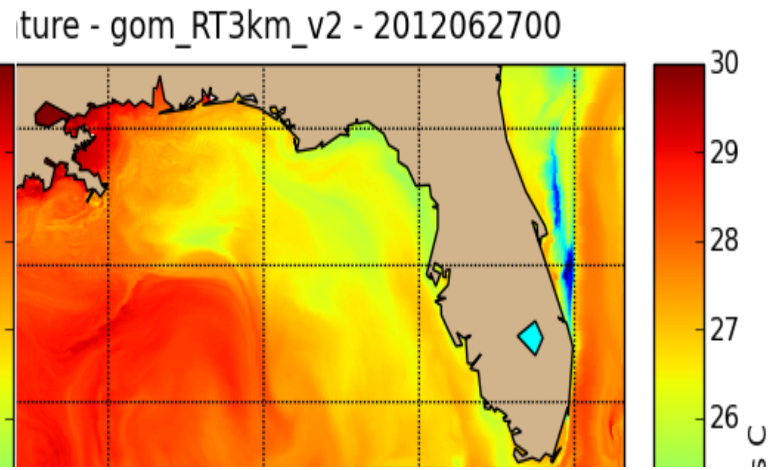


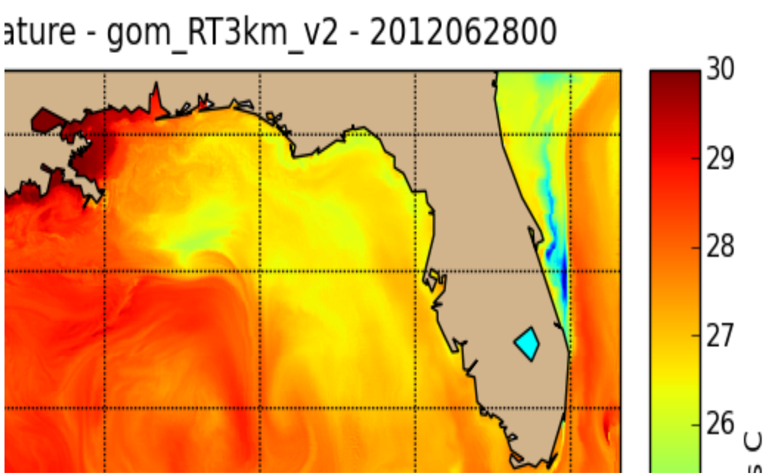
June 28



Are LCS Important in Oceanography?







Out of Flatland – What if there was a 3rd Dimension?

- Mezic and Wiggins (1994)
- Toy Problem

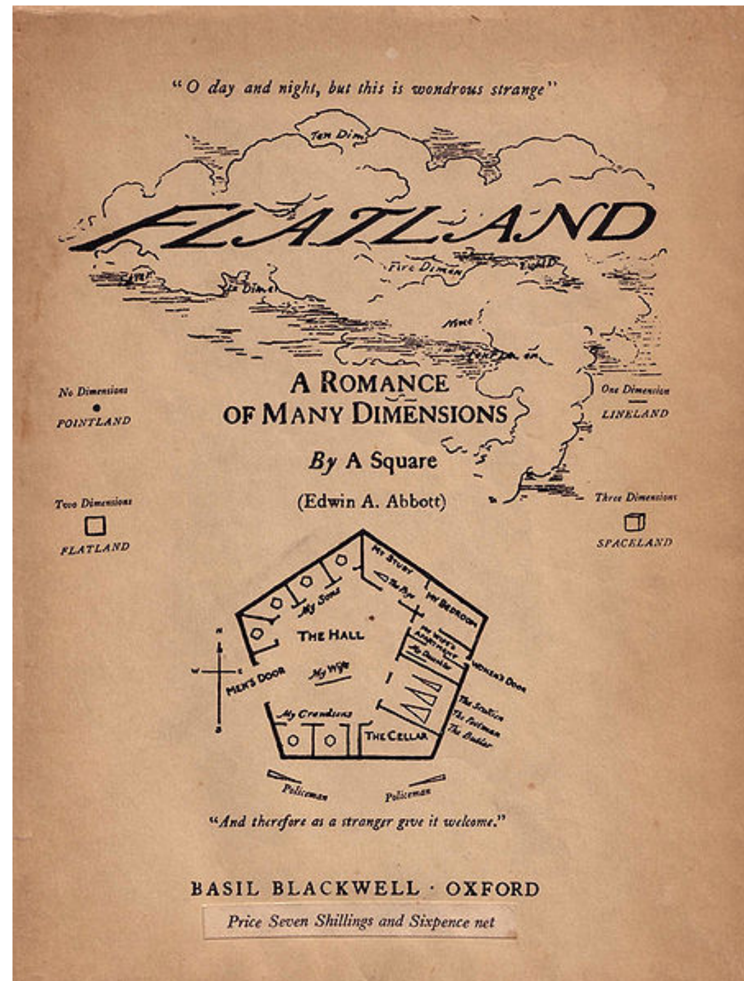
$$\dot{x} = \lambda_1 x - \omega y$$

$$\dot{y} = \omega x + \lambda_2 y$$

$$\dot{z} = -\left(\lambda_1 + \lambda_2\right) z$$

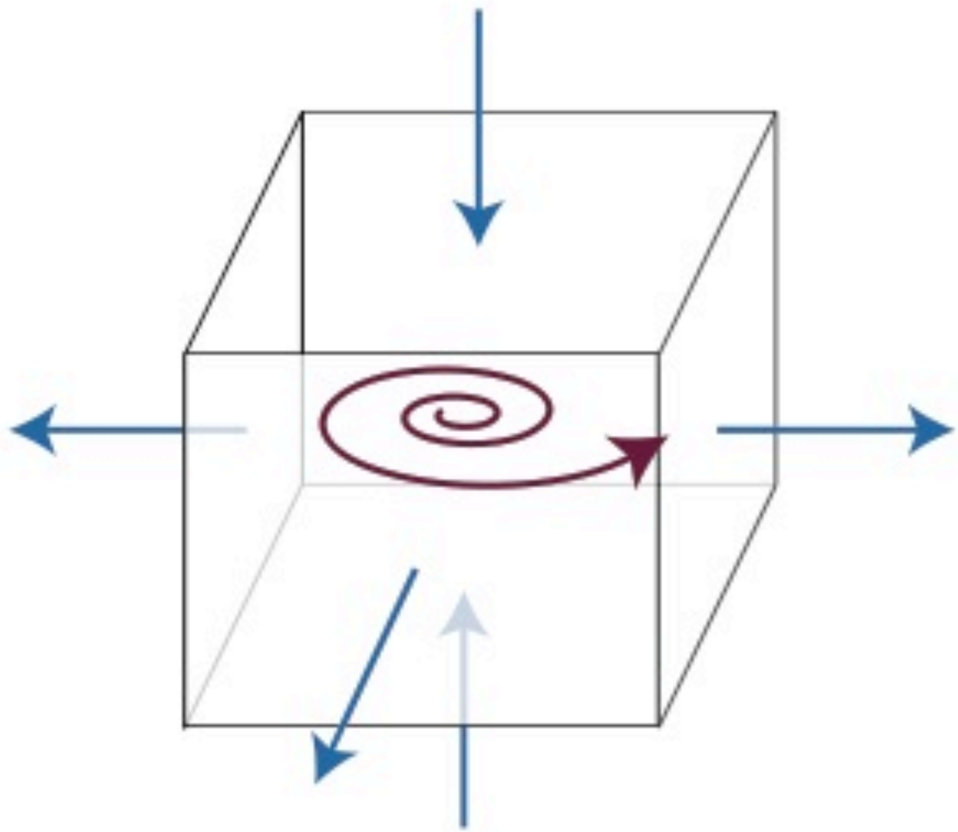
$$0 < \omega^2 - \lambda^2$$

$$\lambda = \frac{\lambda_1 + \lambda_2}{2}$$



Out of Flatland Solution

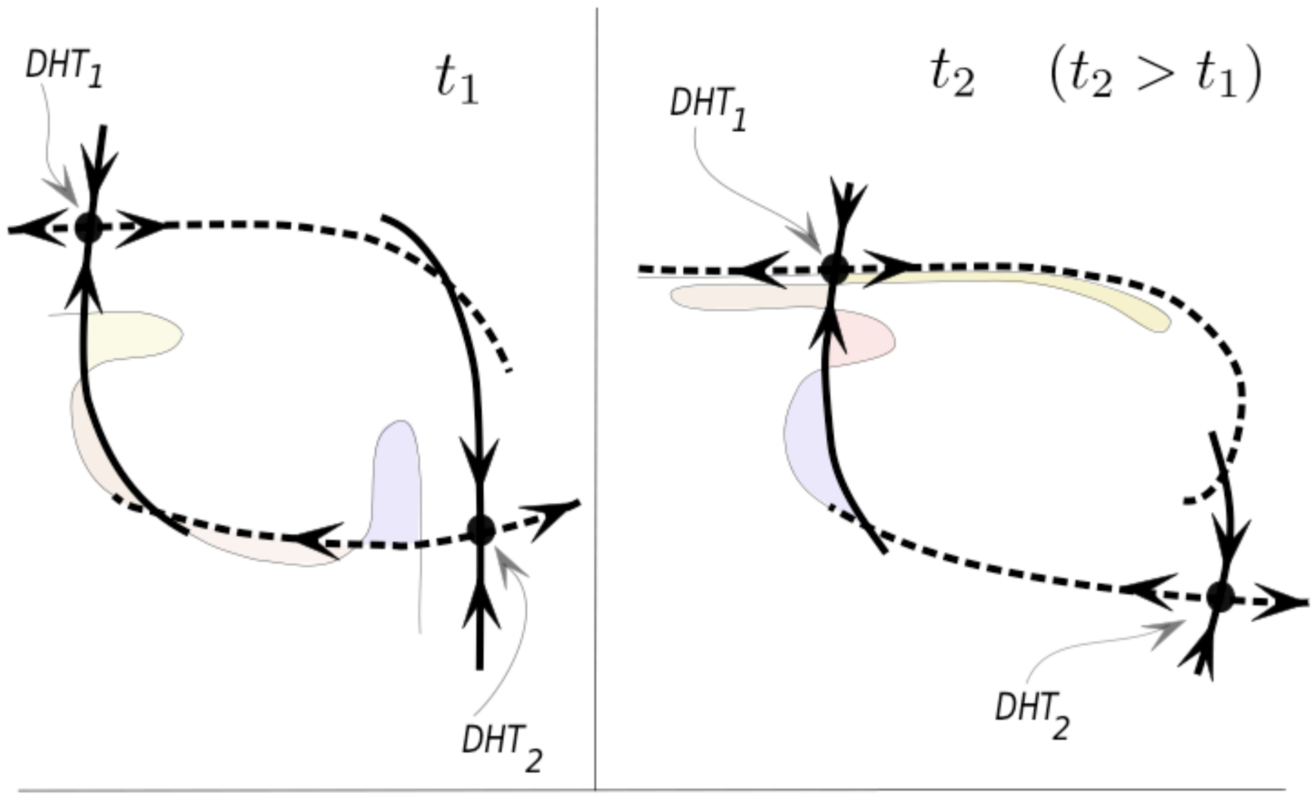
$$
\begin{aligned}
x &= \exp \lambda t \, [X \cos \Omega t - (\lambda_2 X + \omega Y) \sin \Omega t / \Omega] \\
y &= \exp \lambda t \, [Y \cos \Omega t - (\lambda_1 Y - \omega X) \sin \Omega t / \Omega] \\
z &= Z \exp -2\lambda t \\
\Omega^2 &= \omega^2 - \lambda^2
\end{aligned}
$$



