



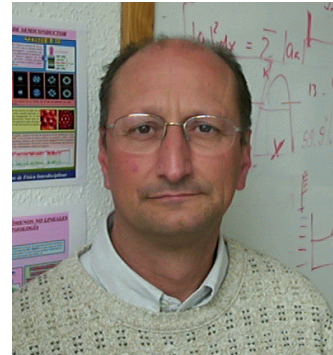
# Localized Structures in the Parametrically Driven CGLE



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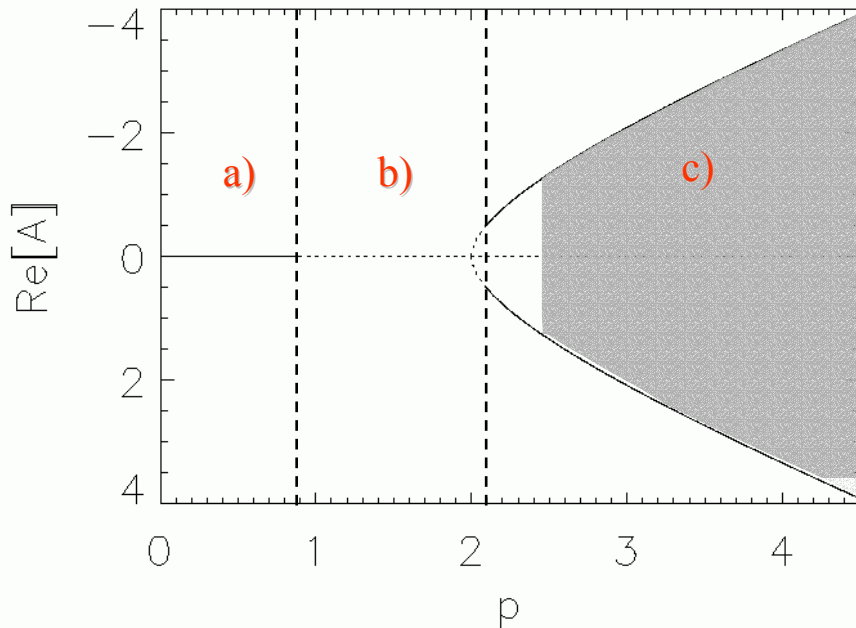


$$\partial_t A = (\mu + i\nu)A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A + pA^*$$

$\alpha \sim \nu \sim p \gg \mu, \beta \rightarrow$  Pattern forming transition

*P. Coulet and K. Emilsson, Physica D, 61, 119 (1992)*

**Homogeneous solutions:**

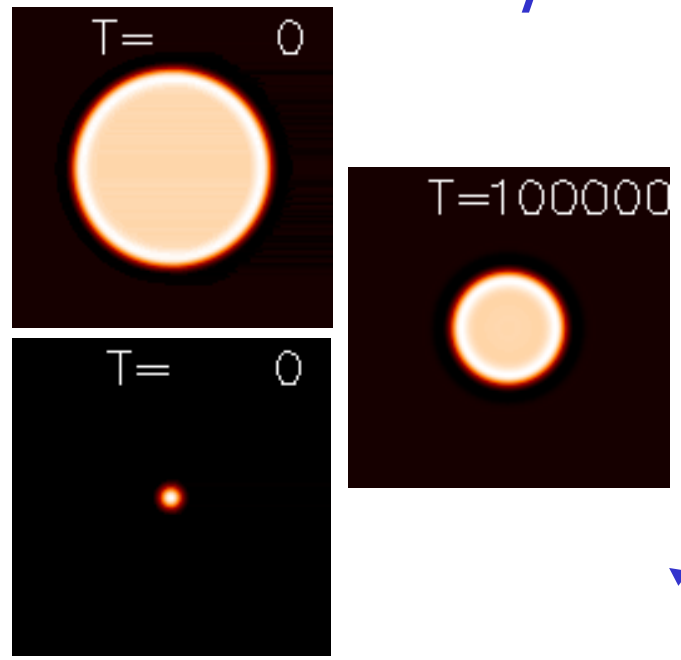


- a) Zero solution
- b) Stripe and hexagonal Patterns
- c) Bistability between homogeneous solutions (frequency locked solutions).

