

PARAMETRICALLY FORCED CGLE

- General framework: spatio-temporal dynamics in systems with **broken symmetry** → nucleation of **spatio-temporal patterns**
- Resonant couplings between the forcing and oscillatory modes may lead to quasi-periodicity, **frequency lockings**, devil's staircases, chaos and intermittency.

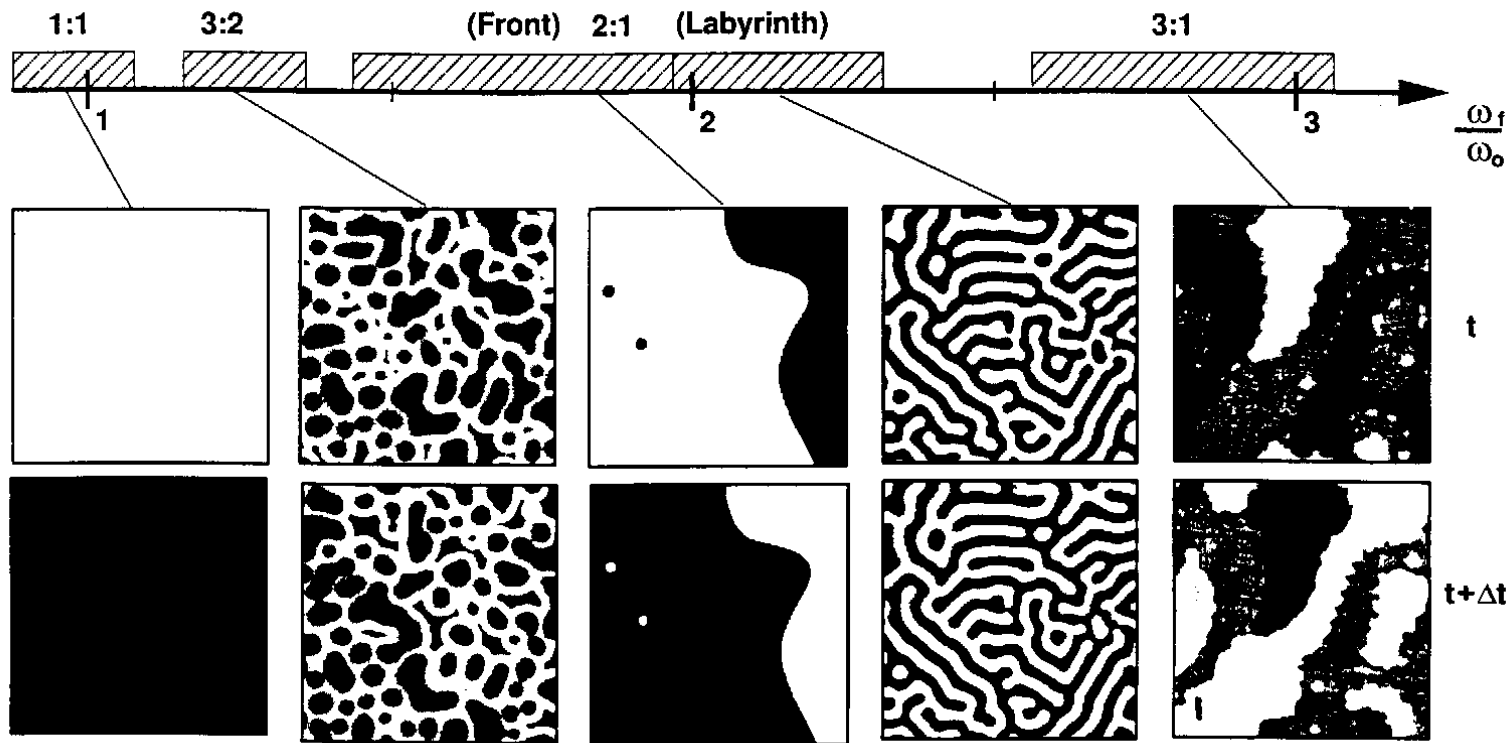


Fig.: Periodically perturbed ruthenium-catalyzed Belousov-Zhabotinsky reaction-diffusion system [A.L. Lin *et. al.*, Proceedings of IMA Workshop on Pattern Formation and Nonlocal Effects, U Minn (1998)]

- Temporal modulation of Hopf bifurcation in extended systems \Rightarrow **CGLE with broken phase symmetry**

$$\partial_t A = \mu A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)A|A|^2 + \gamma_n \bar{A}^{n-1}$$

- $\gamma_n = 0$: *Phase spirals around zeroes of A*

- $\gamma_n > \gamma_c$: *n equivalent states with different fixed phase (frequency locked solutions)*

- AIMS ($n=3$): Study of**

- Analogies with the dynamics of systems with **competing fields**
- Transition between **phase spirals** and **amplitude spirals**

UNIFORM SOLUTIONS

$$R_0 e^{i\Phi_0} \rightarrow \begin{cases} \dot{R}_0 = \mu R_0 - R_0^3 + \gamma R_0^2 \cos 3\Phi_0 \\ \dot{\Phi}_0 = -\beta R_0^2 - \gamma R_0 \sin 3\Phi_0 \end{cases}$$