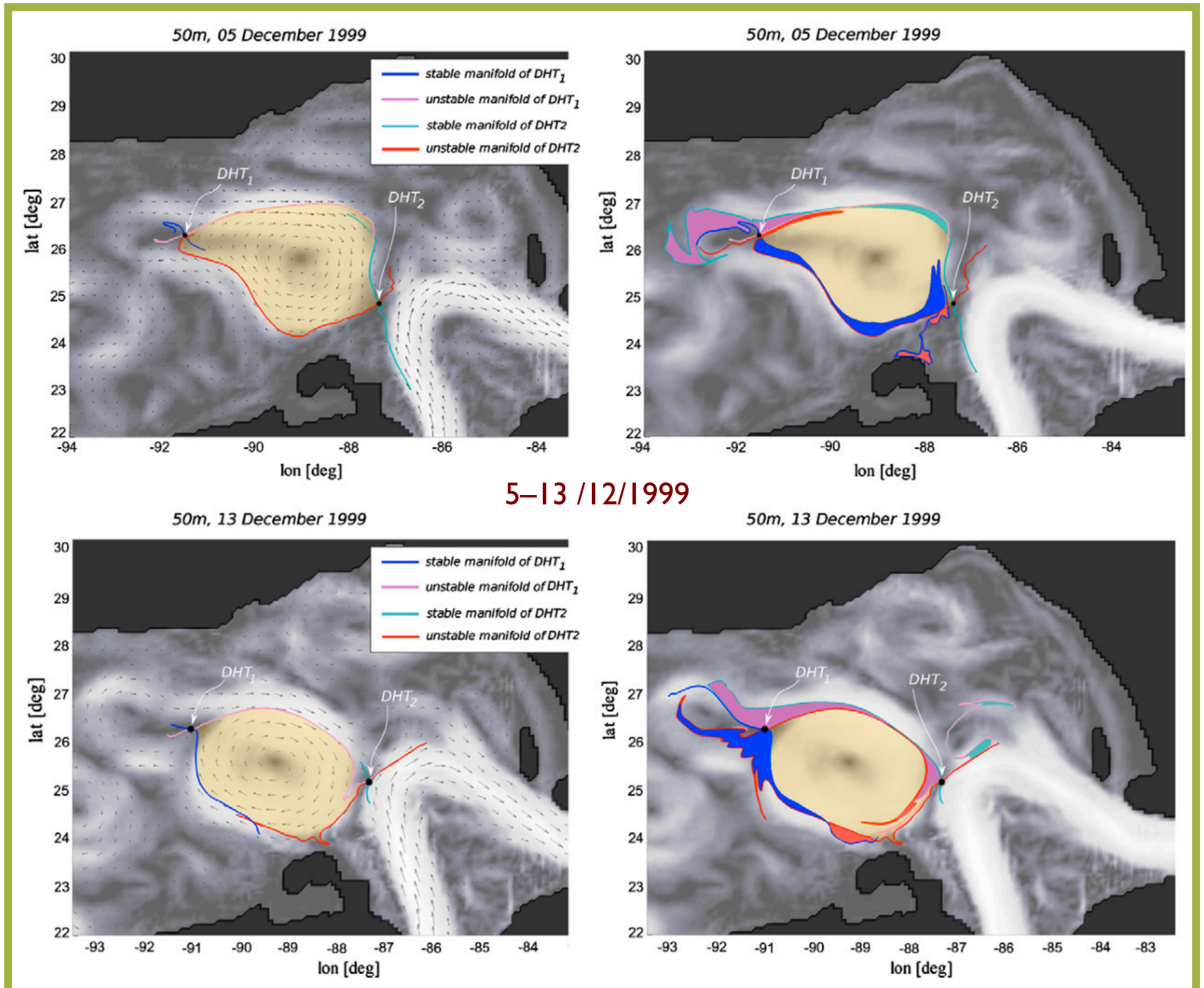
Branicki & Kirwan (IJES 2010)

- FTLE located DHTs
- 1D manifolds using 2D velocities (Ide et al, 2002) from 0 to 250m for eddy Juggernaut
- Stitched 1D manifolds into 2D material surfaces

Dynamical Systems Approach

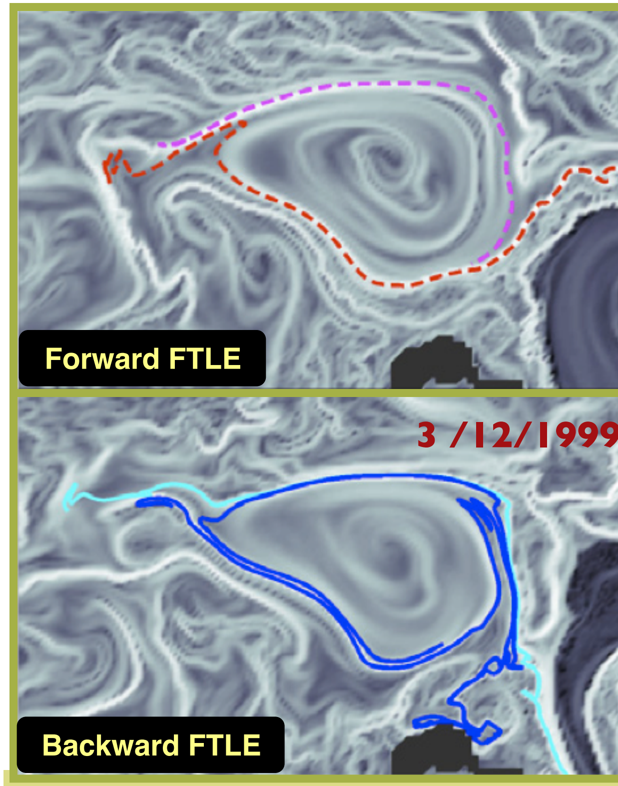


Loop Current Ring Exchange

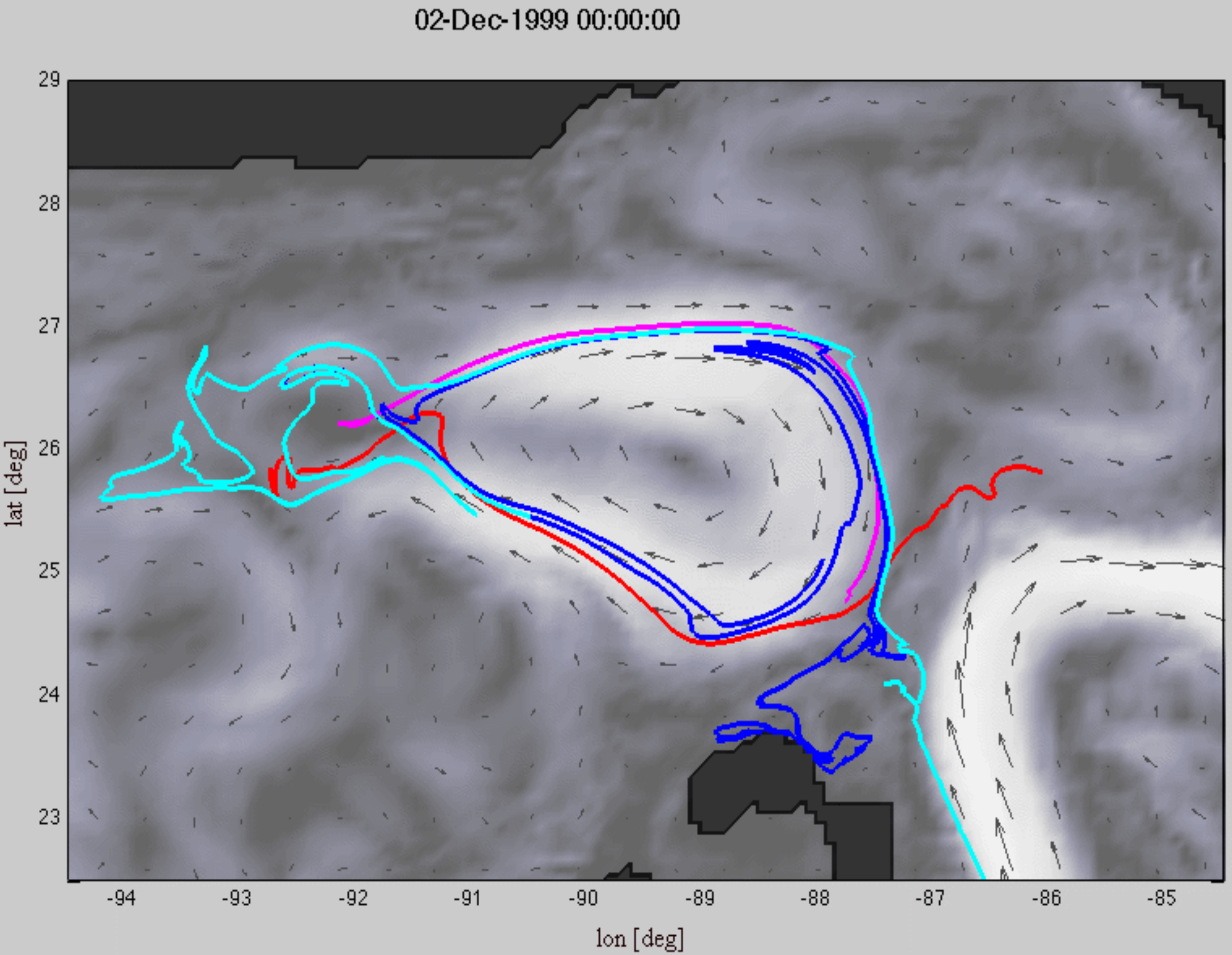


DHTs, manifolds, and lobes near the eddy

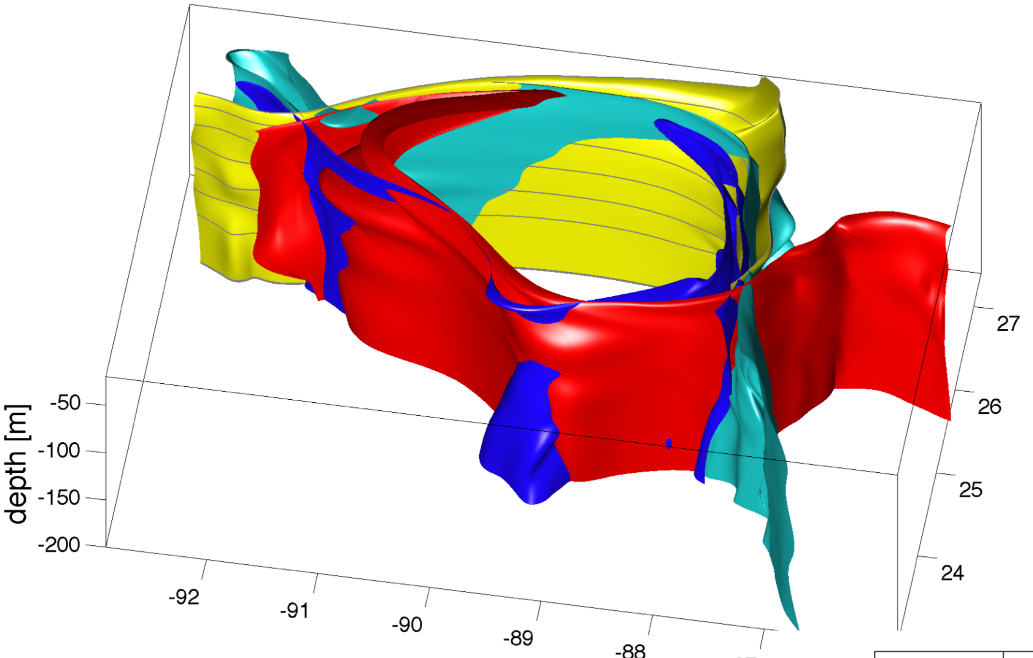
Eddy Juggernaut

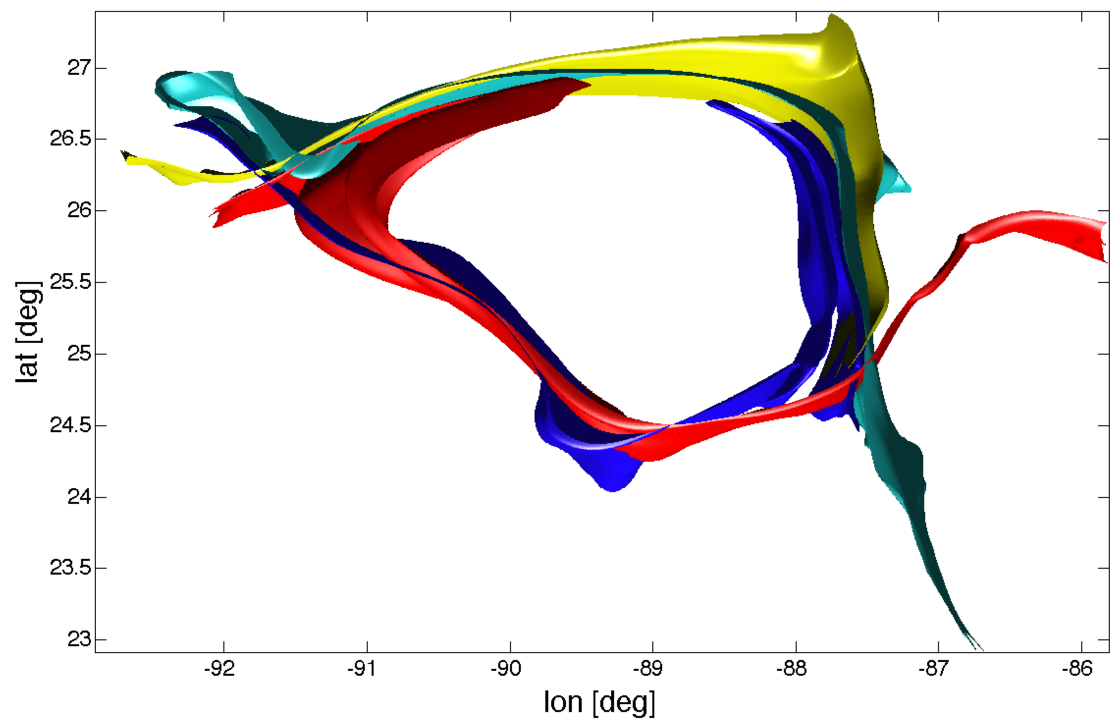


How do manifolds compare with FTLEs ?



02-Dec-1999 00:00:00
29
28
27
26
25
24
23
lat [deg]
-94
-93
-92
-91
-90
-89
-88
-87
-86
-85
lon [deg]


depth [m]
-50
-100
-150
-200
-92
-91
-90
-89
-88
27
26
25
24


lat [deg]
27
26.5
26
25.5
25
24.5
24
23.5
23
-92
-91
-90
-89
-88
-87
-86
lon [deg]

B&K Conclude

- Material surfaces stitched from 2D analysis revealed coherent lobes with depth
- Material surfaces drop nearly vertically. No evidence of eddy lens structure
- Net inflow at bottom, outflow at top

Realistic, or artifacts of stitching and/or data assimilation?