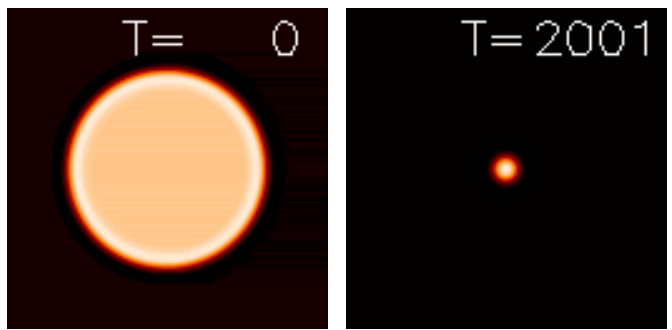
*Stable Ising walls.*

$$\alpha = \nu = 2, \mu = \beta = 0$$

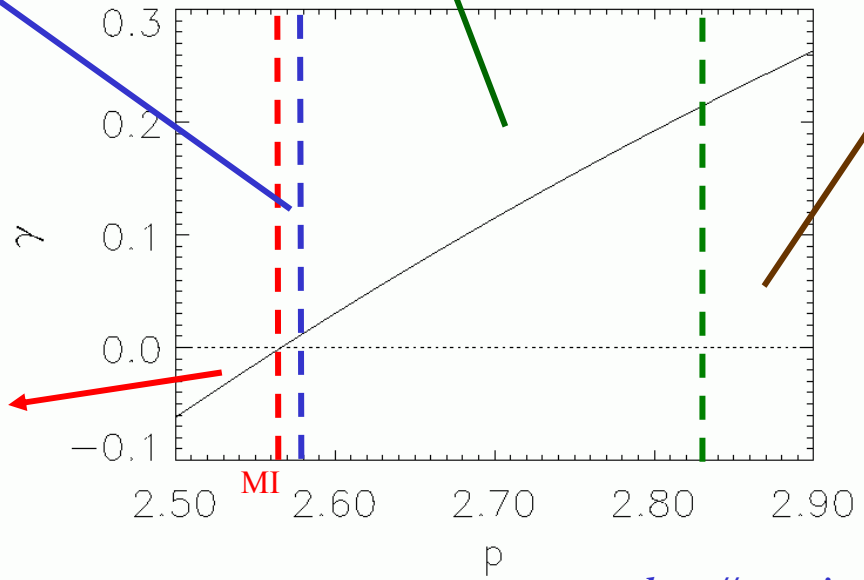
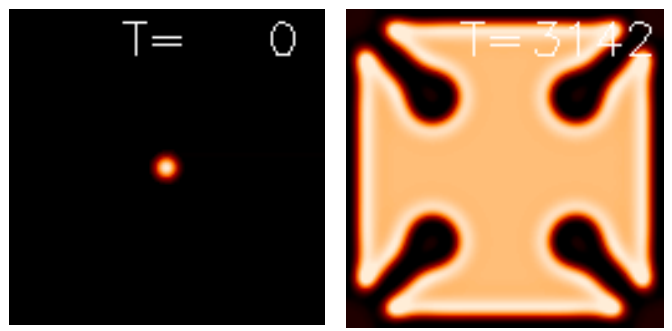
### Stable droplet



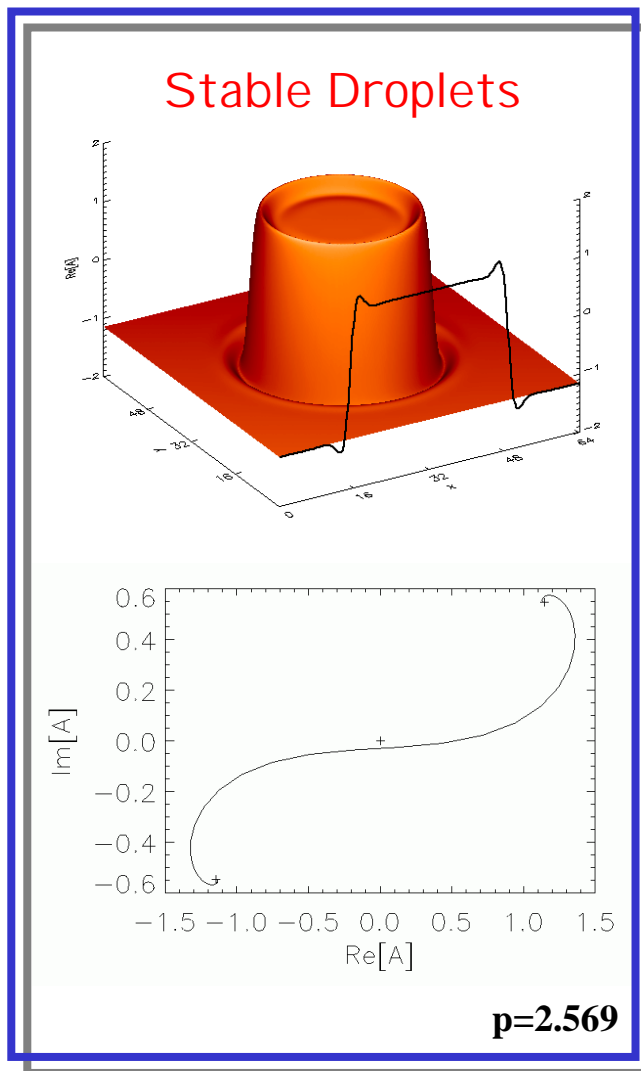
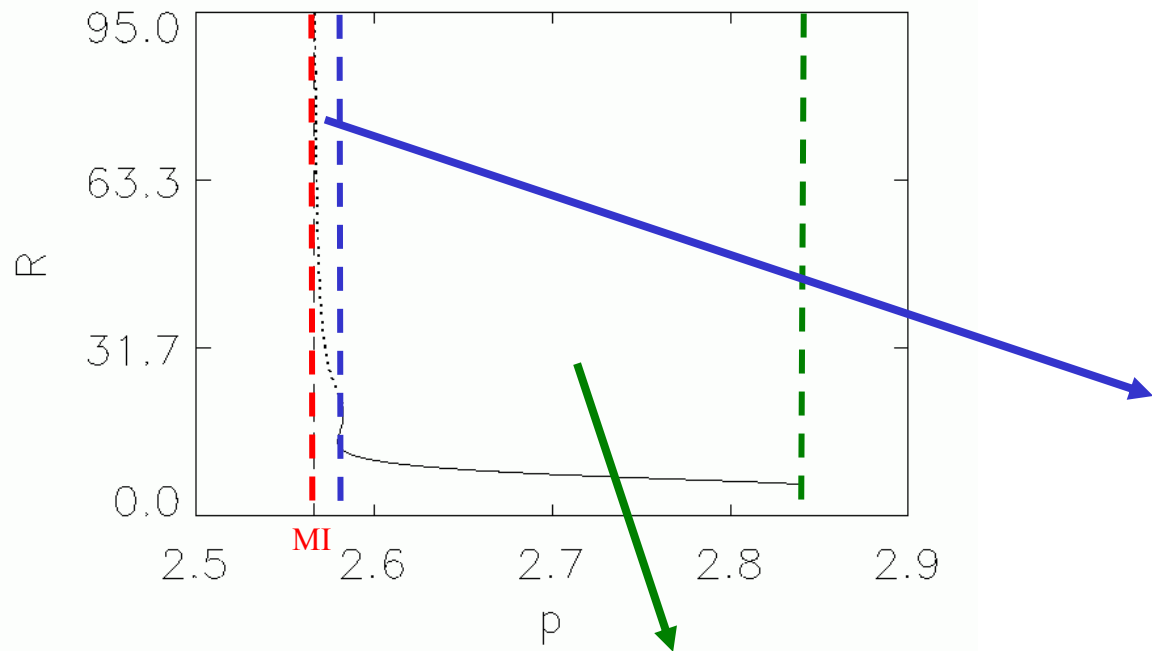
### Localized Structure