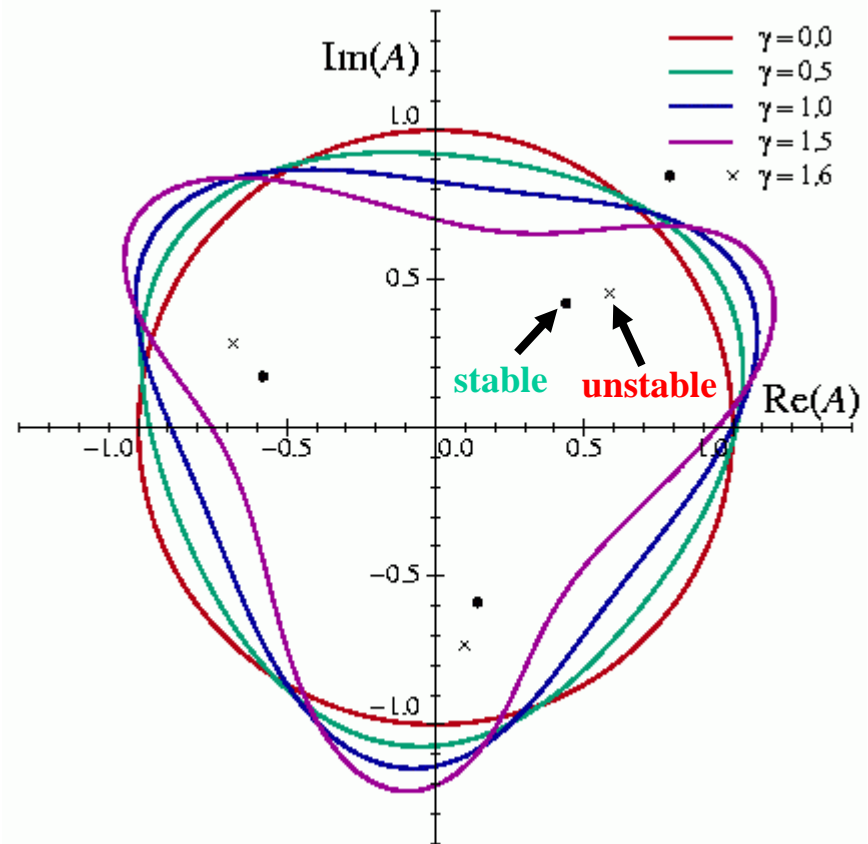
$$R_{\pm}^2 = \frac{1}{2(1+\beta^2)} \left[2\mu + \gamma^2 \pm \sqrt{\gamma^4 + 4\mu\gamma^2 - 4\mu^2\beta^2} \right]$$

$$\cos 3\Phi_0 = \frac{R_0^2 - \mu}{\gamma R_0}, \quad \sin 3\Phi_0 = \frac{-\beta R_0}{\gamma}$$

Stationary solutions exist provided that $\gamma^2 > \gamma_c^2 \Rightarrow$

Critical value: $\gamma_c^2 \equiv 2\mu(\sqrt{1+\beta^2} - 1)$

■ **OSCILLATORY REGIME ($\gamma < \gamma_c$):**
 asymptotic solutions correspond to
 temporal oscillations of limit cycle type



EXCITABLE REGIME ($\gamma > \gamma_c$): asymptotic solutions are fixed points

$$\Phi^u = R_- e^{i\Phi_-^{(k)}}, \quad k=1,2,3$$

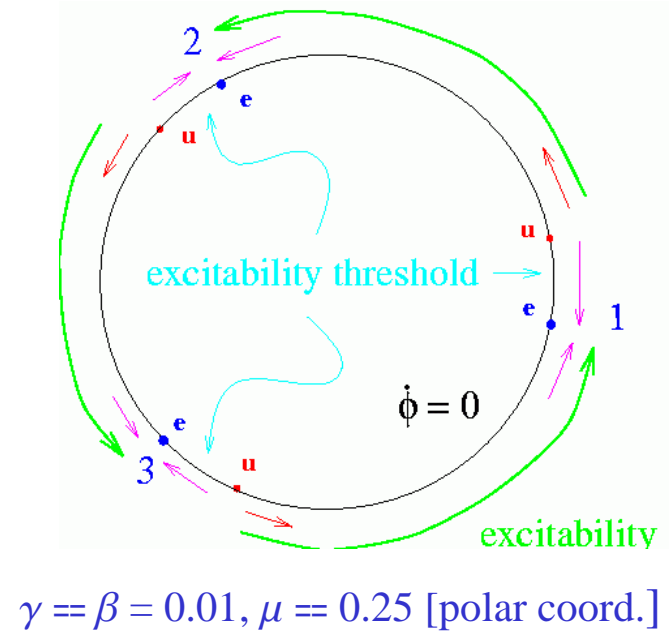
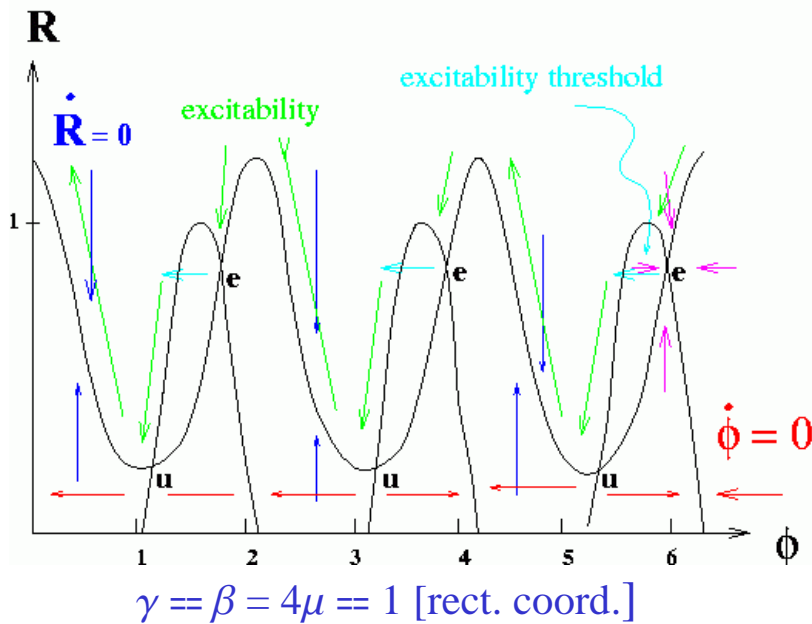
always linearly **unstable**