Feasibility of 3D LCS from 2D Velocities

- Options
  - Follow B & K paradigm
  - Extend FTLE calculations to include vertical shear of horizontal velocities
  - Calculate 3D trajectories using diagnostic vertical velocity
- Strategy
  - Test options with toy models to control vertical velocity and vertical gradients
  - Apply to data-assimilating OGCMs

Incompressible Models

$$\mathbf{v} = \nabla \times \left[-\Psi \mathbf{k} + \nabla \times (\Phi \mathbf{k}) \right]$$

$$\begin{aligned} u &= -\frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Phi}{\partial z \partial x} \\ v &= \frac{\partial \Psi}{\partial x} + \frac{\partial^2 \Phi}{\partial z \partial y} \\ w &= -\nabla_h^2 \Psi \end{aligned}$$

ABC & Quadrupole Flows

ABC

$$\begin{aligned}\Psi &= -\left[C\sin\left(y+f\left(t\right)\right)+B\cos\left(x+f\left(t\right)\right)\right]\\ \Phi &= A\left[-x\cos\left(z+f\left(t\right)\right)+y\sin\left(z+f\left(t\right)\right)\right]-\Psi\end{aligned}$$

Quadrupole

$$\begin{aligned}\Psi &= A\left(z,t\right)\sin\left(\pi x/L_x\right)\sin\left(\pi y/L_y\right)\\ \Phi &= B\left(z,t\right)\cos\left(\pi x/L_x\right)\cos\left(\pi y/L_y\right)\end{aligned}$$

Strain Tensor and Velocity Gradient

Consider

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}\left(\mathbf{x}, t\right), \mathbf{x} \in \Omega \subset \mathcal{R}^3$$

With Solution

$$\mathbf{x} = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{v}\left(\mathbf{x}, \tau\right) d\tau$$

And Strain Tensor

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \mathbf{I} + \int_{t_0}^{t} \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} d\tau$$

Cauchy-Green Tensor and FTLE

Cauchy-Green

$$\mathbf{C} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \right)^T \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \right)$$

FTLE

$$\Lambda \left(t; t_0, \mathbf{x}_0 \right) = \frac{\log \left(\sqrt{\lambda_{max} \left(\mathbf{C} \right)} \right)}{t - t_0}$$

Strain Tensor

3D

$$\begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}$$

2D

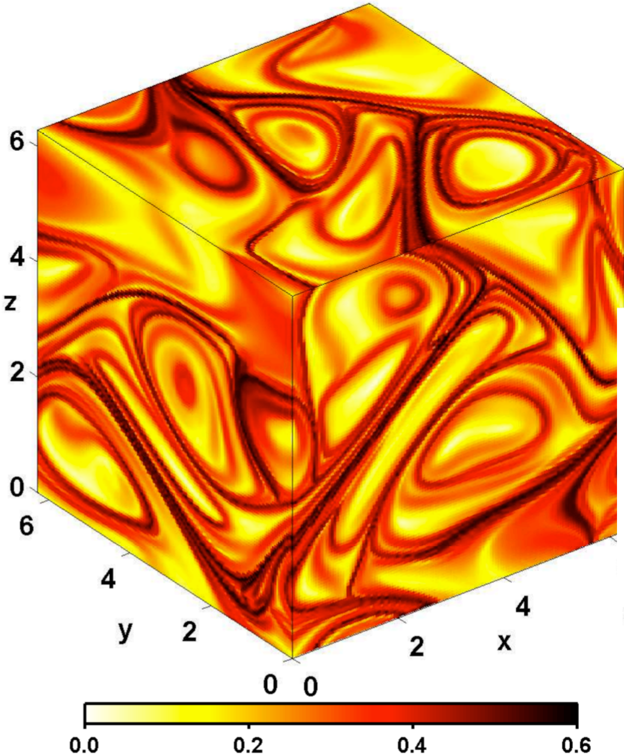
$$\begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} \rightsquigarrow \frac{\partial}{\partial z_0} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = 0$$

$$\nabla_h z = 0$$

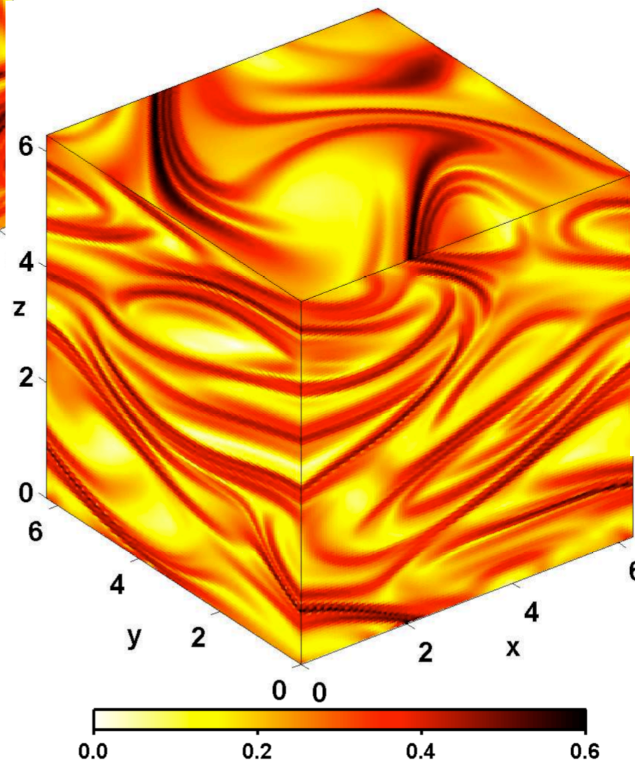
3D2D

$$\begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \frac{\partial}{\partial x_0, \partial y_0} \begin{bmatrix} z \\ w \end{bmatrix} \sim 0$$