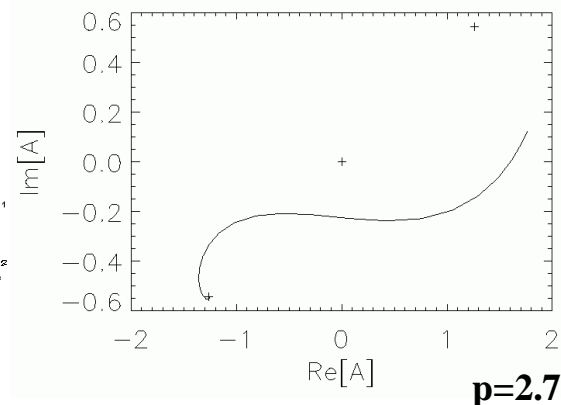
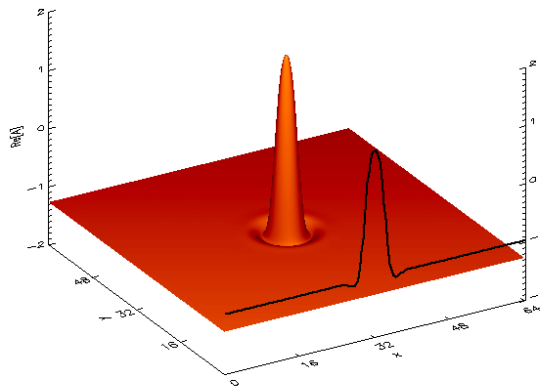
### Exploding droplet



$$\dot{R} = -\gamma \frac{1}{R}$$



### Localized Structures



$$\partial_t \vec{A}(\vec{x}) = D \nabla^2 \vec{A}(\vec{x}) + \vec{W}(\vec{A}(\vec{x}), p)$$