$$\Phi^e = R_+ e^{i\Phi_+^{(k)}}, \quad k=1,2,3$$

$|\beta| < \sqrt{3}$, **stable**

$|\beta| > \sqrt{3}$,

oscillatory unstable ($k=0, \omega \neq 0$)
 if $\gamma_c \approx \gamma_1 < \gamma < \gamma_2$. **Stable for $\gamma > \gamma_2$**



$$\beta, \gamma \ll \mu \Rightarrow \Delta\Phi = \frac{\pi}{3} - \frac{2}{3} \frac{\gamma_c}{\gamma} \quad (\text{excitability threshold})$$

PHASE APPROXIMATION

OSCILLATORY REGIME ($\gamma < \gamma_c$)

$$\left. \begin{aligned} R &= R_0(t) + \rho(r, t) \\ \Phi &= \Phi_0(t + \phi(r, t)) \end{aligned} \right\}$$

Adiabatic
elimination
of $\rho(r, t)$

Phase dynamics

$$\partial_t \phi = (1 + \alpha \bar{\beta}) \nabla^2 \phi + \kappa (\nabla \phi)^2 + \Lambda$$

$$\kappa = \frac{\int_0^T dt \left(\frac{2\beta R_0 + \gamma \sin 3\Phi_0}{2R_0 - \gamma \cos 3\Phi_0} - \alpha \right) \Phi_0^3}{\int_0^T dt \Phi_0^2}, \quad \bar{\beta} = \frac{\int_0^T dt \frac{2\beta R_0 + \gamma \sin 3\Phi_0}{2R_0 - \gamma \cos 3\Phi_0} \Phi_0^2}{\int_0^T dt \Phi_0^2}$$

$\gamma \rightarrow 0 \Rightarrow$ Burgers equation: $\partial_t \bar{\phi} = (1 + \alpha \bar{\beta}) \nabla^2 \bar{\phi} + (\alpha - \beta) (\nabla \bar{\phi})^2 + \Lambda$, $\bar{\phi} = \beta \mu \phi$

$1 + \alpha \bar{\beta} > 0$: stable (phase) spirals waves, $q \propto \kappa$

$1 + \alpha \bar{\beta} < 0$: turbulence regimes may be expected

System presents the same complexity and the same spatio-temporal behavior than self-oscillating systems

EXCITABLE REGIME ($\gamma > \gamma_c$)

If $\gamma, \beta \ll \mu, \beta \ll 1 \implies R^2 \cong \mu, \gamma_c \cong |\beta| \sqrt{\mu}$:

$$\partial_t \Phi = -\sqrt{\mu}(\gamma_c + \gamma \sin 3\Phi) + (1 + \alpha \beta) \nabla^2 \Phi - (\alpha - \beta) (\nabla \Phi)^2 + \frac{\alpha^2 (1 + \beta^2)}{2\mu} \nabla^4 \Phi$$

- $\alpha = \beta = 0$: **relaxational dynamics. Isolated 1D front at rest**
 $\alpha = \beta \neq 0$: **relaxational dynamics but 1D front moves**
 $\alpha \neq \beta$: **nonpotential front motion**
- Φ small \implies **damped Kuramoto-Sivashinsky phase equation.**
Frequency locked solutions stable regardless of the sign of $1 + \alpha \beta$
 \implies **no pattern forming instabilities.**