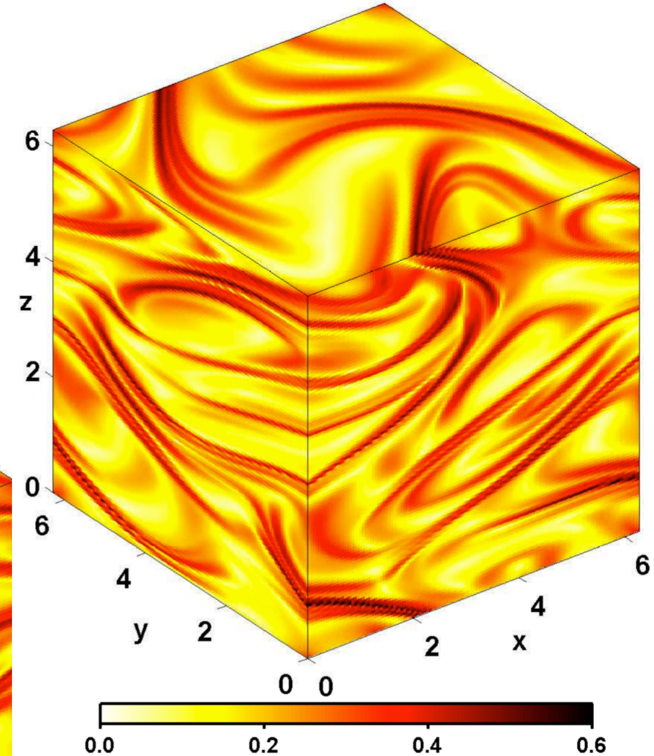
ABC Flow



3D

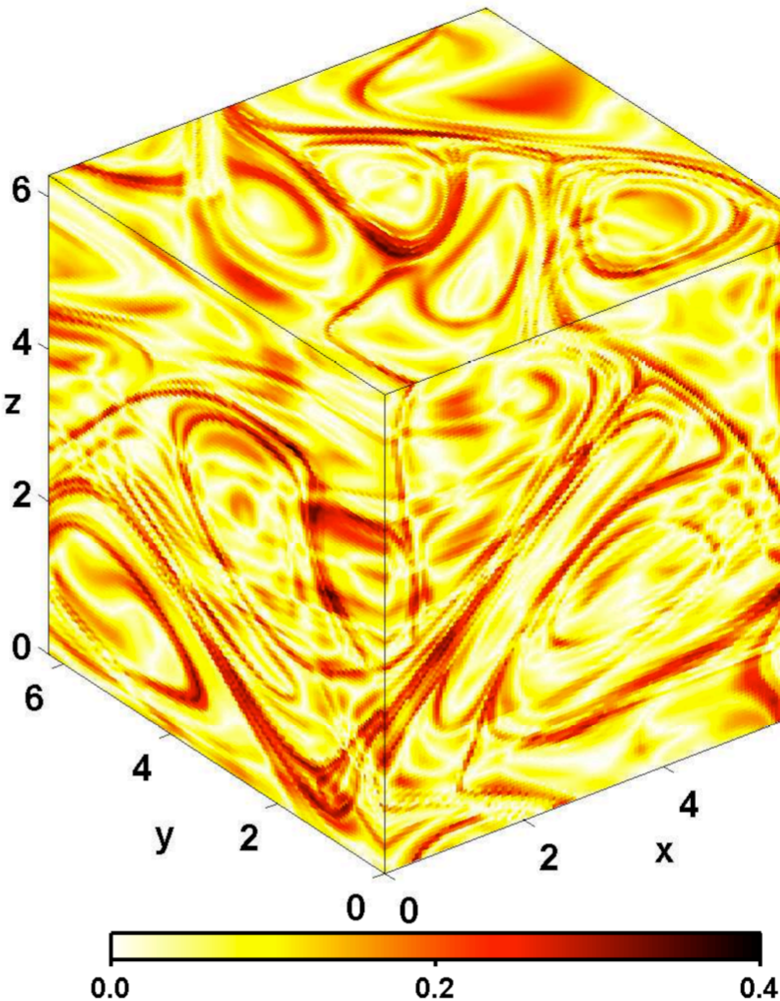


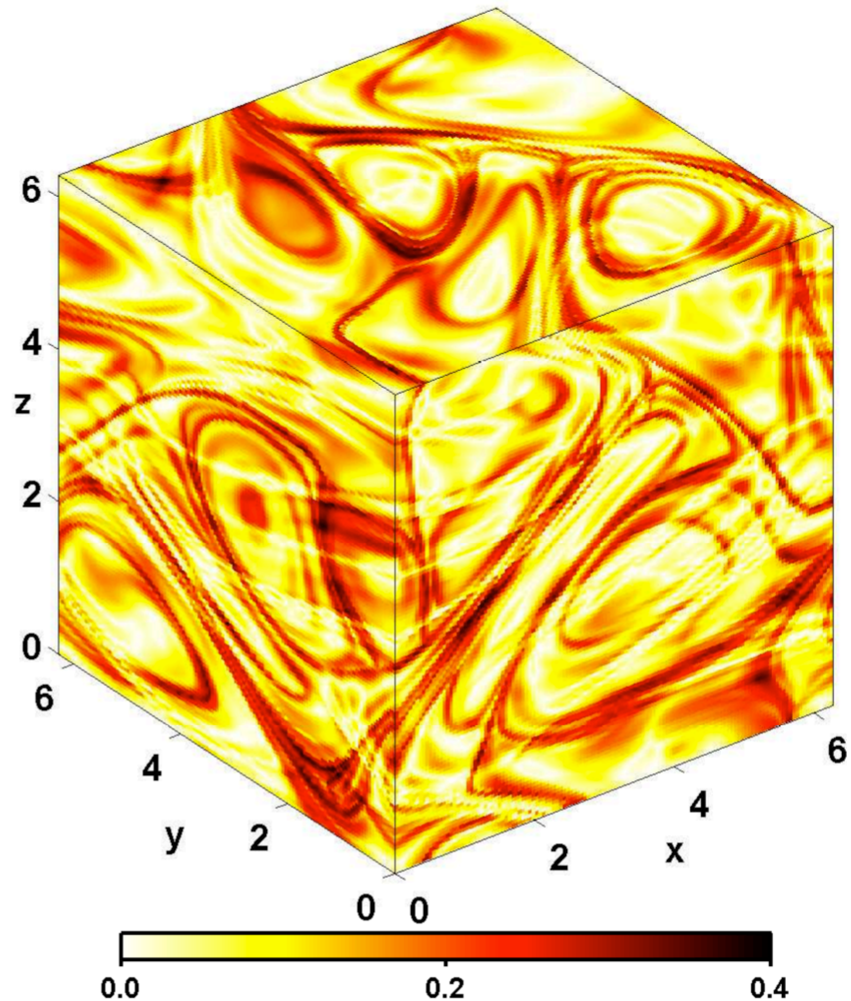
3D2D



2D

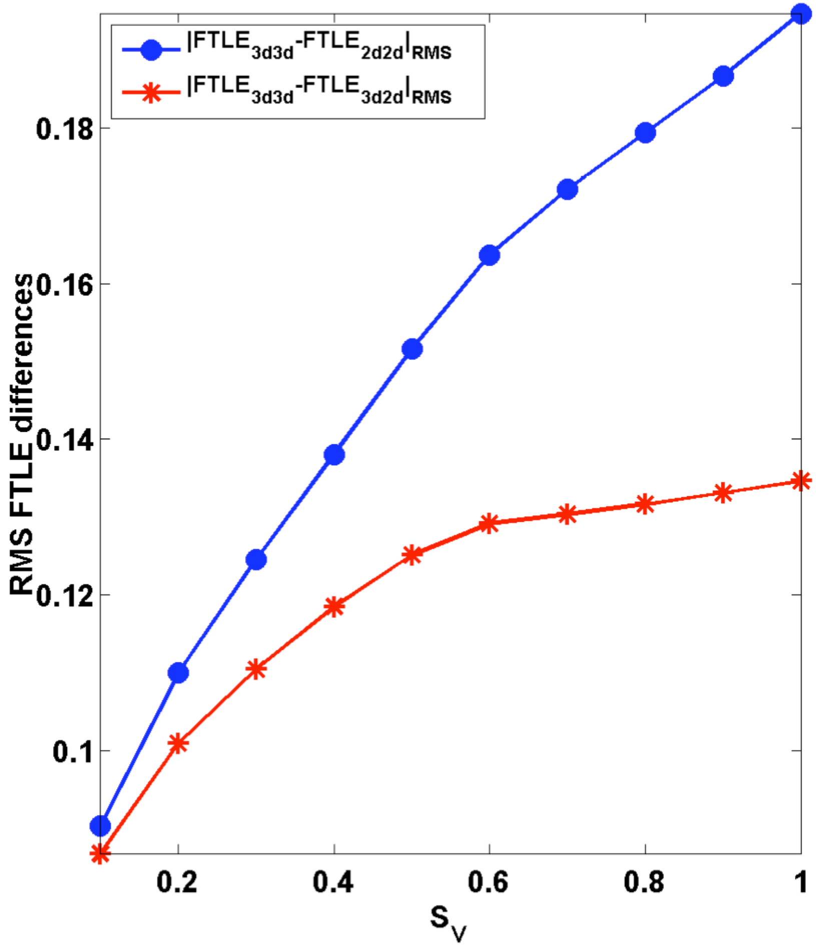
FTLE Differences





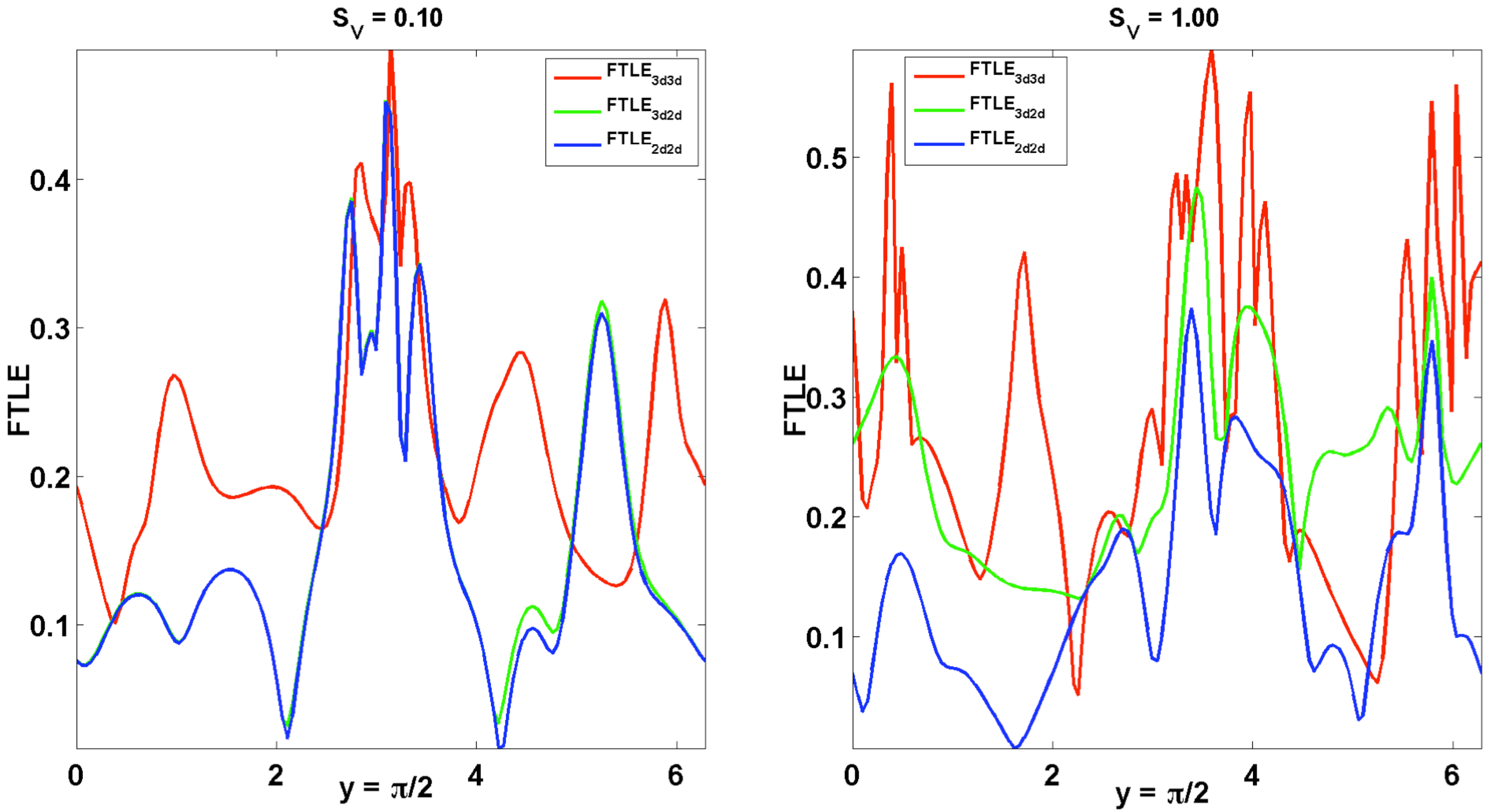
RMS FTLE Differences

$$S_V = \sqrt{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}$$

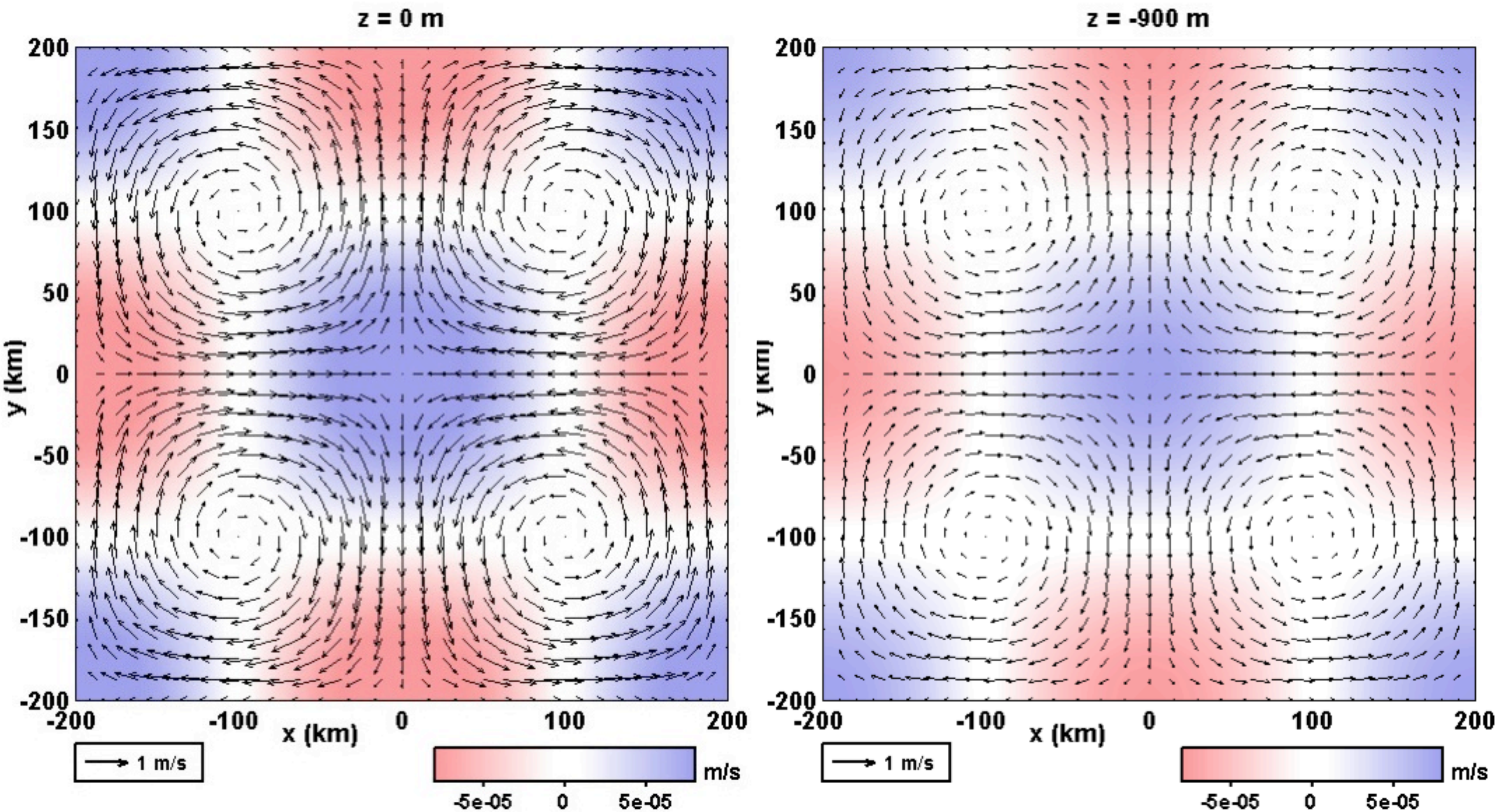


It Doesn't Look Good for Reduced Representations of Cauchy - Green

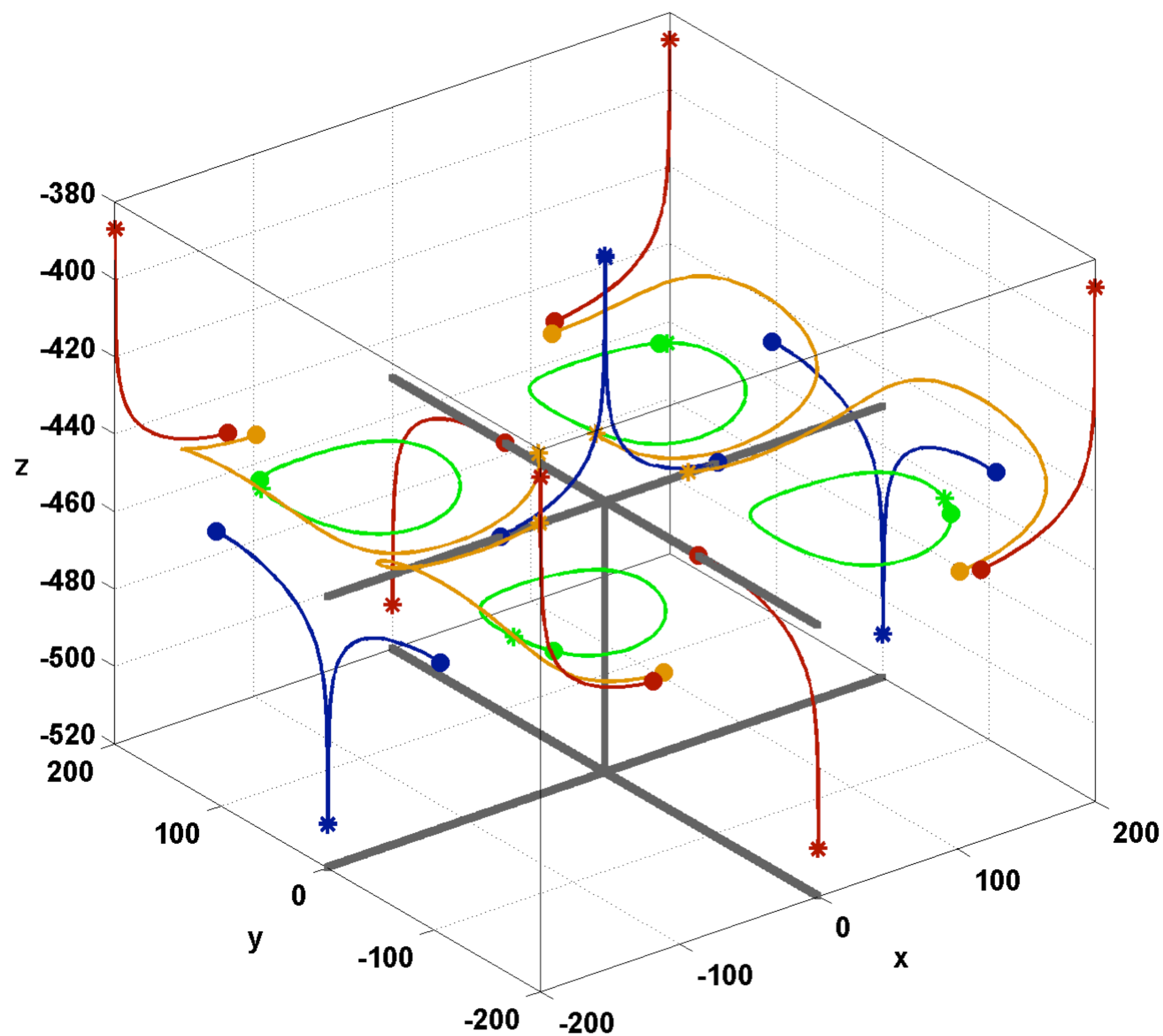
$$S_V = \sqrt{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}$$