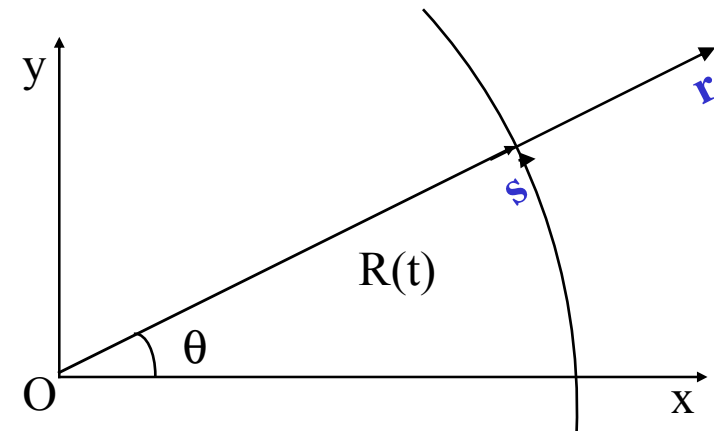
- $\vec{A}$  is a real vector field (For the PCGLE  $\vec{A} = \begin{pmatrix} \text{Re}[A] \\ \text{Im}[A] \end{pmatrix}$  and  $D = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$ ).
- $\vec{W}(\vec{A})$  is a local nonlinear function of the fields.
- There exists a discrete symmetry  $\mathbf{Z}$  of  $\vec{W}(\vec{A})$  such that there are two equivalent stable homogeneous solutions (For the PCGLE  $\mathbf{Z}: A \rightarrow -A$ ).
- Parameter region with stable d=1 | sing walls  $\vec{A}_0(x, p)$ .

## Local Coordinates $r, s$

$$\partial_t \rightarrow -v \partial_r \quad (v \equiv \dot{R})$$

$$\nabla^2 \rightarrow \partial_r^2 + \frac{\kappa}{1+r\kappa} \partial_r + \frac{\kappa^2}{(1+r\kappa)^2} \partial_\theta^2 \quad (\kappa \equiv \frac{1}{R})$$

*E. Meron, Phys. Rep. 218 (1992)*

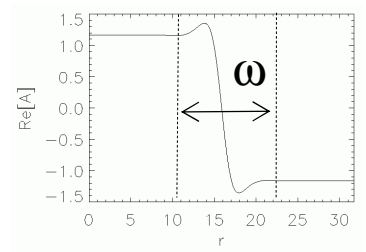


$$D\partial_r^2 \vec{A}(r) + (v + \frac{\kappa}{1+r\kappa} D)\partial_r \vec{A}(r) + \vec{W}(\vec{A}(r), p) = 0$$

Linearizing around the 1D profile:  $\vec{A}(r) = \vec{A}_0 + \vec{A}_1$

$$M\vec{A}_1 = -(v + \kappa D)\partial_r \vec{A}_0, \quad M \equiv D\partial_r^2 + \frac{\delta \vec{W}}{\delta \vec{A}} \Big|_{\vec{A}_0}$$

Gently curved fronts  
 $\kappa\omega \ll 1$



$\vec{e}_0$  Goldstone Mode:  $\vec{e}_0 \equiv \partial_r \vec{A}_0, M\vec{e}_0 = 0; M^+ \vec{a}_0 = 0$

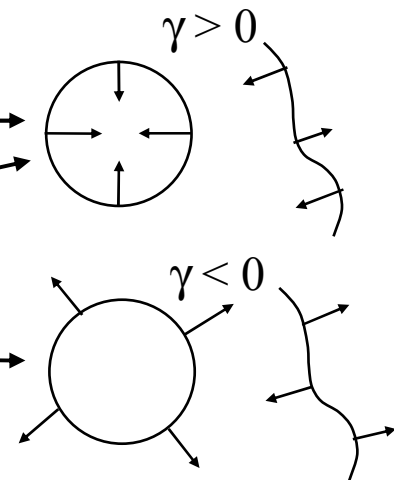
Solvability condition:

$$v = -\gamma(p)\kappa$$

$$\gamma(p) \equiv \frac{\int_{-\infty}^{\infty} \vec{a}_0^+ \cdot D\vec{e}_0 dr}{\int_{-\infty}^{\infty} \vec{a}_0^+ \cdot \vec{e}_0 dr}$$

Circular domain:  $\dot{R} = -\gamma \frac{1}{R}$

- $D=dI \Rightarrow \gamma = d > 0, v = -d\kappa$   $\gamma > 0$  Droplet shrinking
- $\vec{A}_1 \propto \vec{e}_0, \text{ no deformation}$   $\gamma > 0$  Droplet shrinking
- $D=dI+C \Rightarrow \gamma = \gamma(p), v = -\gamma\kappa$   $\gamma = 0$  Modulational instability
- $\vec{A}_1 \not\propto \vec{e}_0, \text{ deformation } \propto \kappa, C$   $\gamma < 0$  Droplet growth



Radial symmetry  $\rightarrow D\partial_r^2 \vec{A}(r) + \left(-\frac{\dot{\kappa}}{\kappa^2} + \frac{\kappa}{1+r\kappa} D\right) \partial_r \vec{A}(r) + \vec{W}(\vec{A}(r), p) = 0$

$$p = p_c + \varepsilon p_1, \quad \gamma(p_c) = 0, \quad \varepsilon \ll 1,$$

$$\vec{A} = \vec{A}_0 + \varepsilon^{1/2} \vec{A}_1 + \varepsilon \vec{A}_2 + \varepsilon^{3/2} \vec{A}_3,$$

$$\kappa = \varepsilon^{1/2} \kappa_1, \quad \partial_t = \varepsilon^2 \partial_T$$

$$\Rightarrow O(\varepsilon^{3/2}) \quad \frac{\partial_T \kappa_1}{\kappa_1^2} = c_1 p_1 \kappa_1 + c_3 \kappa_1^3$$