FRONTS AND SPIRALS

1D

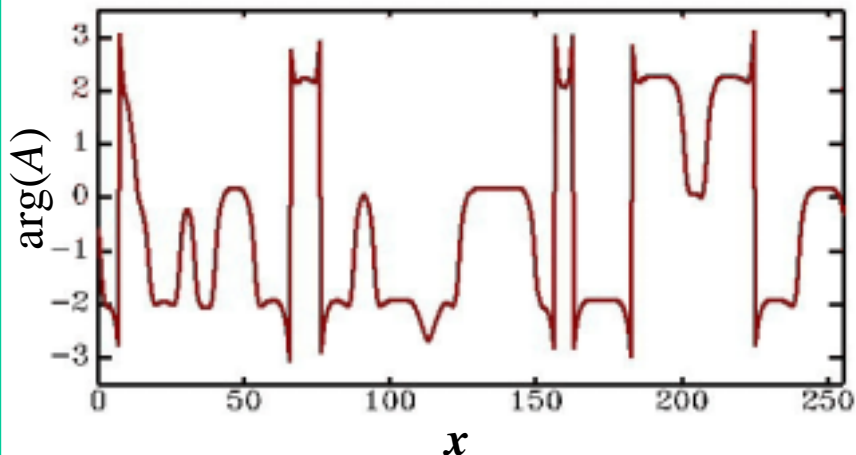
- Isolated kink moves at a constant velocity due to non-potential dynamics

$$v = \frac{2\pi\gamma_c\sqrt{\mu} + 3(\alpha - \beta)\int_{-\infty}^{+\infty}(\partial_x\Phi_{ij}^0)^3}{3\int_{-\infty}^{+\infty}(\partial_x\Phi_{ij}^0)^2}$$

- $\alpha > \beta$, $\Phi_1^e < \Phi_2^e < \Phi_3^e$

$$\Phi_{12}, \Phi_{23}, \Phi_{31} \longrightarrow v$$

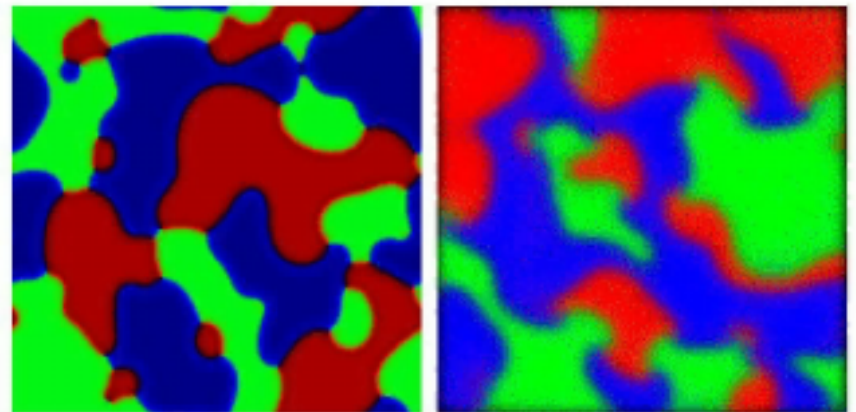
$$\Phi_{21}, \Phi_{13}, \Phi_{32} \longleftarrow v$$



2D

- Formation of **vertex points**
- Rotation of interfaces around vertices \Rightarrow **no coarsening**
- Analogies with systems with **competing fields**