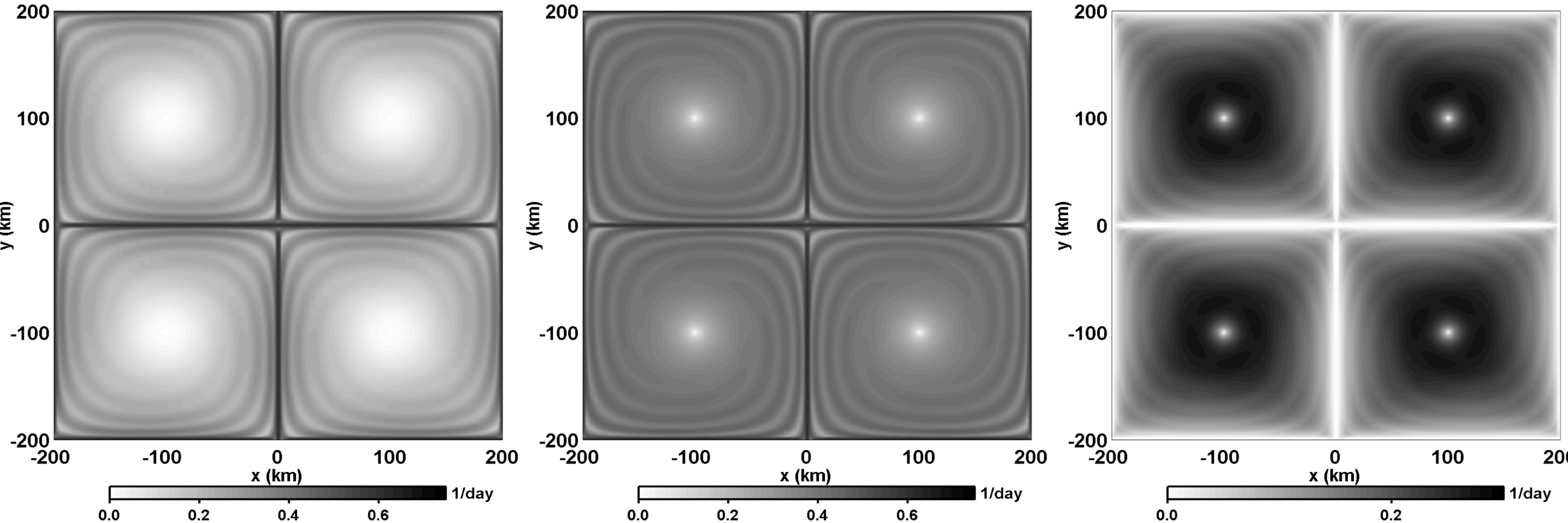
Quadrupole - Eulerian View



Lagrangian View at - 450m



FTLE Comparison – 450m



3D

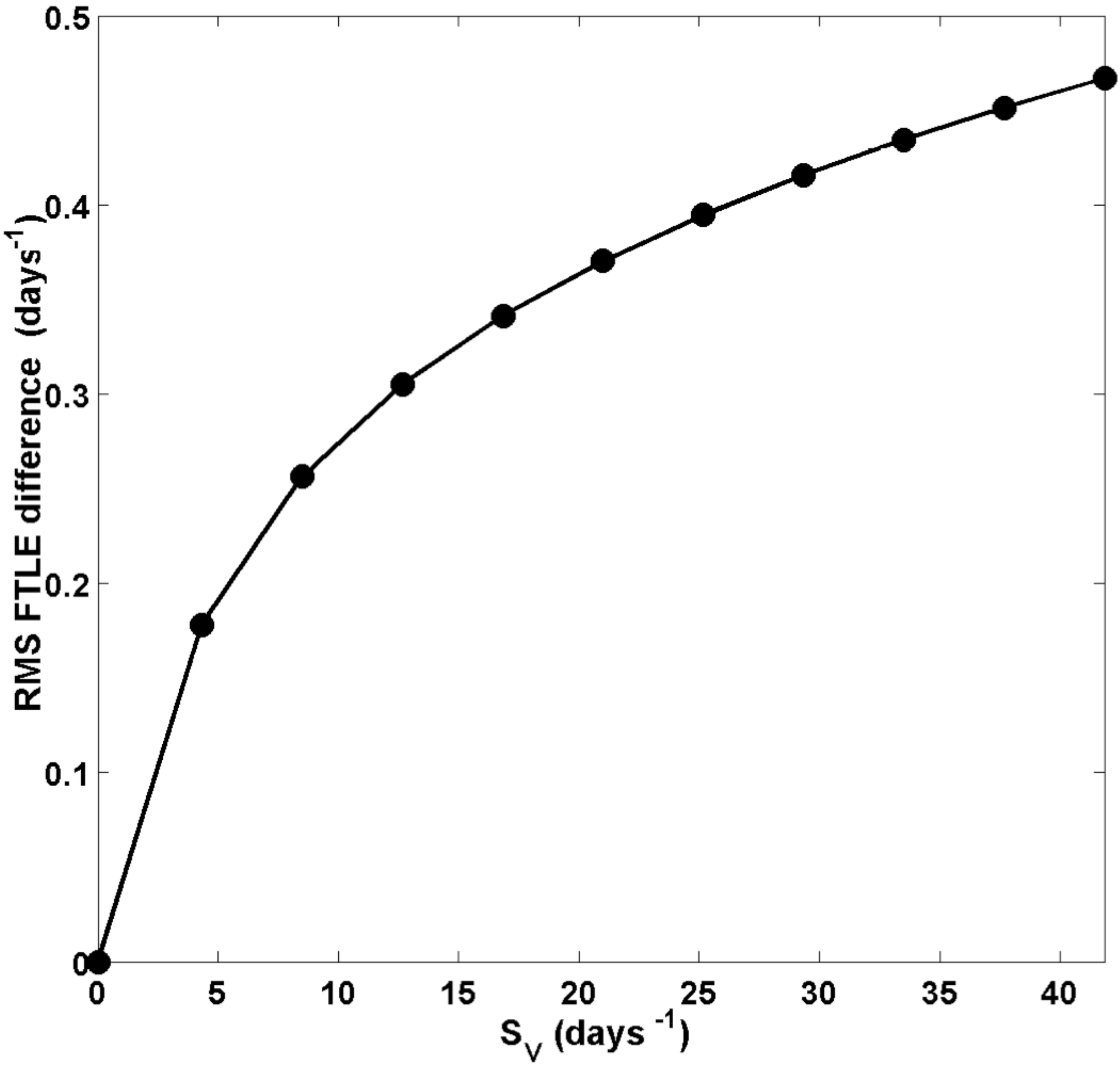
3D2D

Difference

Effect of Vertical Shear on FTLE

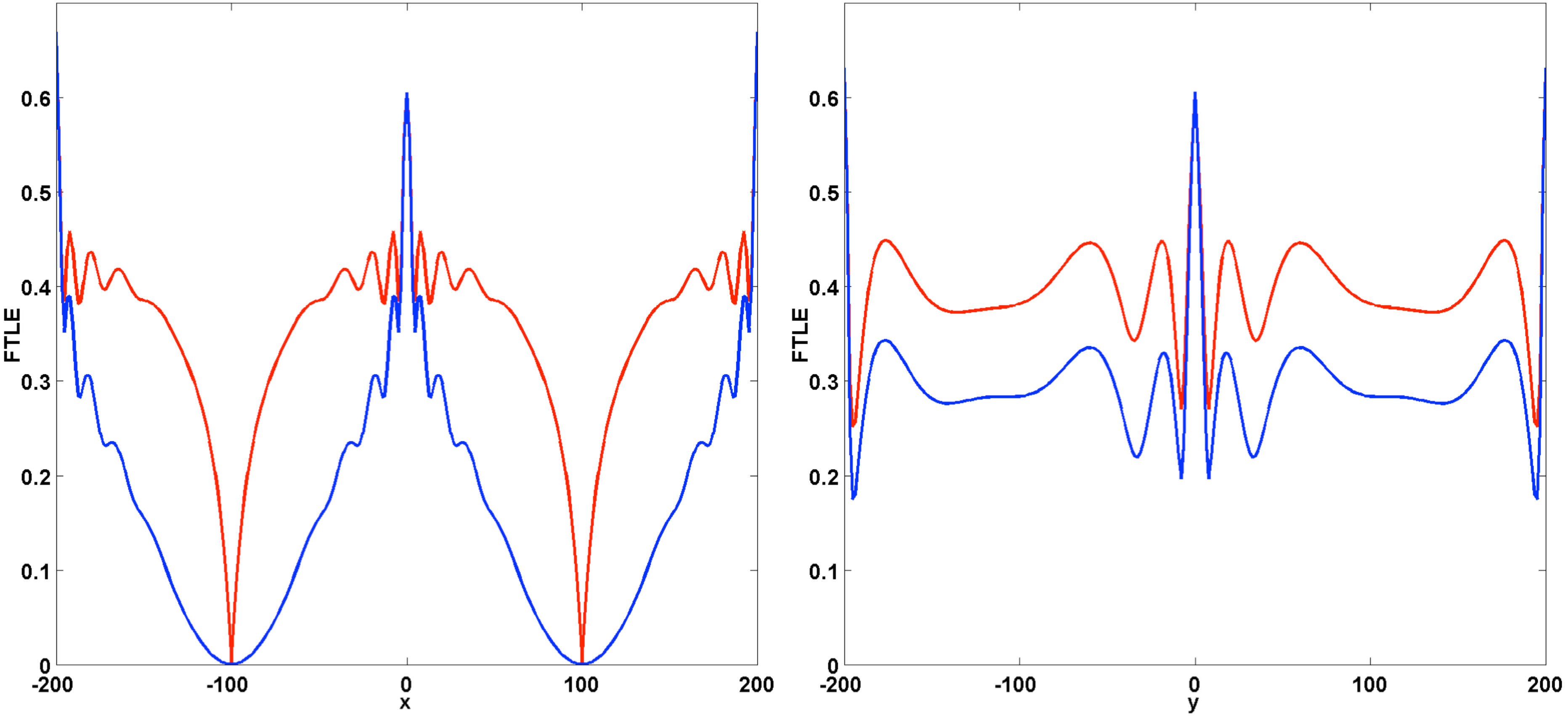
$$S_V = \sqrt{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}$$

$$S_V \leq 2N$$



FTLE Comparison

Figure 4. Profiles of FTLE3d (red) and FTLE2d (blue) at z = - 450 m. Left: Profile along y = - 100 km; right: Profile along x = - 12.5 km.



What to Expect in Ocean

For large FTLE $\partial x \approx \partial y = \triangle_H \approx 50$ km and $\partial z = \triangle_V \approx 0.4$km

And $\partial x_0 \approx \partial y_0 = \triangle_{H_0} \approx 5$km and $\partial z_0 \approx \triangle_{V_0} \approx 0.03$km