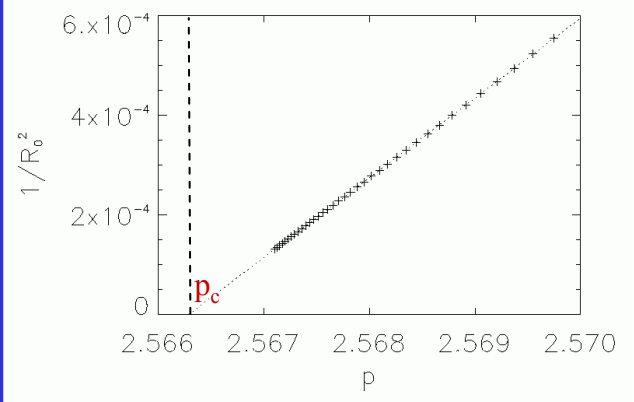
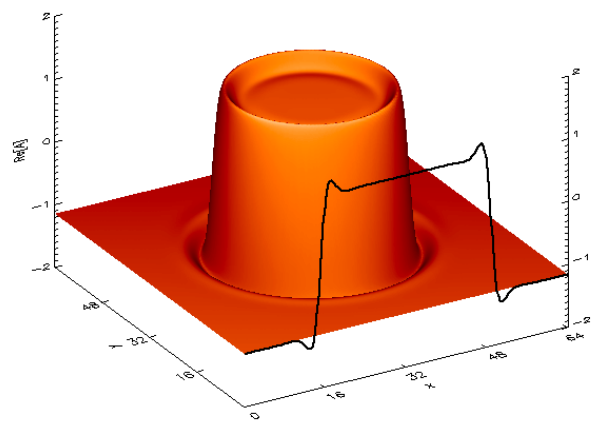
$$c_1 > 0, c_3 < 0$$

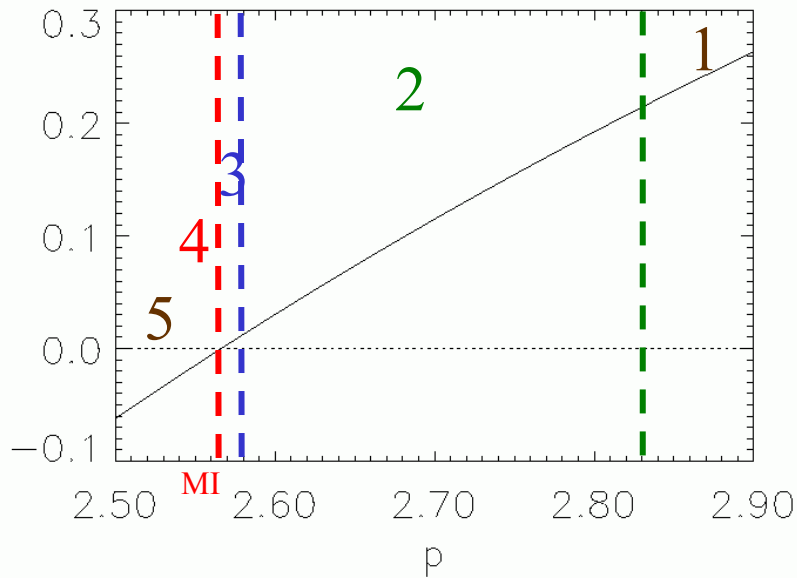
$$\dot{R} = -c_1 (p - p_c) \frac{1}{R} - c_3 \frac{1}{R^3}$$

 $\Rightarrow$ 

Stable Droplet

$$R_0 = \frac{1}{\sqrt{p - p_c}} \sqrt{\frac{-c_3}{c_1}}$$





$$1, 5 \rightarrow \dot{R} = -\gamma \frac{1}{R}, \quad R \approx t^{1/2} \quad \text{shrinking, growth}$$