3ω Forced CGLE Busse-Heikes Model



NUMERICAL SIMULATIONS

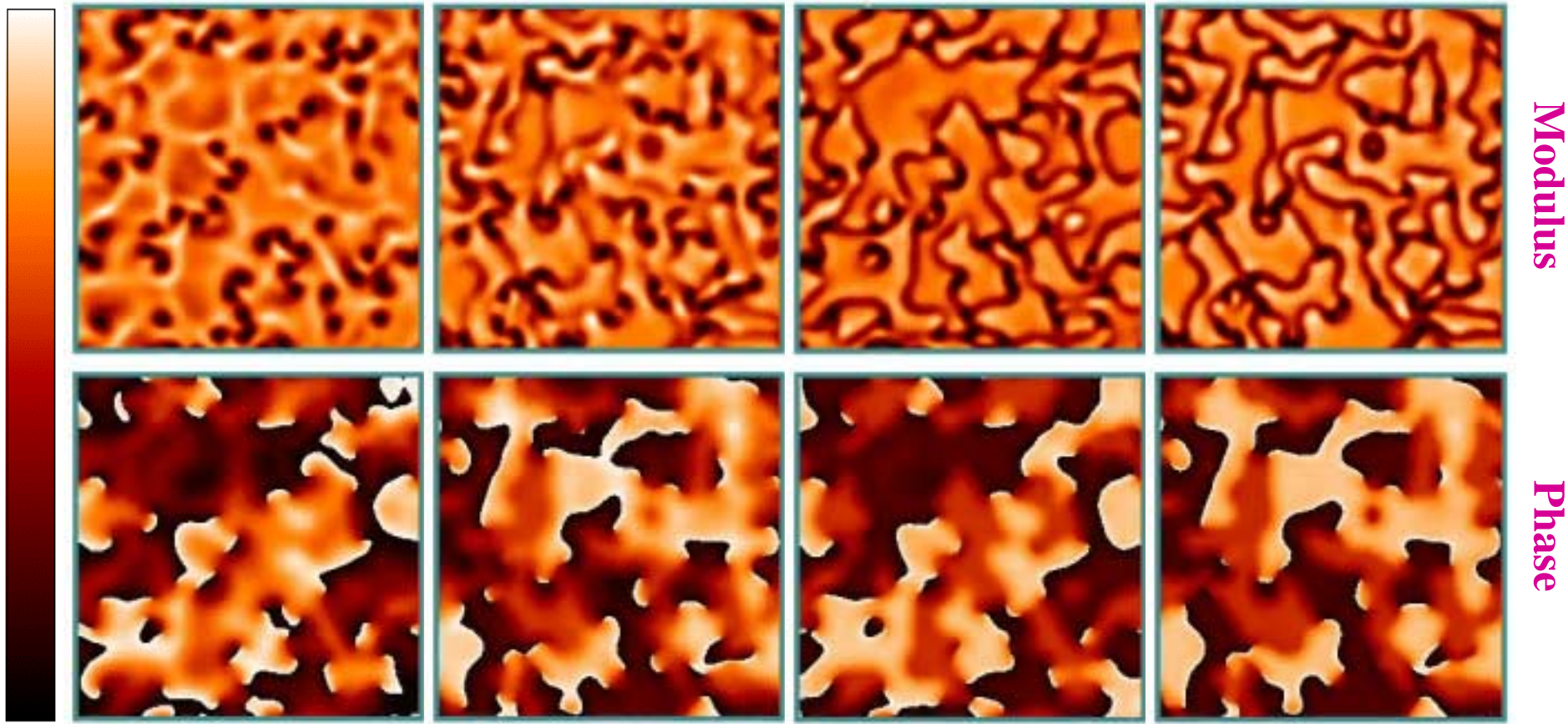
❖ PHASE APPROXIMATION VALID

μ	α	β	γ_c	$1+\alpha\beta$	REGIME ($\gamma = 0$)
1	2	-0.2	0.199	>0	<i>Frozen States I</i>
1	5.5	-0.2	0.199	<0	<i>Frozen States II</i>

❖ PHASE APPROXIMATION INVALID

μ	α	β	γ_c	$1+\alpha\beta$	REGIME ($\gamma = 0$)
1	2.0	-1.00	0.910	<0	<i>Defect Turbulence I</i>
1	0.0	-1.80	1.450	>0	<i>Defect Turbulence II</i>
1	2.0	-0.76	0.720	<0	<i>Phase Turbulence</i>