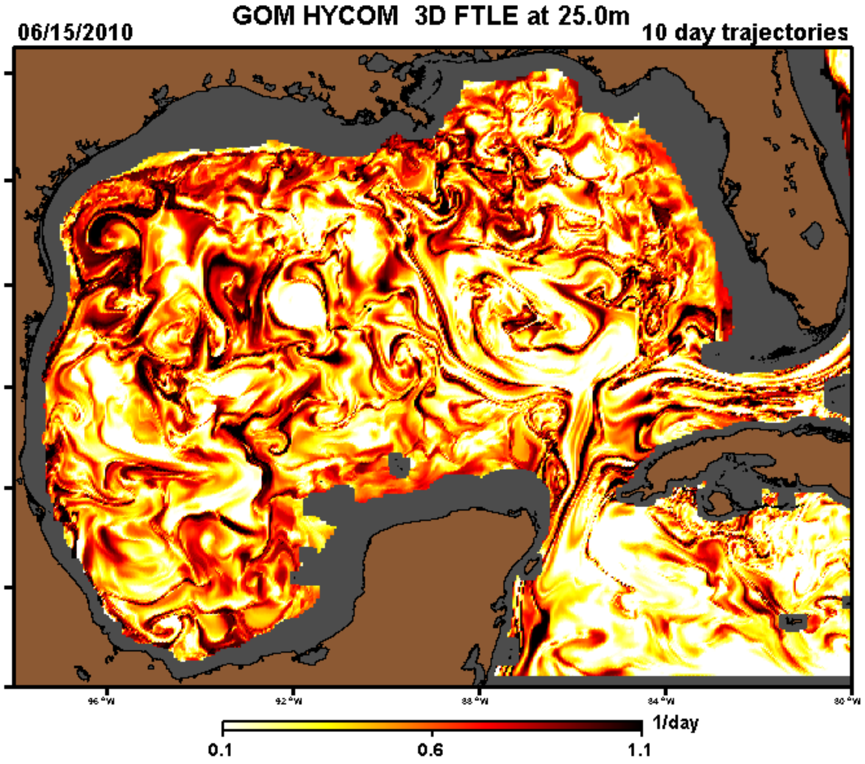
Then

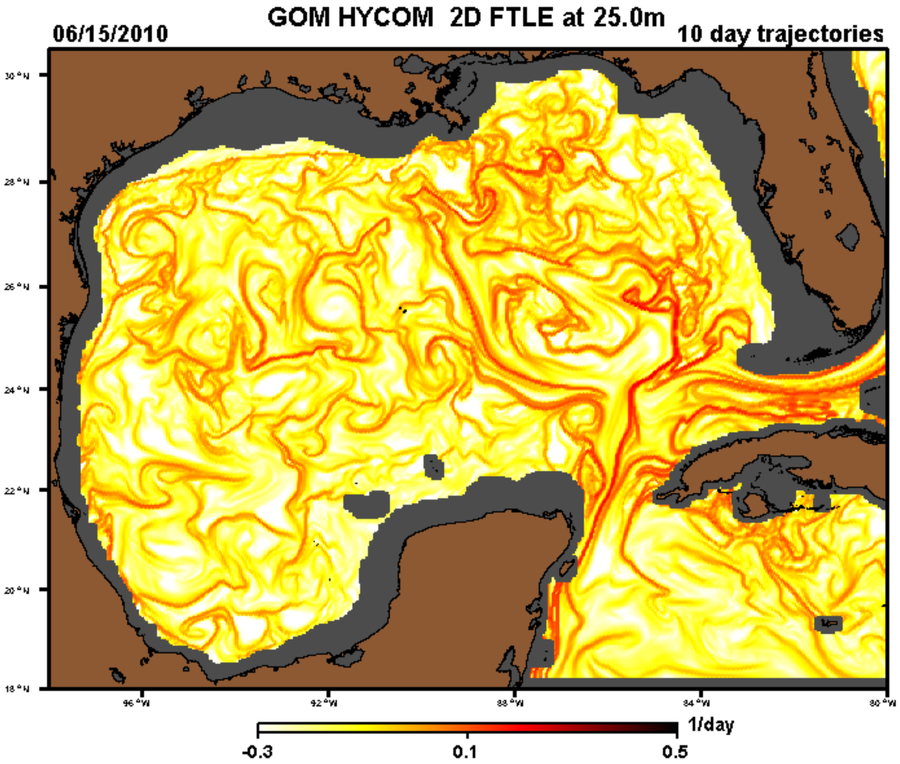
$$\begin{bmatrix} \frac{\partial x}{\partial x_0} = a \approx 10 & \frac{\partial x}{\partial y_0} = b \approx 10 & \frac{\partial x}{\partial z_0} = \Gamma \approx 1.7 \cdot 10^3 \\ \frac{\partial y}{\partial x_0} = c \approx 10 & \frac{\partial y}{\partial y_0} = d \approx 10 & \frac{\partial y}{\partial z_0} = \Gamma \approx 1.7 \cdot 10^3 \\ \frac{\partial z}{\partial x_0} = \epsilon \approx 4 \cdot 10^{-2} & \frac{\partial z}{\partial y_0} = \epsilon \approx 4 \cdot 10^{-2} & \frac{\partial z}{\partial z_0} = e \approx 1.4 \cdot 10. \end{bmatrix}$$

And

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right)^T \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right) = \begin{bmatrix} a^2 + c^2 & ab + cd & \Gamma(a+b) \\ ab + cd & b^2 + d^2 & \Gamma(b+d) \\ \Gamma(a+c) & \Gamma(b+d) & 2\Gamma^2 + e^2 \end{bmatrix}$$

Application to GoM HYCOM





What We Get

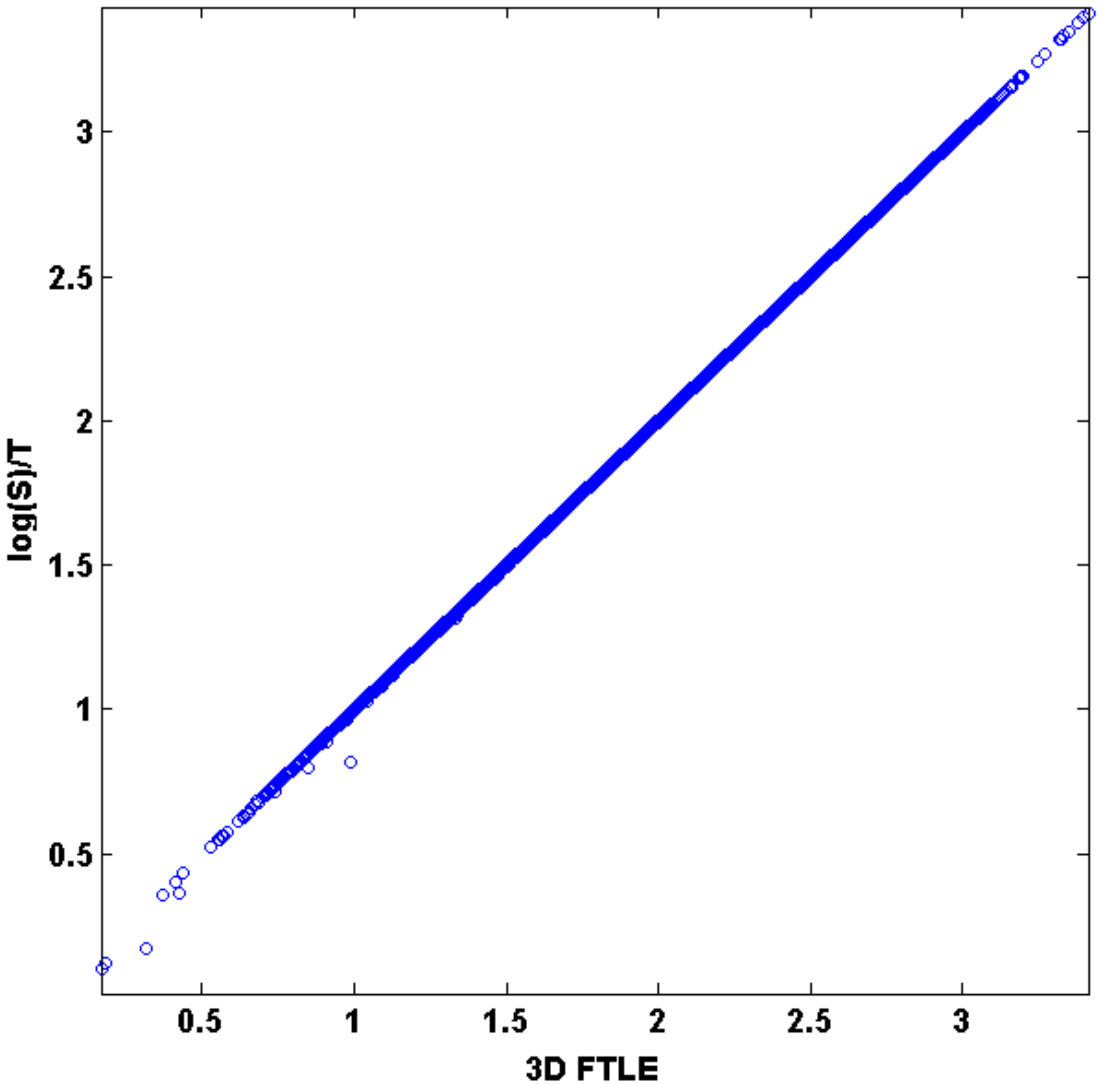
$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \begin{bmatrix} 42 & 39 & 4057 \\ 21 & 17 & 1753 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices borderline ill-conditioned.

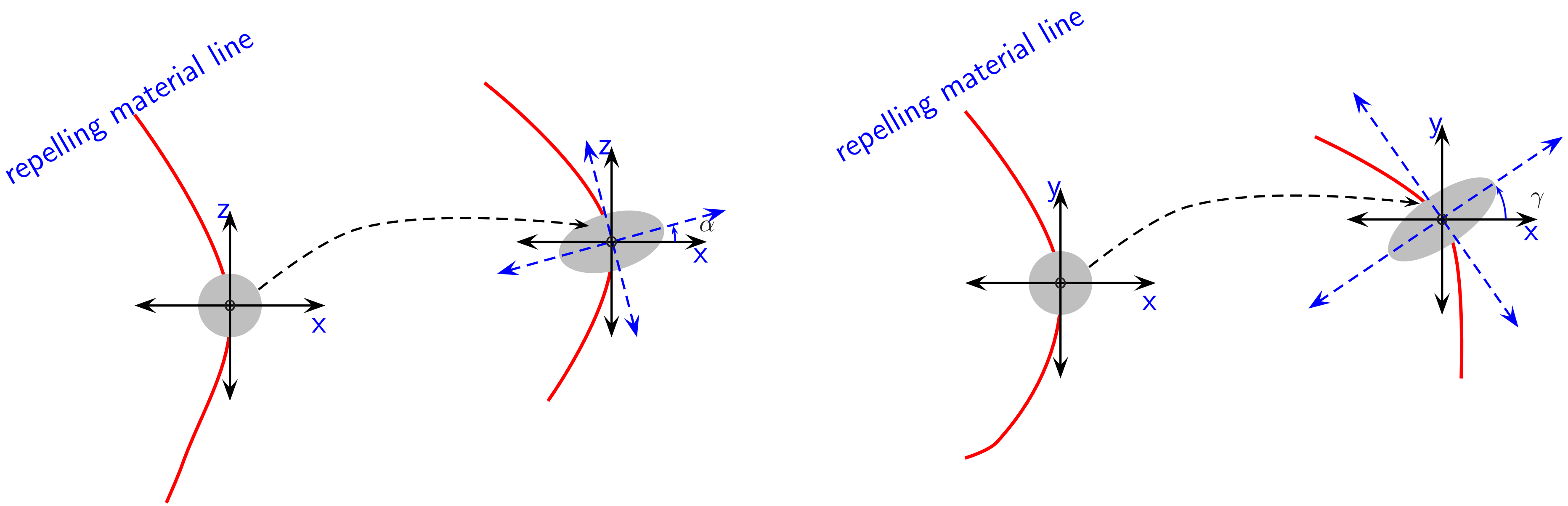
But only need largest eigenvalue, which is nearly:

$$C_{33} = 1 + \left(\frac{\partial x}{\partial z_0}\right)^2 + \left(\frac{\partial y}{\partial z_0}\right)^2 = 6.5 \cdot 10^4$$