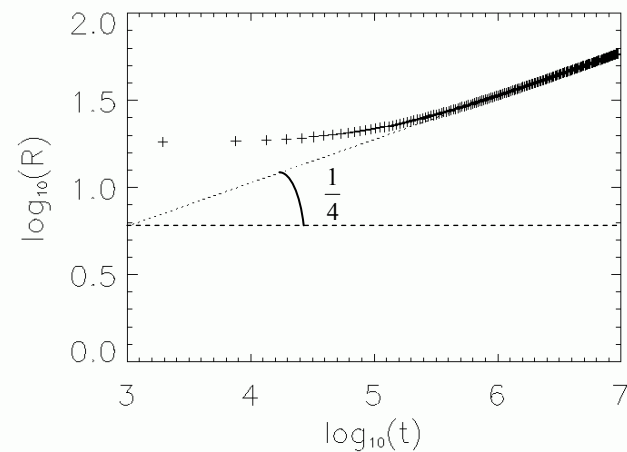
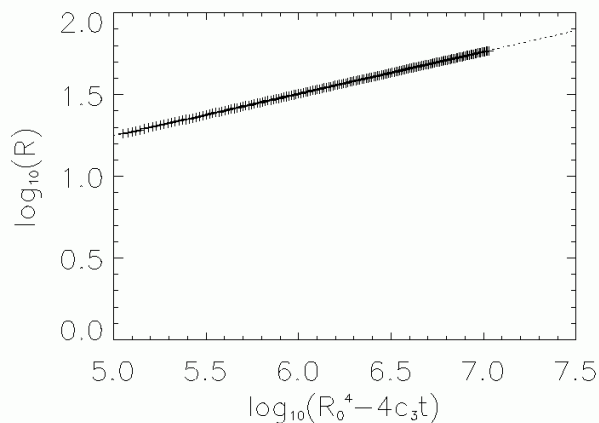
$$2 \rightarrow \dot{R} = -\gamma \frac{1}{R}, \quad R \approx t^{-1/2} \quad \text{until LS forms}$$

$$3 \rightarrow \dot{R} = -\gamma \left( \frac{1}{R} - \frac{R_0^2}{R^3} \right) \quad \text{no power law}$$

$$4 \rightarrow \dot{R} = -c_3 \frac{1}{R^3},$$

$$R \approx t^{1/4}$$

modulational instability,  $\gamma = 0$

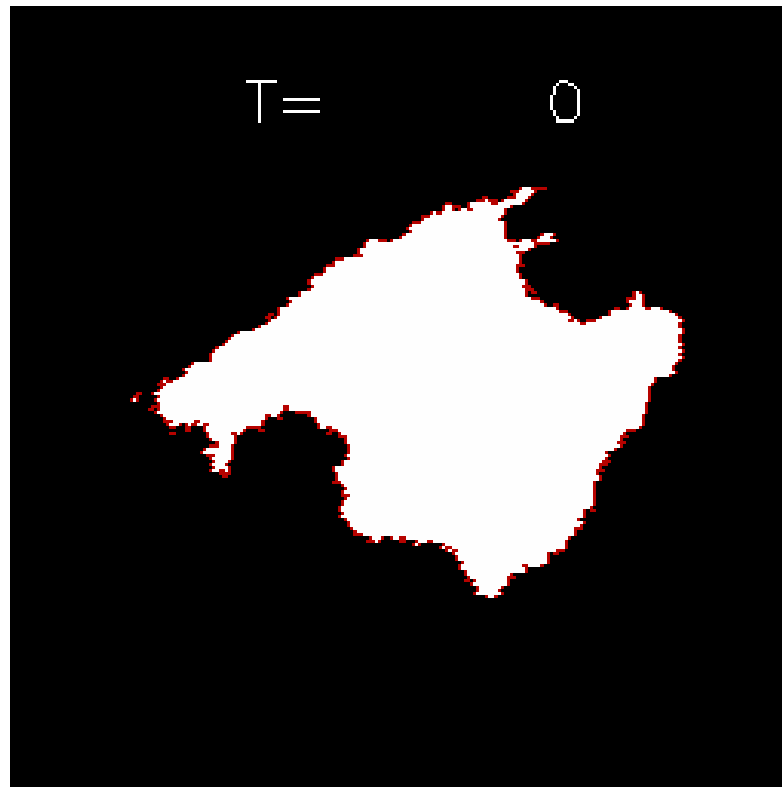




Normal front velocity:  $v = -c_1(p - p_c)\kappa - c_2\kappa^2\partial_\theta^2\kappa - c_3\kappa^3$   $c_1 > 0, c_2, c_3 < 0$

Shrinking term  $O(\kappa)$       Reduction of curvature differences  $O(\kappa^2)$       Exploding term  $O(\kappa^3)$

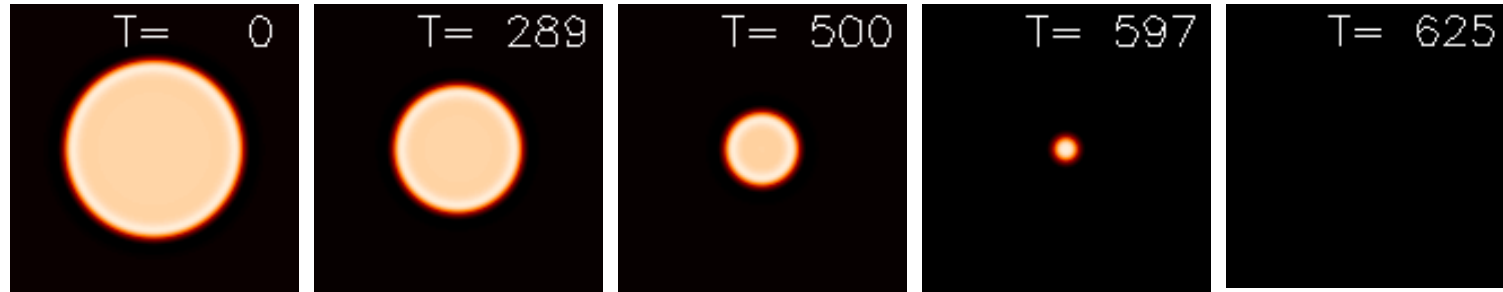
At the Modulational Instability  $p = p_c$ :  $v = -c_2\kappa^2\partial_\theta^2\kappa - c_3\kappa^3$



