NUMERICAL MODELLING OF PRESSURE GRADIENT EFFECTS ON OCEANIC TURBULENT MIXING LAYERS

T. CHACÓN-REBOLLO, M. GÓMEZ-MÁRMOL, S. RUBINO

(chacon@us.es, macarena@us.es, samuele@us.es)

Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla

INTRODUCTION

The **Sea Surface Temperature** is closely related to different aspects of the oceanic biosystems and global climate changes. For this reason, its right computation is of primary importance, especially in Tropical regions, where the high temperatures generate surface turbulent mixing layers.

This problem has been analyzed by the physical oceanography community since the early *80's*. Several PDEs models, known as standard, have been formulated (*cf.* [5], [3]). Turbulent mixing-layer models are usually 1D algebraic closure models, where the turbulent viscosity and diffusion are parametrized by the **gradient Richardson number**.

Our main purpose is to analyze a mixing-layer model, proposed by R. Lewandowski (IRMAR, Université de Rennes 1), with imposed pressure gradients, usually neglected in the standard mixing-layer models. We perform a non-linear asymptotic analysis proving that the discrete time iterates approach the continuous equilibria. We also show the convenience of including pressure gradients in the model for 1D initial conditions with horizontal small perturbations.

RICHARDSON NUMBER-BASED 1D EDDY DIFFUSION MODEL

We assume that the mixing layer is strongly dominated by vertical fluxes, and that the flow is turbulent, so that we represent respectively the mean velocity and density of the fluid by:

$$\mathbf{U} = (u(z,t), v(z,t), w(z,t)), \quad \rho = \rho(z,t).$$

Note that the Coriolis force is negligible in Equatorial regions (*cf.* [6]). Moreover, we assume that the vertical equilibrium has been reached:

$$w(z,t) = 0, \ \forall z \in [-h, 0], \ \forall t \geq 0.$$

The problem for the analysis of the mixing layer becomes:

$$\begin{cases} \partial_t u - \partial_z(\nu_1 \partial_z u) = -D_1, \\ \partial_t v - \partial_z(\nu_1 \partial_z v) = -D_2, \quad \text{for } t \geq 0 \text{ and } -h \leq z \leq 0, \\ \partial_t \rho - \partial_z(\nu_2 \partial_z \rho) = 0, \end{cases} \tag{1}$$

where $\nu_1 = a_1 + \nu_{T1}$, $\nu_2 = a_2 + \nu_{T2}$ respectively are the total viscosity and diffusion, and (D_1, D_2) is the horizontal pressure gradient $\nabla_H p$, that we assume to be known. Let us complete model (1) with initial and boundary conditions:

$$\begin{cases} u = u_b, \ v = v_b, \ \rho = \rho_b \ \text{at the depth } z = -h, \\ \nu_1 \partial_z u = \dfrac{\rho_a}{\rho_r} V_x, \ \nu_1 \partial_z v = \dfrac{\rho_a}{\rho_r} V_y, \ \nu_2 \partial_z \rho = Q \ \text{at the surface } z = 0, \\ u = u_0, \ v = v_0, \ \rho = \rho_0 \ \text{at initial time } t = 0, \end{cases} \tag{2}$$

where ρ_a is the air density, ρ_r is a reference density, V_x and V_y are respectively the stress exerted by the zonal and the meridional wind-stress, and Q represents thermodynamic fluxes. To describe the modelling of the eddy coefficients ν_{T1} and ν_{T2}, we consider a non-standard model (*cf.* [1]) that brings to positive eddy diffusion ν_{T2} for negative gradient Richardson number R:

$$\nu_1 = f_1(R) = a_1 + \frac{b_1}{(1+5R)^2}, \quad \nu_2 = f_2(R) = a_2 + \frac{f_1(R)}{(1+5R)^2},$$

$$R = -\frac{g}{\rho_r} \frac{\partial_z \rho}{(\partial_z u)^2 + (\partial_z v)^2}.$$

NUMERICAL DISCRETIZATION AND RESULTS

Numerical approximation

We discretize the IBV problem (1)-(2) by linear piecewise FE. We set $I = (-h, 0)$, and we construct the FE space:

$$V_\Delta = \{ w_\Delta \in C^0(\overline{I}) \,|\, w_{\Delta|_{(z_{i-1}, z_i)}} \text{ is affine}, i = 1, \ldots, N; \ w_\Delta(-h) = 0 \}. \tag{3}$$

To discretize the equation for u, for instance, we consider the method:

$$\text{Obtain} \quad u_\Delta \in u_b + V_\Delta \quad \text{s.t.}$$

$$\int_{-h}^0 \frac{u_\Delta^{n+1} - u_\Delta^n}{\Delta t} w_\Delta + \int_{-h}^0 f_1(R_\Delta^{n+\gamma}) \partial_z u_\Delta^{n+1} \partial_z w_\Delta = L(w_\Delta), \quad \forall w_\Delta \in V_\Delta, \tag{4}$$

where $\gamma = 0$ for a semi-implicit method, and $\gamma = 1$ for a fully implicit method. This scheme is replaced in practice by a finite difference discretization obtained by numerical integration. We consider similar discretizations for v and ρ. These discretizations are stable under the following:

Hypothesis 1: $f_1, f_2 \in C^1(\mathbb{R}^3)$, and:

$$\exists \quad 0 < \nu \leq M \quad \text{s.t.} \quad \nu \leq f_1(R), f_2(R) \leq M, \quad \forall R \in \mathbb{R}.$$

Analysis of discrete equilibria

To analyze the discrete equilibrium solutions, we reformulate the discrete steady problem as an equivalent system of algebraic equations for the unknowns:

$$\alpha_\Delta = \partial_z u_\Delta, \quad \beta_\Delta = \partial_z v_\Delta, \quad \theta_\Delta = \partial_z \rho_\Delta.$$

Theorem 0.1 *Under Hypothesis 1, the steady problem admits, for small enough data, a unique solution that verifies:*

$$C_1 \leq |\alpha_\Delta|, |\beta_\Delta|, |\theta_\Delta| \leq C_2, \tag{5}$$

for some positive constants C_1, C_2 depending on the data (cf. [2]).

Theorem 0.2 *Under Hypothesis 1, for small enough data, the sequence $\{(u_\Delta, v_\Delta, \rho_\Delta)\}_{\Delta z > 0}$ is strongly convergent in $[H^1(I)]^3$ to a weak solution (u, v, ρ) of the steady version of problem (1)-(2) (cf. [2]).*

To prove that the continuous equilibria are well approximated by the solution of the evolutive discrete problem (4), we assume the following:

Hypothesis 2: $f_1, f_2 \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$.

Theorem 0.3 *Assume that Hypotheses 1 and 2 hold. Then, for small enough data, the implicit version of the discrete method (4) is asymptotically stable, in the sense that:*

$$\limsup_{n \to +\infty} ||\mathbf{U}_\Delta^n - \Pi_\Delta \mathbf{U}^e||_{L^2(I)} \leq C \Delta z,$$

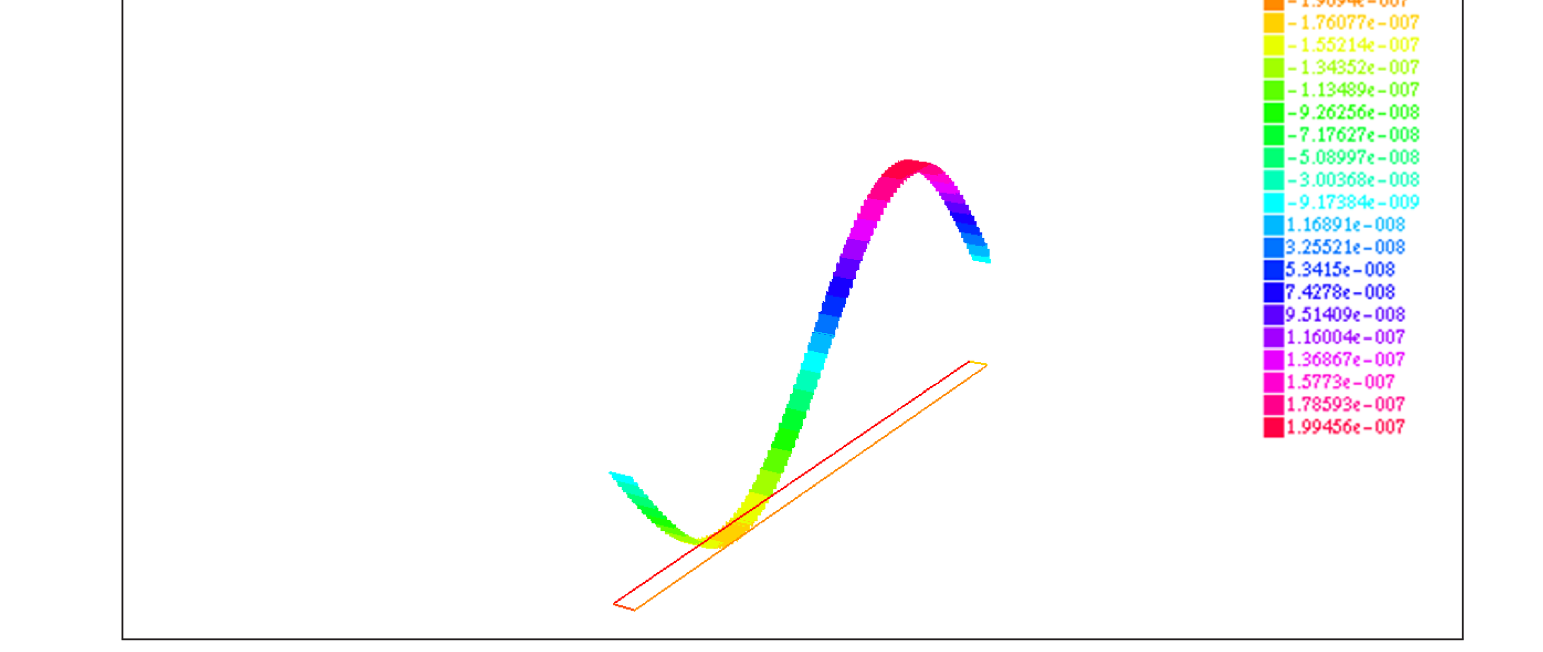
where C is a positive constant, and Π_Δ denotes the orthogonal $L^2(I)$ projection onto the space V_Δ (cf. [2]).