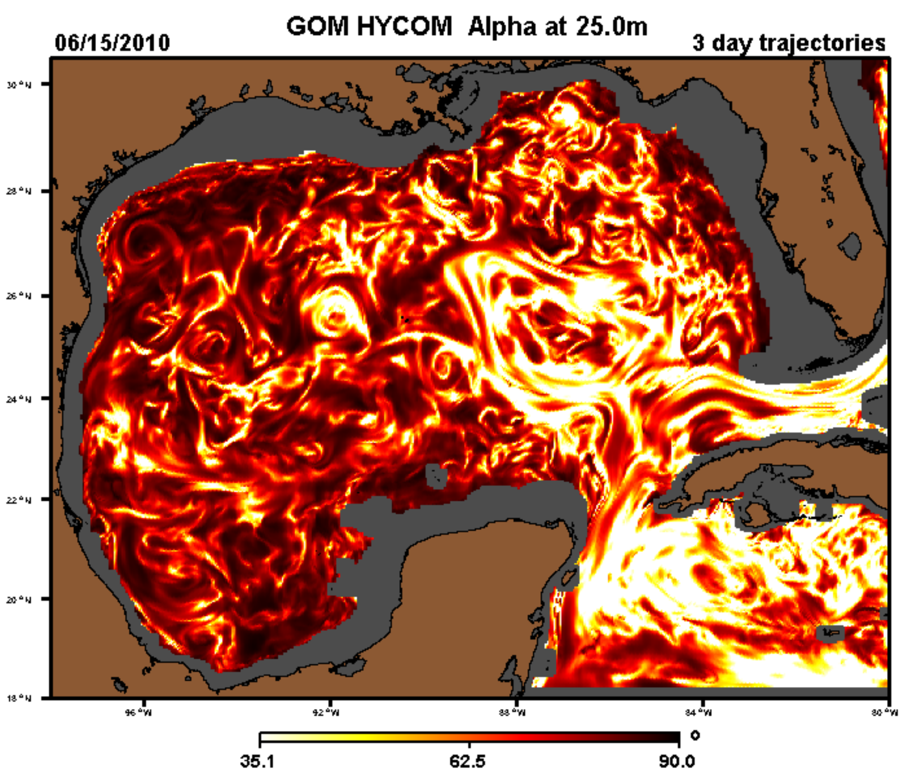
3D2D FTLE vs C_{33}

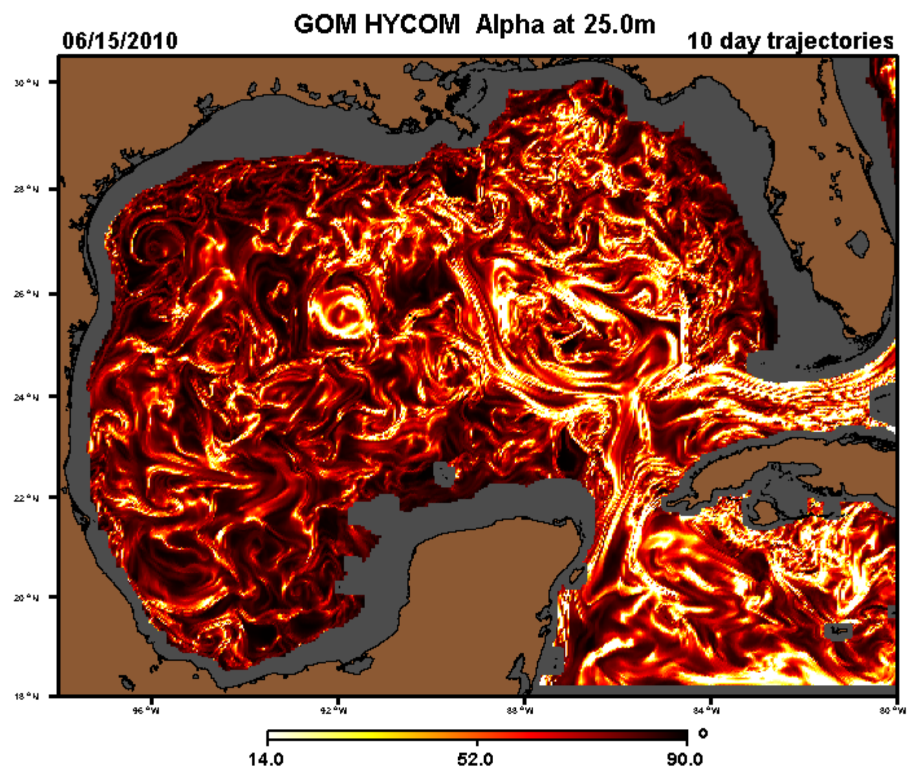


3D2D Eigendirections

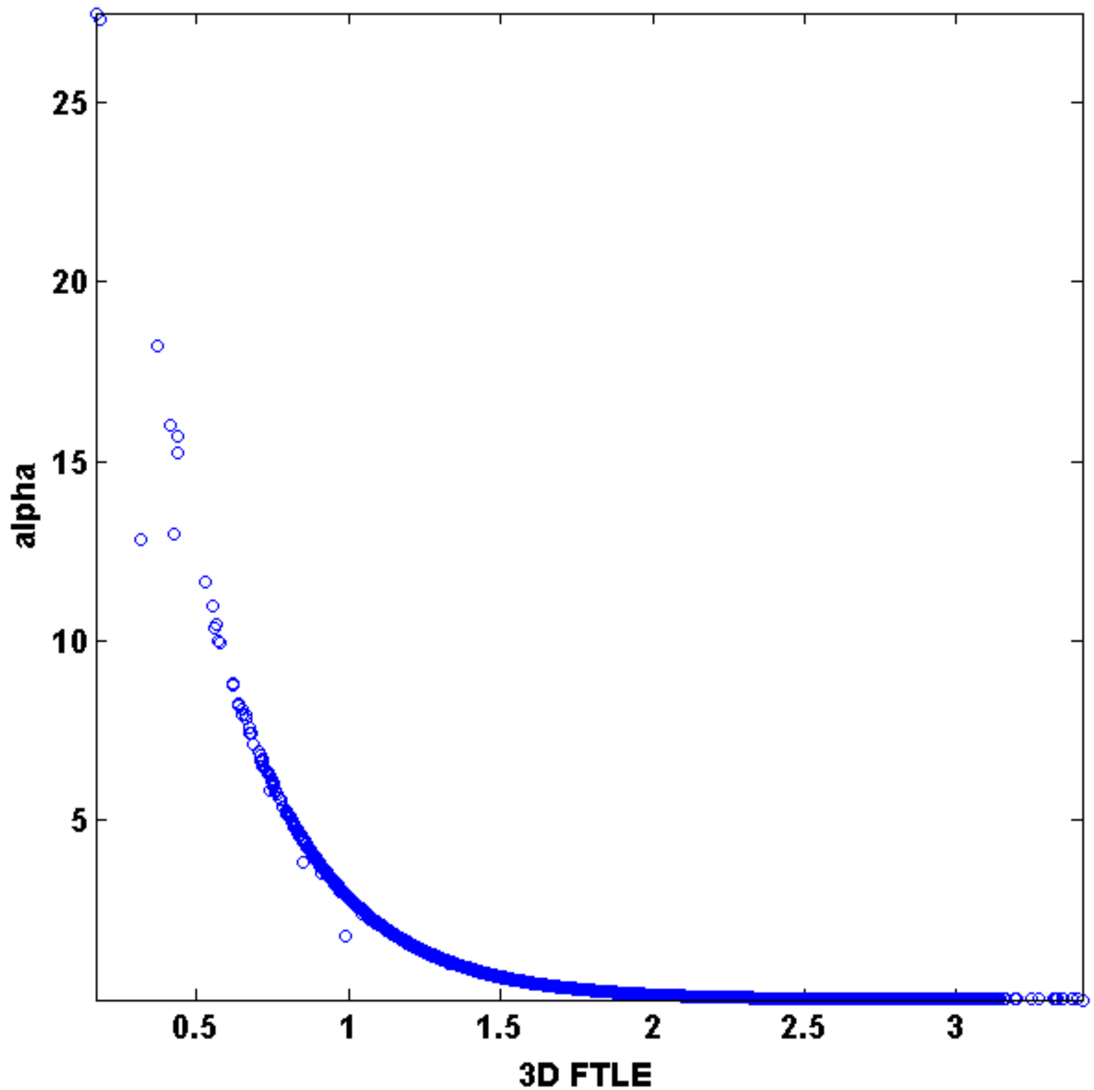


Vertical Angle of Max Stretch

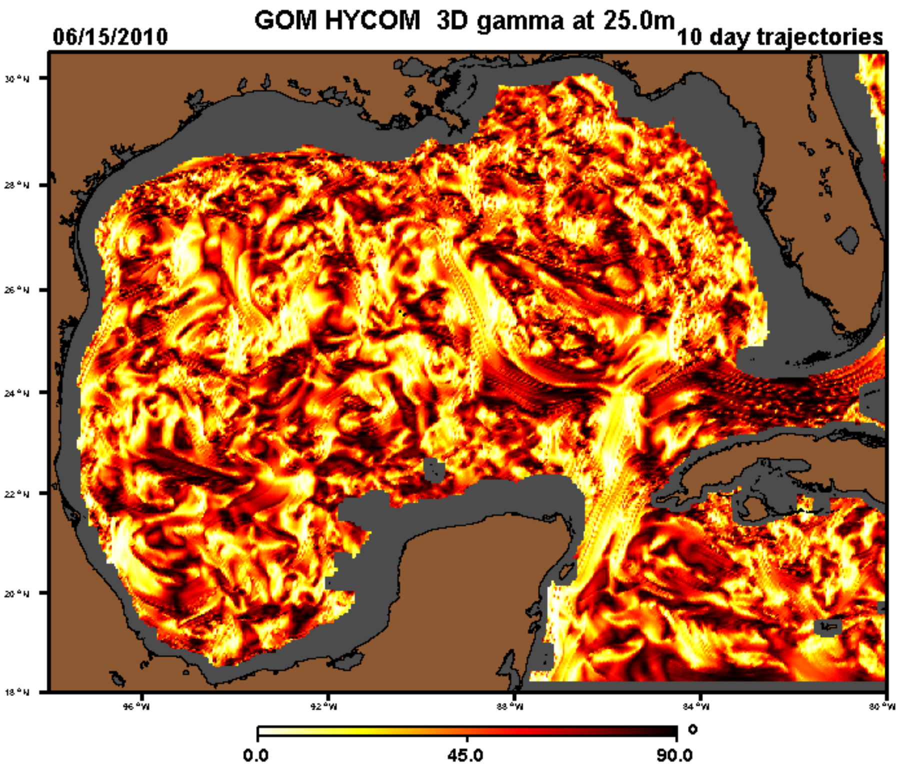


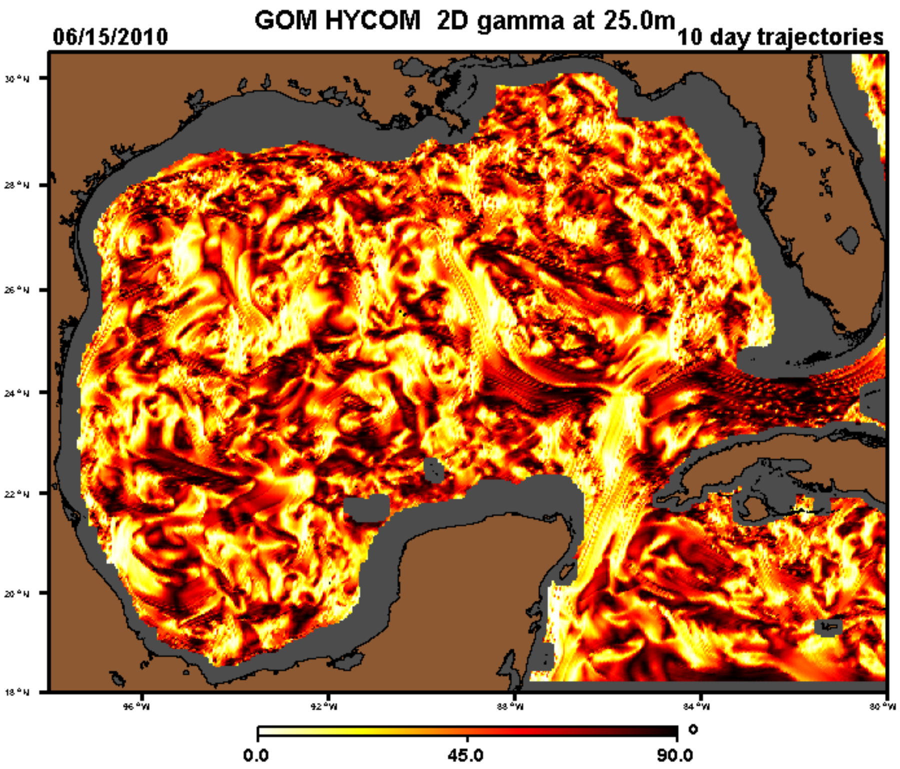


3D2D FTLE Directions



Horizontal Angle of Max Stretch





Discussion

- In the ocean
  - 2D velocity fields capture essence of 3D FTLE fields
  - Large stretch is nearly horizontal

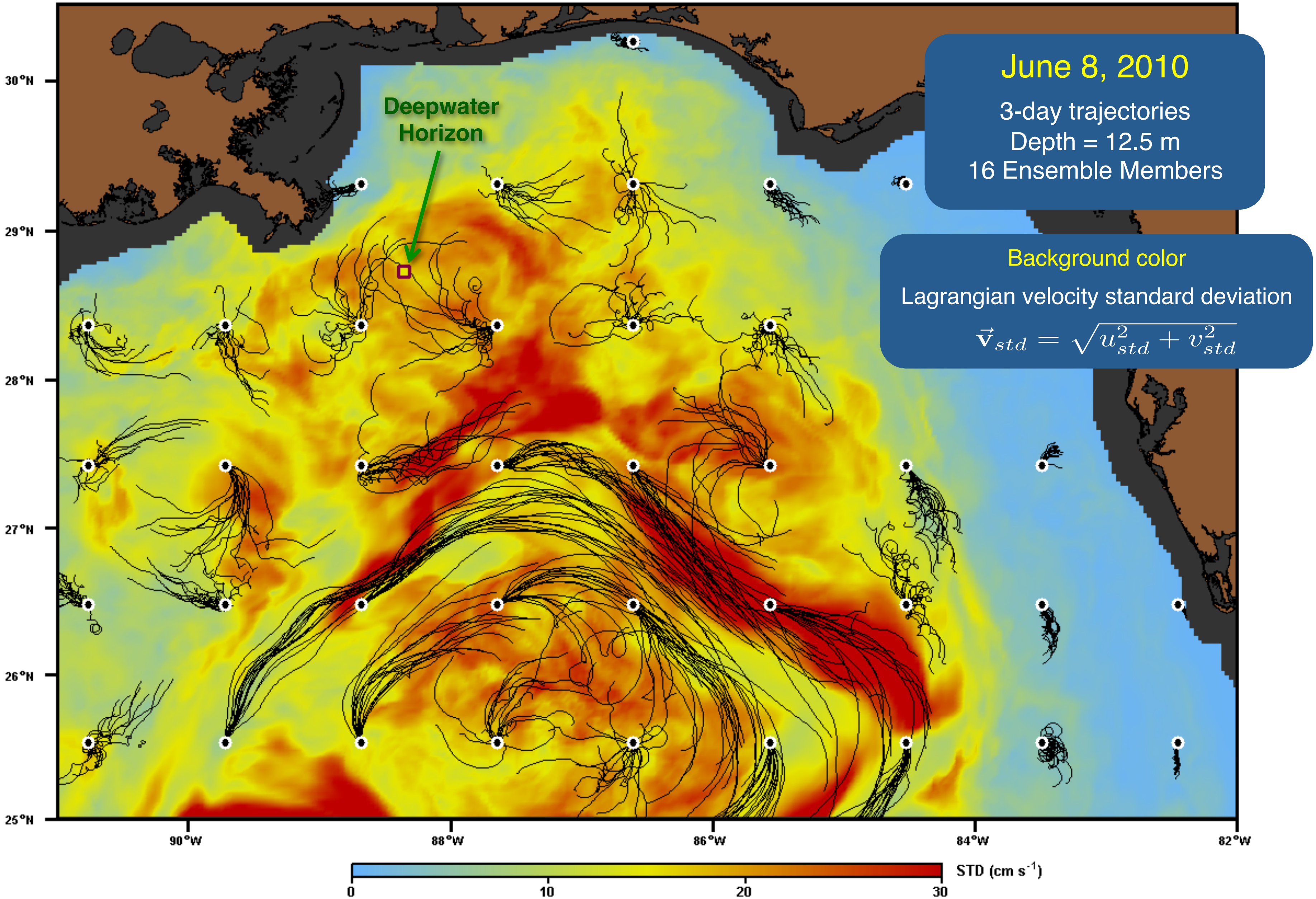
- In GoM 3D FTLE Surrogate: $S_V = \left(\partial x/\partial z_0\right)^2 + \left(\partial y/\partial z_0\right)^2$