● Coarsening

$$p = 3.0$$

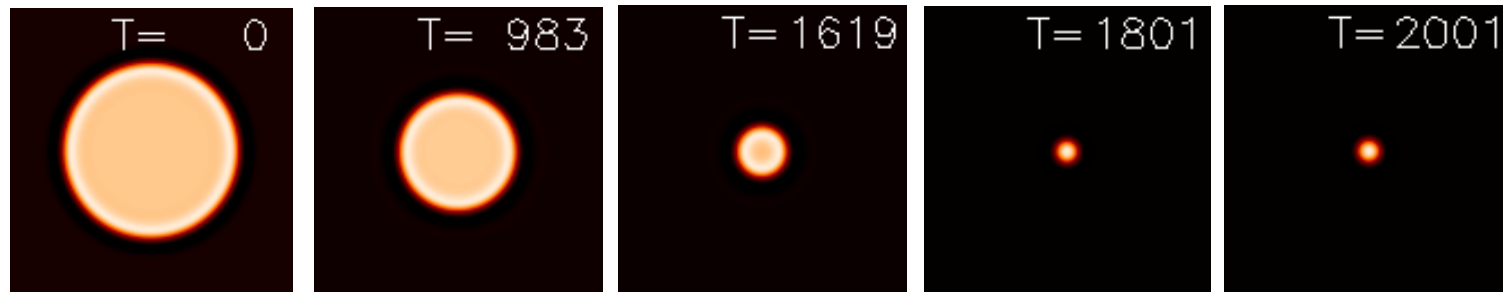
$$\gamma(p) = 0.3$$



● Localized Structure

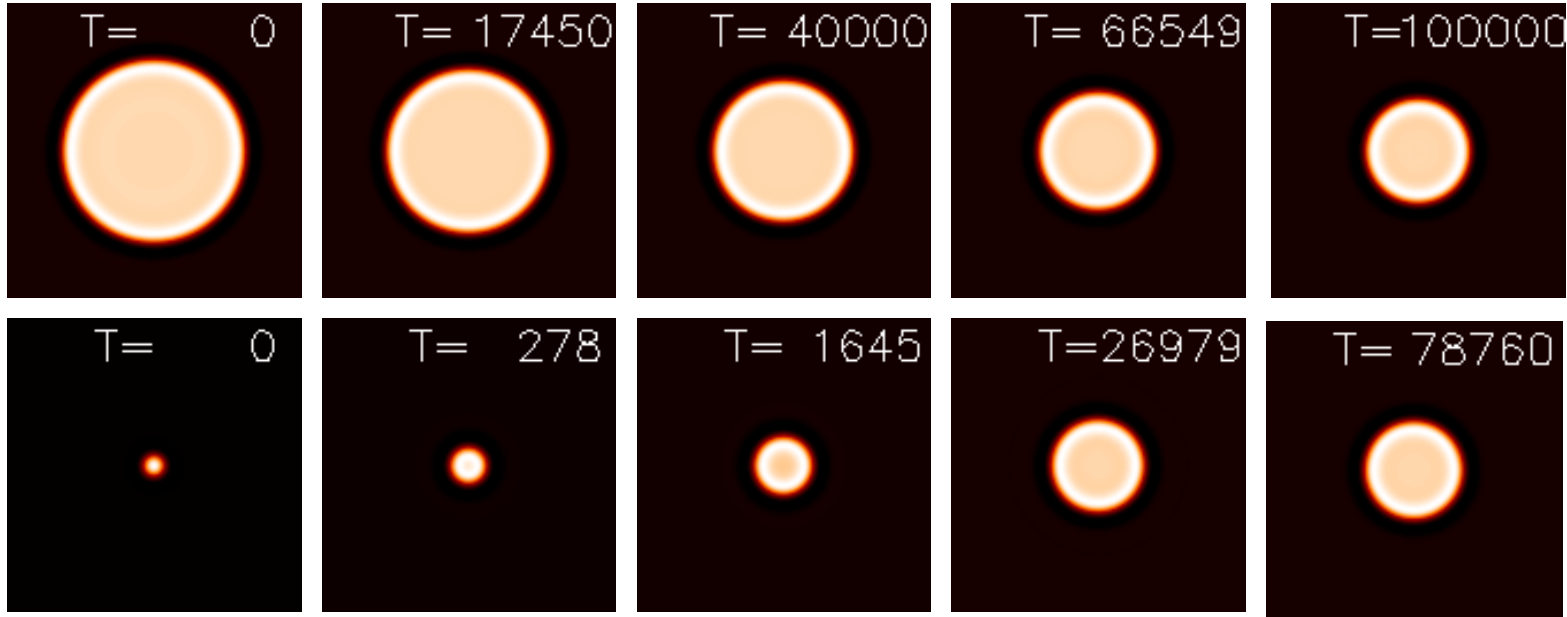
$$p = 2.7$$

$$\gamma(p) = 0.1$$



● Stable Droplet