Frozen States I: $\mu = 1, \alpha = 2, \beta = -0.2, \gamma_c = 0.199$



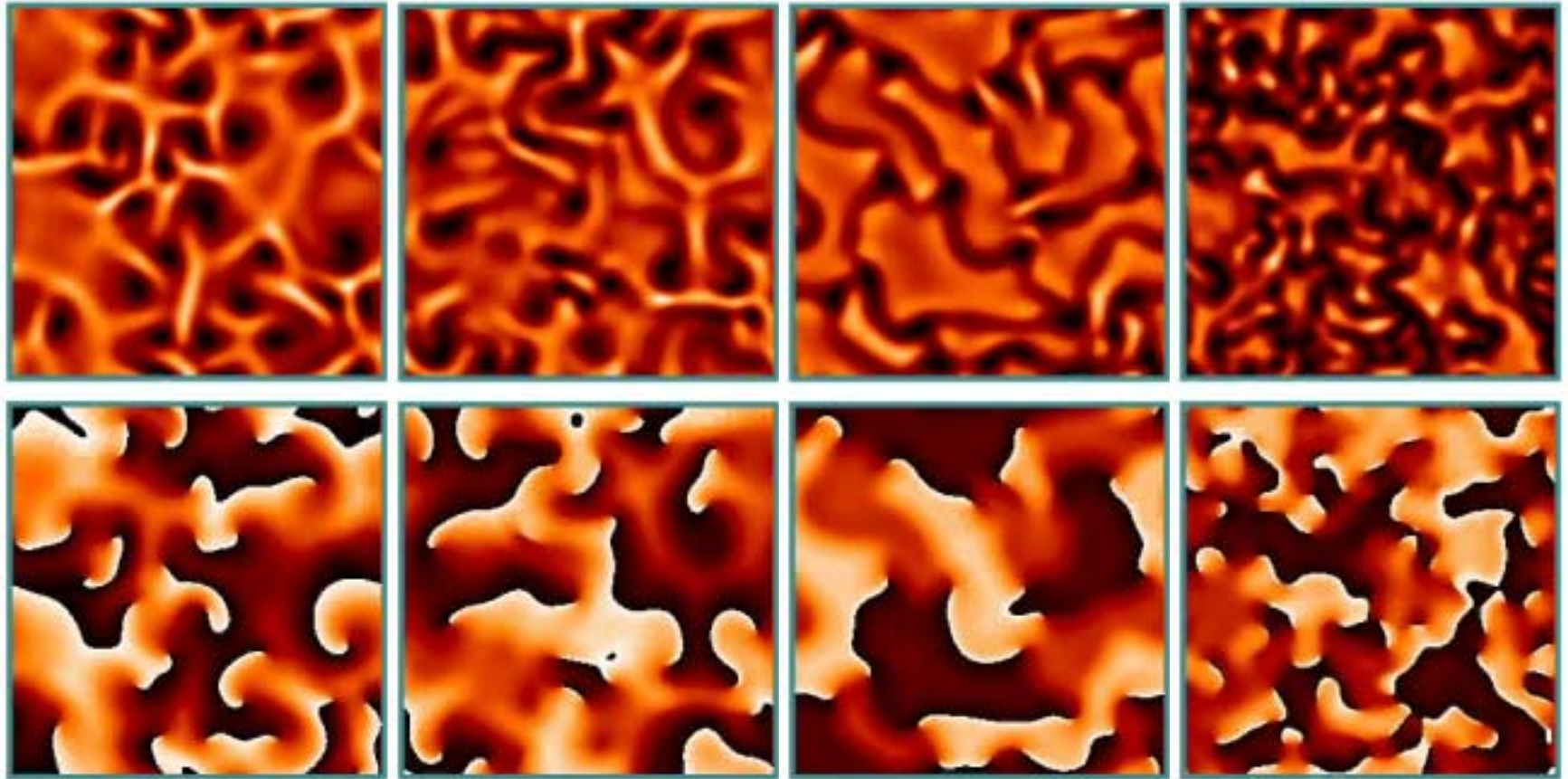
$\gamma = 0$

$\gamma < \gamma_c$

$\gamma \approx \gamma_c$

$\gamma > \gamma_c$

Frozen States II: $\mu = 1, \alpha = 5.5, \beta = -0.2, \gamma_c = 0.199$



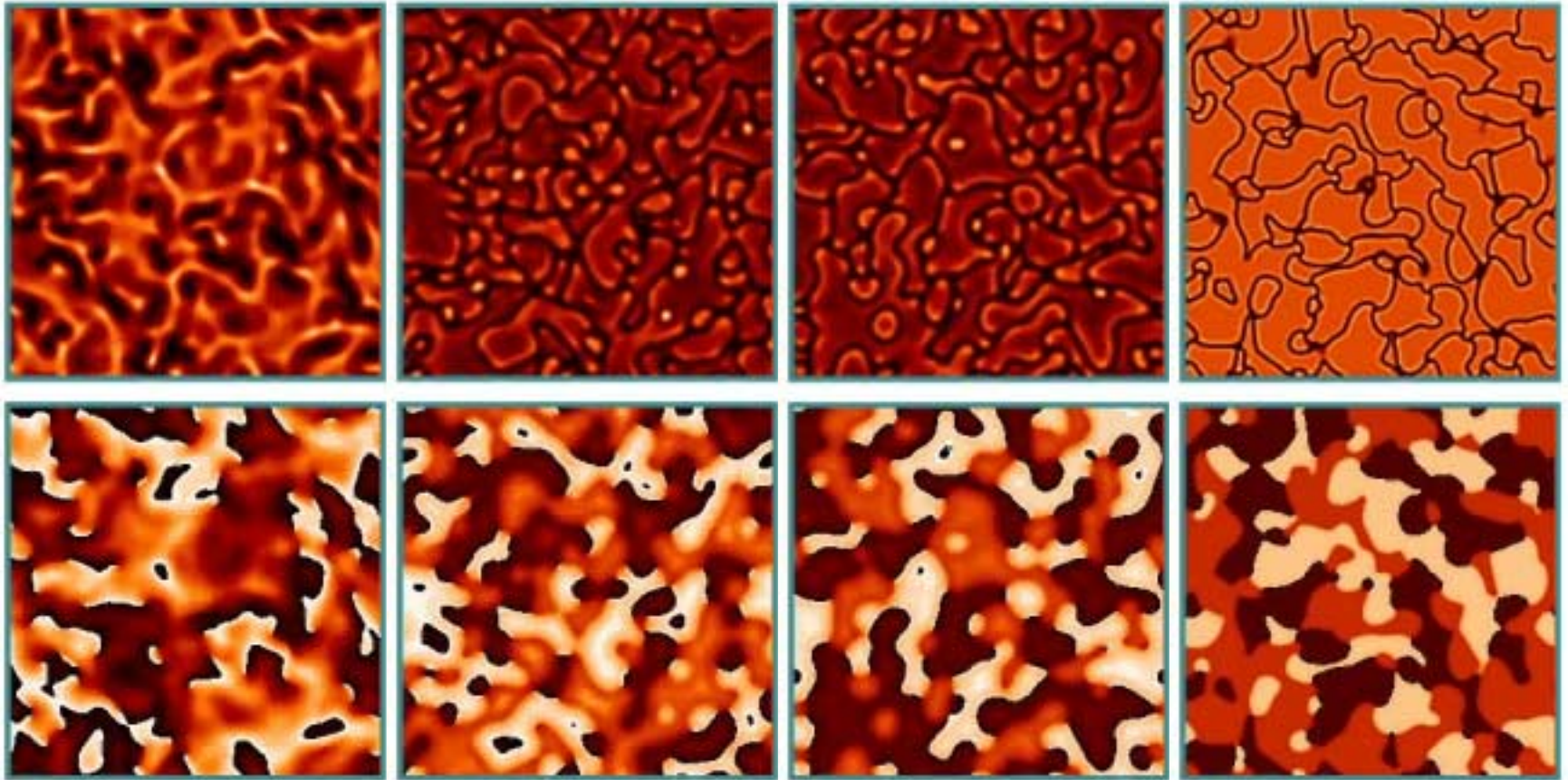
$\gamma = 0$

$\gamma < \gamma_c$

$\gamma \approx \gamma_c$

$\gamma > \gamma_c$

Defect Turbulence I: $\mu = 1, \alpha = 2, \beta = -1.0, \gamma_c = 0.910$



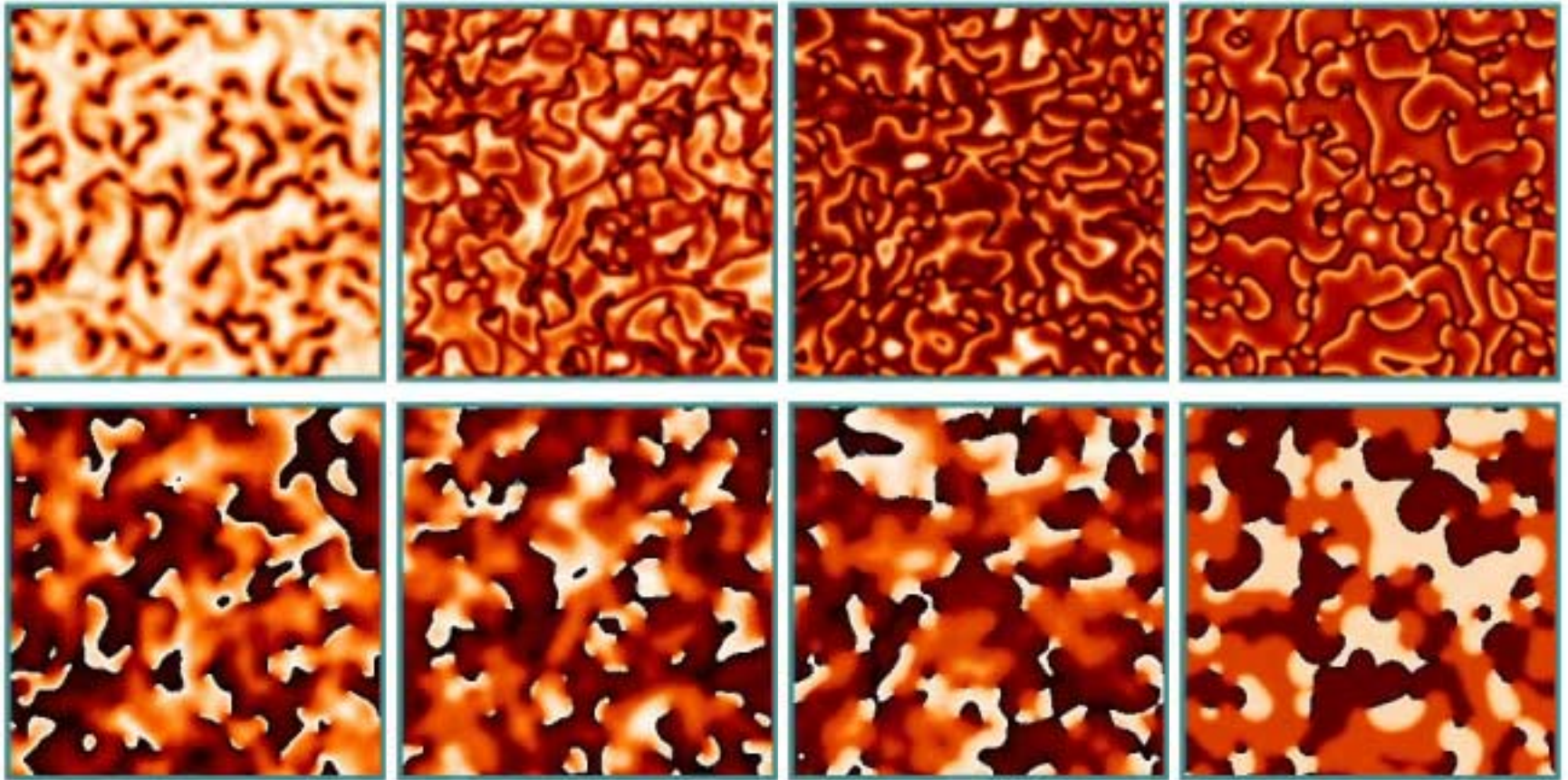