- Next
  - Test S_V with other metrics
  - Construct 2D transport barriers from data-assimilating OGCMs
  - **No more Mr Nice Guy on 2D pictures!**
  - Eddy formation, census, and 3D transport

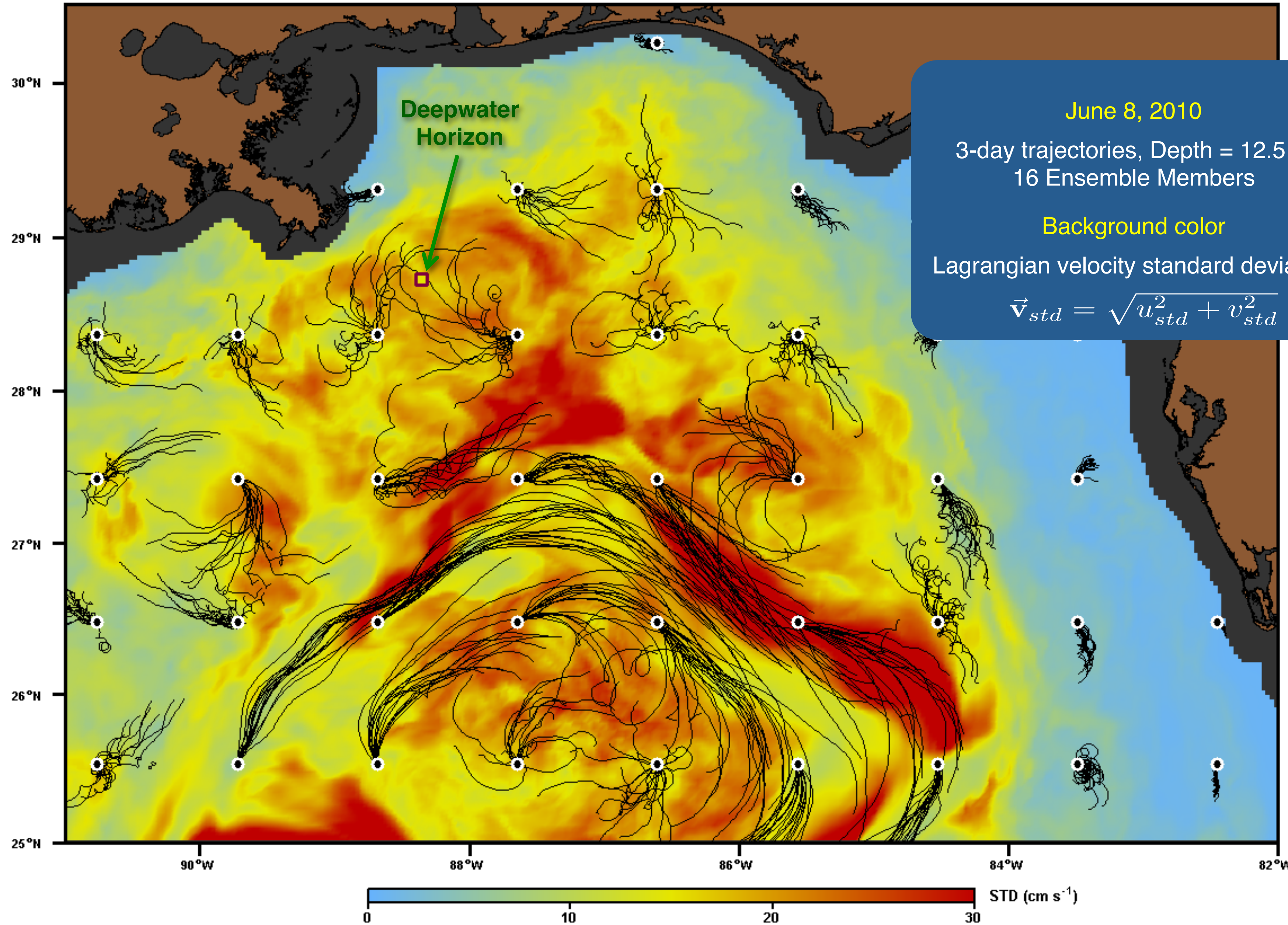
Lagrangian Predictability

Ensembles

RELO in the Northern GoM: Ensemble Spread



RELO in the Northern GoM: Ensemble Spread



June 8, 2010

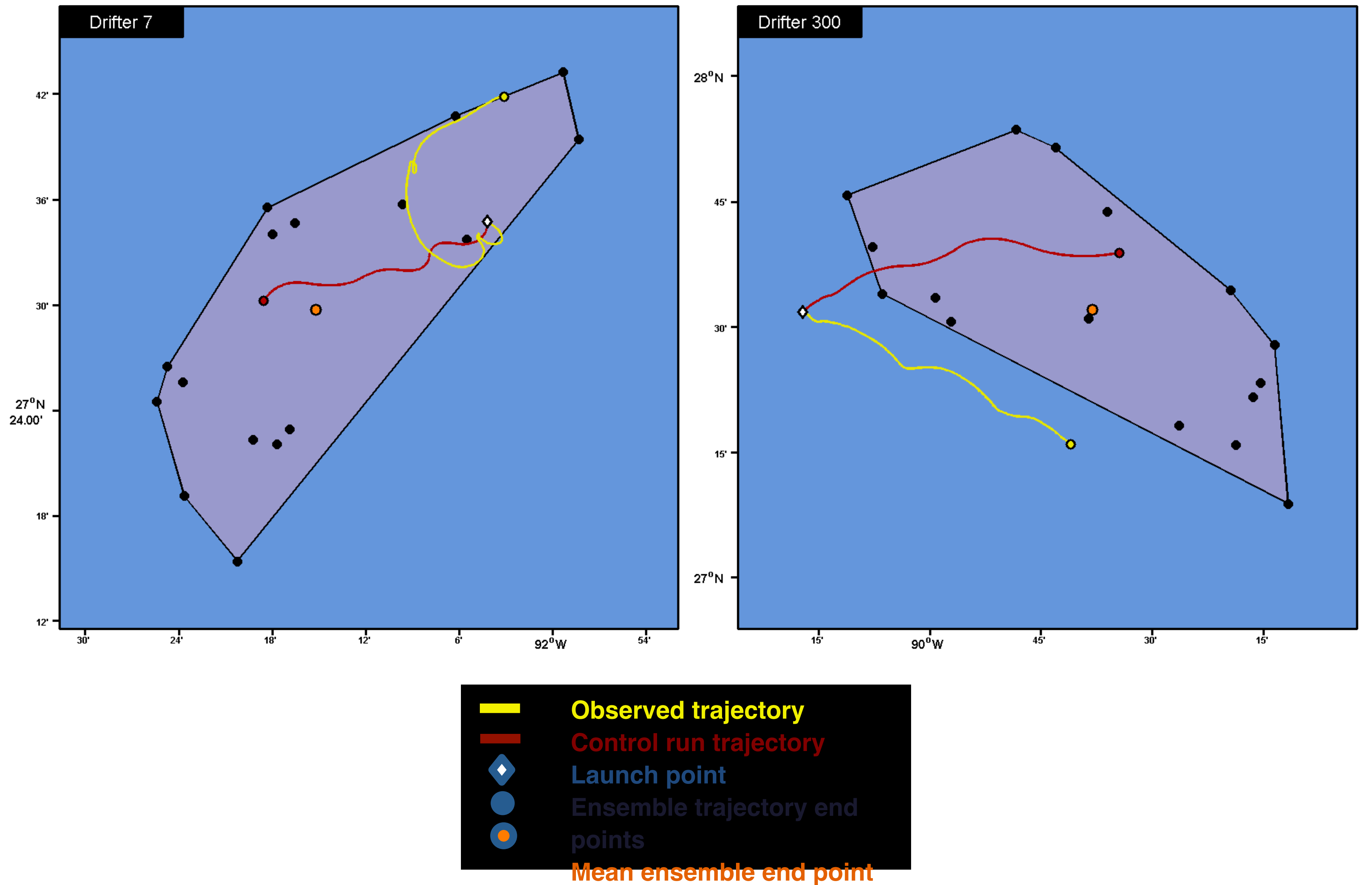
3-day trajectories, Depth = 12.5 m
16 Ensemble Members

Background color

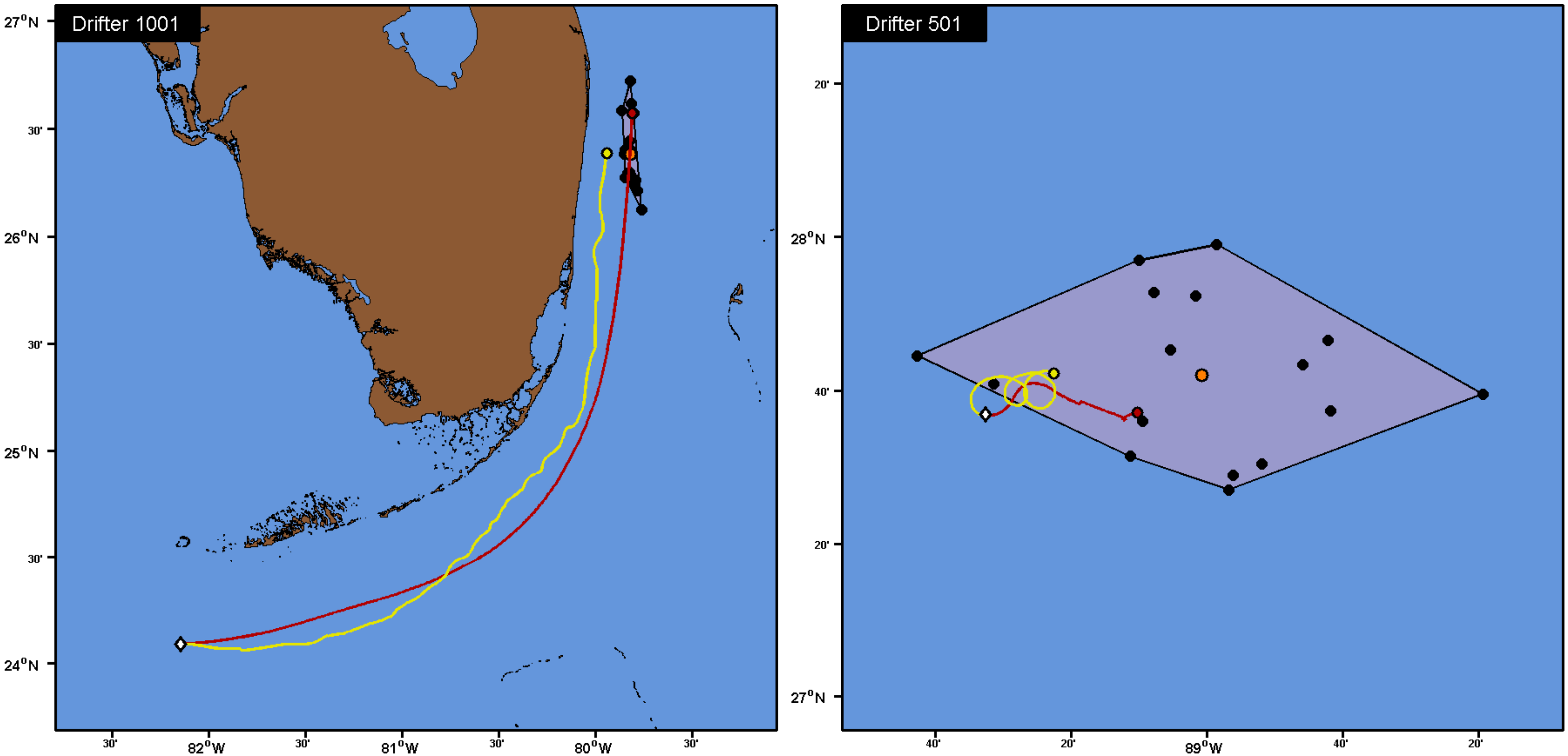
Lagrangian velocity standard deviation

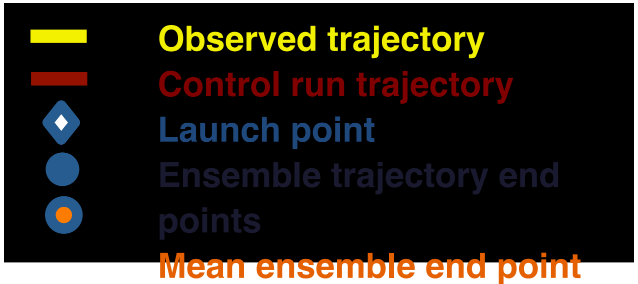
$$\vec{\mathbf{v}}_{std} = \sqrt{u_{std}^2 + v_{std}^2}$$

Example RELO Ensemble



Example RELO Ensemble





All is Based on Data Assimilating Models – How Good are They?