$p = 2.569$   
 $\gamma(p) = 0.01$

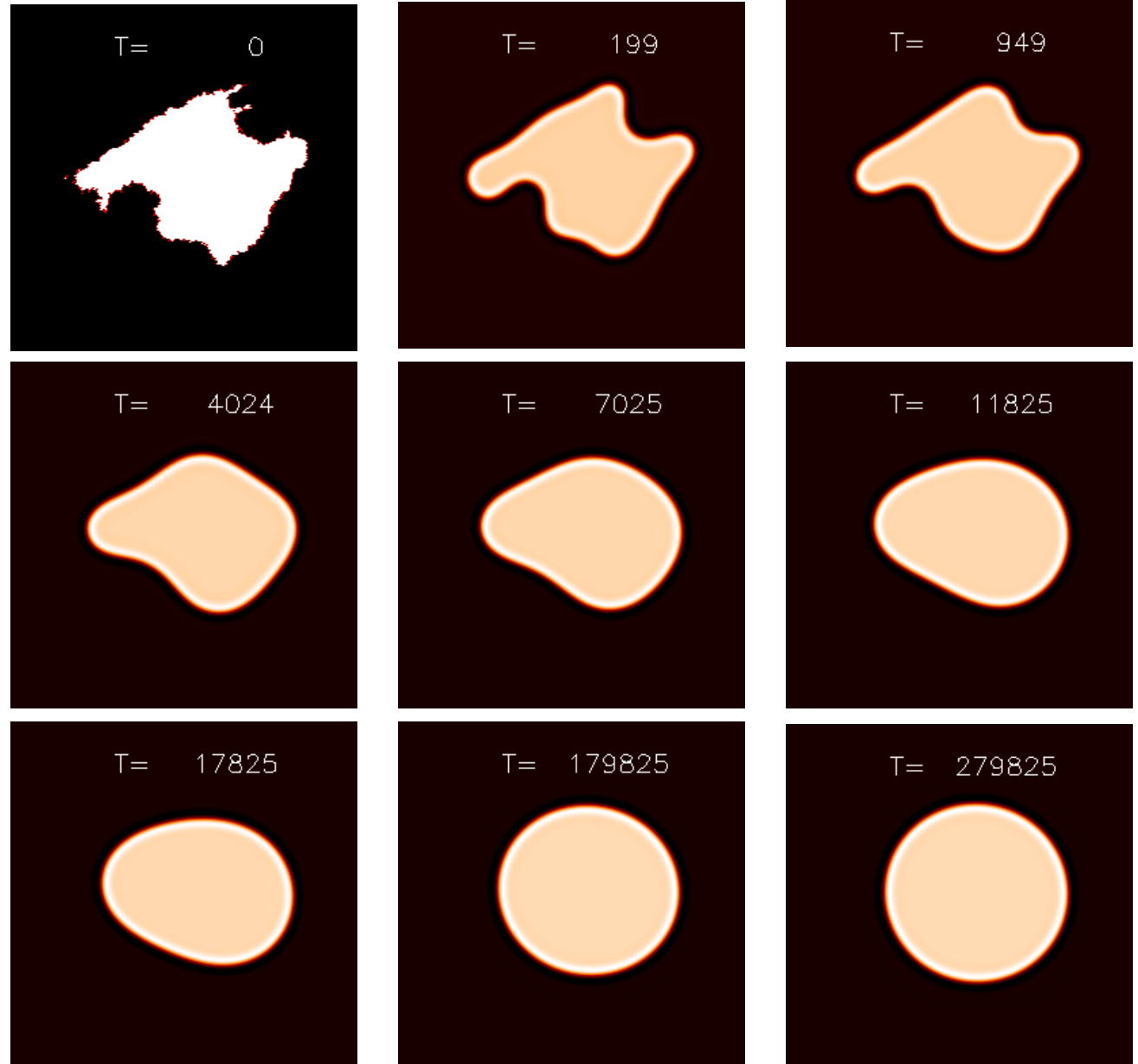


● Exploding Droplet

$p = 2.4$   
 $\gamma(p) = -0.1$



# Dynamics of an Arbitrary Domain



$$p = p_c$$

$$\gamma(p) = 0.0$$