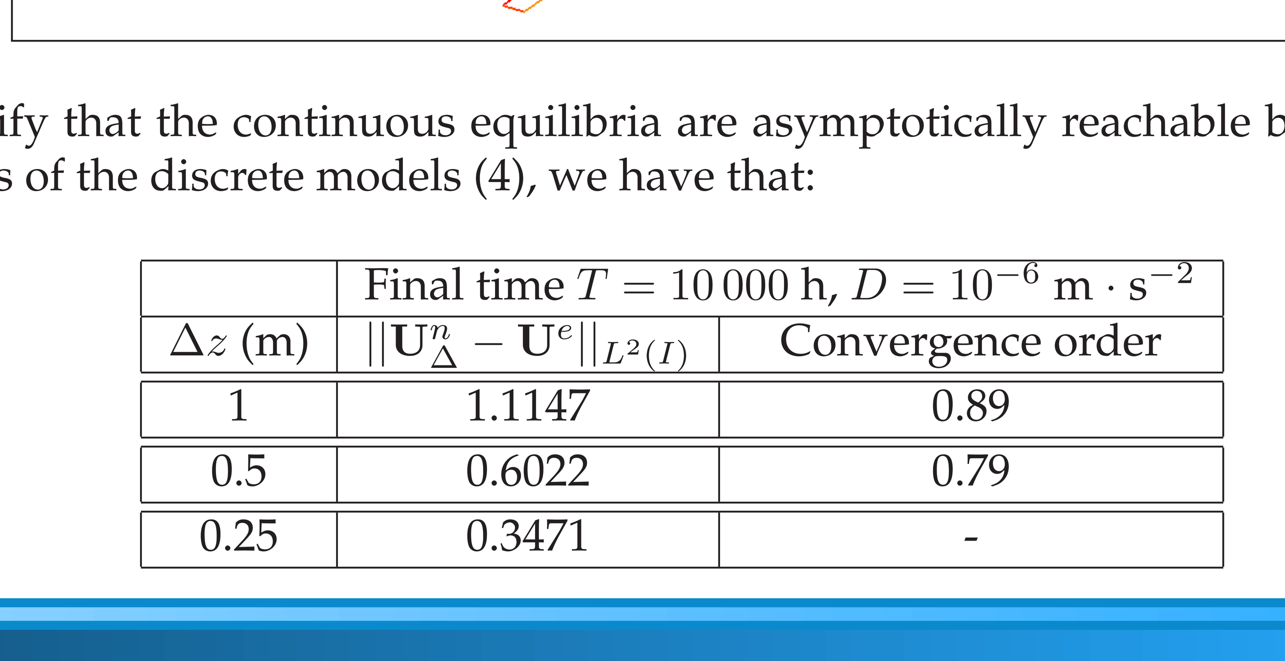
Numerical tests

To test the convenience of including pressure gradients in model (1)-(2), we propose to compare it with a commonly used model in ocean flow modelling: the **Primitive Equations** (2D). For this model, we use 2D initial conditions, which are the sum of 1D zonal velocity and density profiles, and an horizontal perturbation:

$$u(0) = u_0(z) + \lambda \sigma(x), \quad \rho(0) = \rho_0(z) + \lambda \sigma(x). \tag{6}$$

The initial 1D zonal velocity and density profiles correspond to data taken from the **Tropical Atmosphere Ocean** array (*cf.* [4]). For the boundary conditions, we impose typical values for the Equatorial Pacific region called **West-Pacific Warm Pool** (*cf.* [3]).





To verify that the continuous equilibria are asymptotically reachable by the time iterates of the discrete models (4), we have that:

	Final time $T = 10\,000$ h, $D = 10^{-6}$ m $\cdot$ s^{-2}	
Δz (m)	$\|\|\mathbf{U}_\Delta^n - \mathbf{U}^e\|\|_{L^2(I)}$	Convergence order
1	1.1147	0.89
0.5	0.6022	0.79
0.25	0.3471	-

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