$\gamma = 0$

$\gamma < \gamma_c$

$\gamma \approx \gamma_c$

$\gamma > \gamma_c$

Defect Turbulence II: $\mu = 1, \alpha = 0, \beta = -1.8, \gamma_c = 1.447$



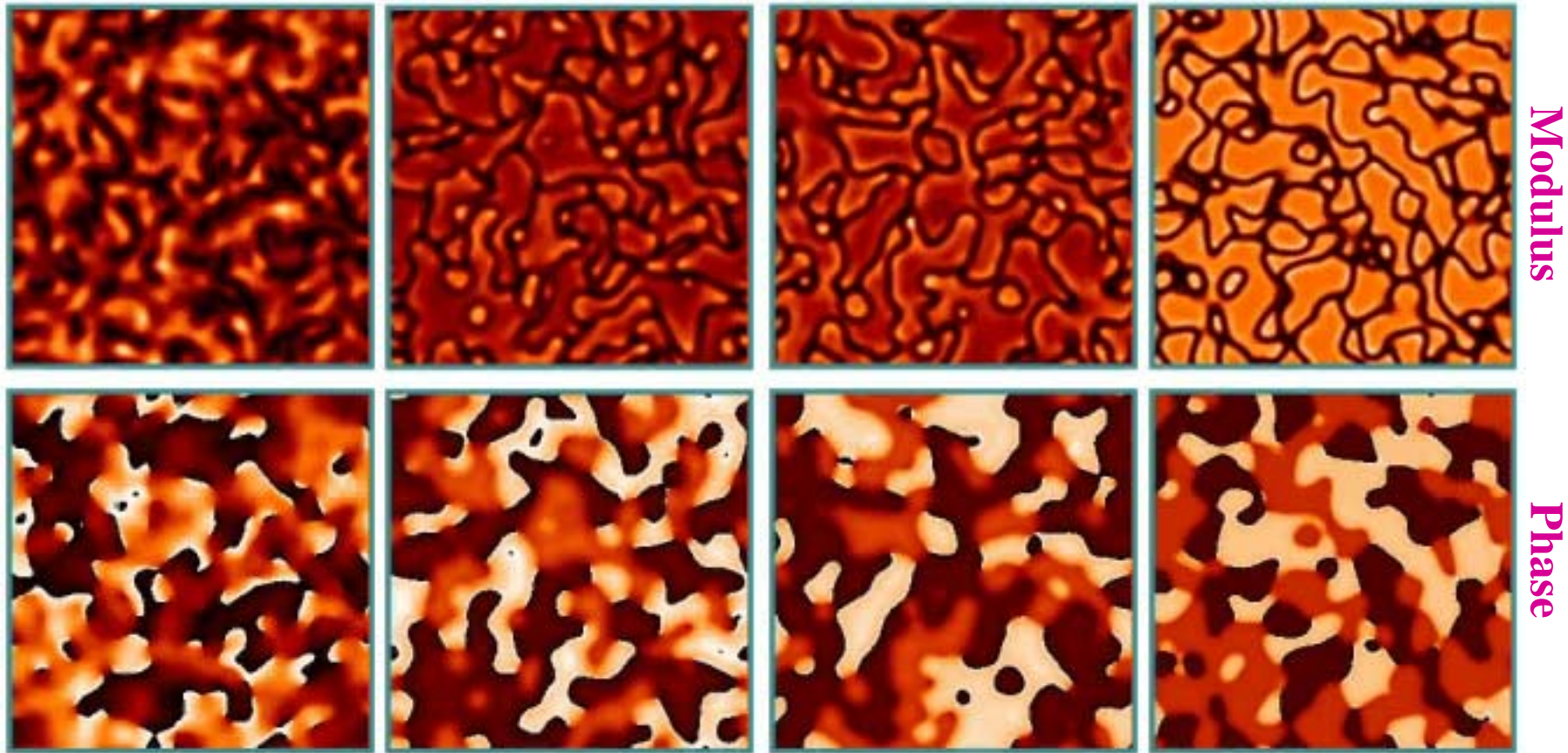
$\gamma = 0$

$\gamma < \gamma_c$

$\gamma \approx \gamma_c$

$\gamma > \gamma_c$

Phase Turbulence : $\mu = 1, \alpha = 2, \beta = -0.76, \gamma_c = 0.72$



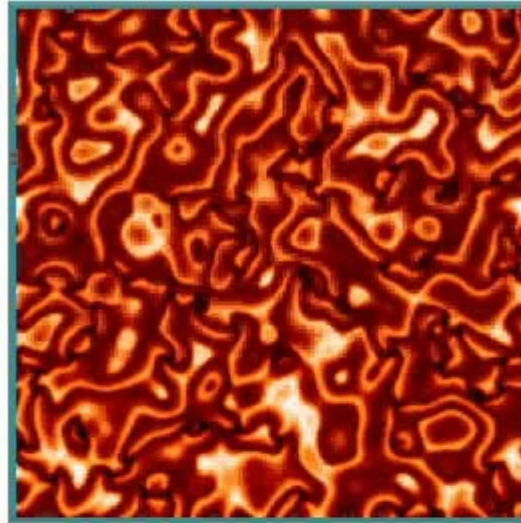
$\gamma = 0$

$\gamma < \gamma_c$

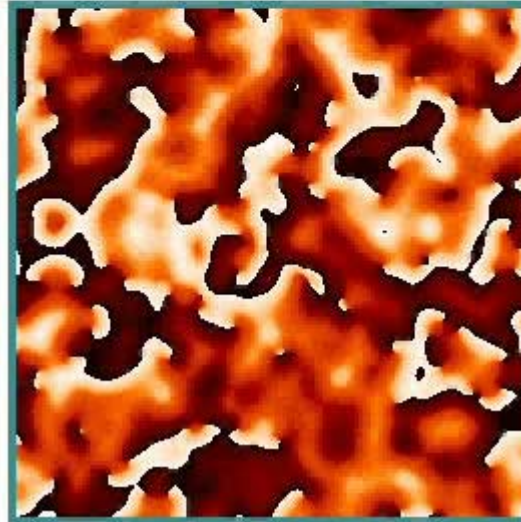
$\gamma \approx \gamma_c$

$\gamma > \gamma_c$

Oscillatory instability at $k = 0$: $\mu = 1, \alpha = 0, \beta = -3.0, \gamma_c = 2.080$



Modulus



Phase

$\gamma = 2.150$

MAIN NUMERICAL RESULTS

- $\gamma \simeq \gamma_c$:
 - ◆ Transition observed when **phase approximation** holds
 - ◆ Other kind of structures may appear (targets, ...)
- $\gamma < \gamma_c$: Amplitude spirals may form. Complex patterns may appear
- $\gamma \gg \gamma_c$: Excitable regime is attained **regardless of** the phase approximation