

BUSSE-HEIKES MODEL

THEORETICAL MODEL

$$\partial_t A_1 = \mathcal{L}_1 A_1 + A_1 [1 - |A_1|^2 - g_+ |A_2|^2 - g_- |A_3|^2]$$

$$\partial_t A_2 = \mathcal{L}_2 A_2 + A_2 [1 - |A_2|^2 - g_+ |A_3|^2 - g_- |A_1|^2]$$

$$\partial_t A_3 = \mathcal{L}_3 A_3 + A_3 [1 - |A_3|^2 - g_+ |A_1|^2 - g_- |A_2|^2]$$

$$g_{\pm} = 1 + \mu_{\pm} \delta$$

$$\mathcal{L}_j = \nabla^2, \partial_{x_j}^2$$

$$\partial_t \rho_A = - \frac{\delta F_{\text{BH}}}{\delta A} + \delta f(\rho_A)$$

$$\rho_A = (A_1, A_2, A_3)$$

\Rightarrow

$\delta = 0 \rightarrow$ **potential dynamics**

$$\delta \int dr \frac{\delta F_{\text{BH}}}{\delta A^*} \cdot f(\rho_A)^* + \text{c.c.} = 0 \rightarrow$$

nonrelaxational potential flow

$\delta \neq 0$

otherwise \rightarrow nonpotential

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ZERO-DIMENSIONAL SYSTEMS

$$\dot{\mathcal{L}}_j = 0, \quad A_j = \sqrt{R_j(t)} e^{i\theta_j(t)}, \quad j = 1, 2, 3$$



$$\left. \begin{aligned} \dot{R}_1(t) &= 2R_1 [1 - R_1 - (1 + \mu + \delta)R_2 - (1 + \mu - \delta)R_3] \\ \dot{R}_2(t) &= 2R_2 [1 - R_2 - (1 + \mu + \delta)R_3 - (1 + \mu - \delta)R_1] \\ \dot{R}_3(t) &= 2R_3 [1 - R_3 - (1 + \mu + \delta)R_1 - (1 + \mu - \delta)R_2] \end{aligned} \right\}, \quad \left. \begin{aligned} \dot{\theta}_1 &= 0 \\ \dot{\theta}_2 &= 0 \\ \dot{\theta}_3 &= 0 \end{aligned} \right\}$$

- Equations for the **real** variables $\{R_1, R_2, R_3\}$ instead of **complex** variables $\{A_1, A_2, A_3\}$
- Similar sets of equations proposed to study **population competition dynamics**

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Stationary Solutions

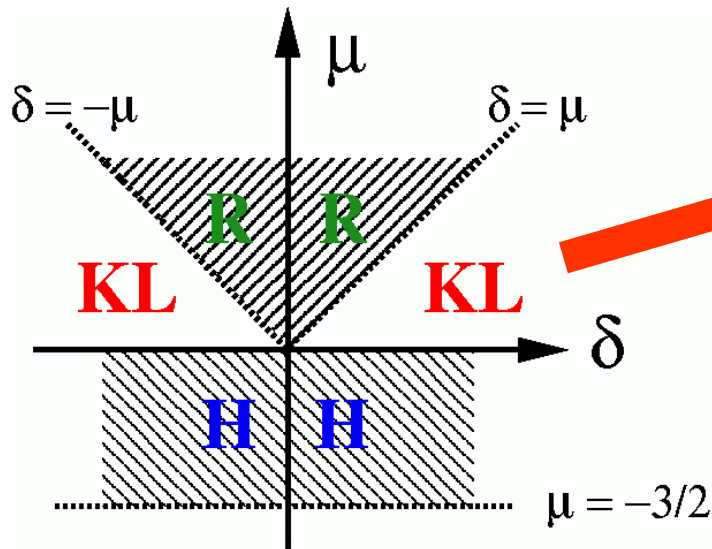
• **Null:** $(R_1, R_2, R_3) = (0,0,0)$

• **Rhombus :** $(R_1, R_2, R_3) = \left(\left\{ \frac{\mu + \delta}{\mu(\mu + 2) - \delta^2}, \frac{\mu - \delta}{\mu(\mu + 2) - \delta^2}, 0 \right\}, K \right)$

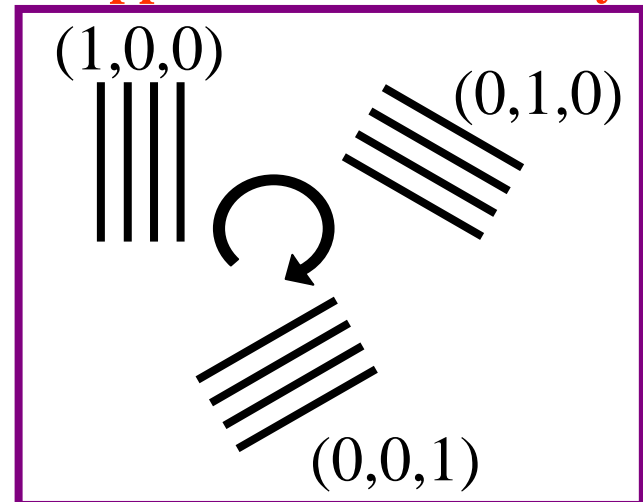
• **Rolls R:** $(R_1, R_2, R_3) = \{(1,0,0), \dots\}$

• **Hexagon H:** $(R_1, R_2, R_3) = 1/(3+2\mu)$

➔ **Stable** somewhere in (μ, δ) parameter space



Küppers-Lortz instability



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The Case $\mu = 0$

- Orthogonality condition holds ($\delta \mu = 0$) \Rightarrow **nonrelaxational potential flow**

$$\dot{R}_j = \boxed{-\frac{\partial V}{\partial R_j}} + \boxed{\delta f_j}, \quad j = 1, 2, 3$$

Relaxational part \swarrow Residual dynamics

$$V = -(R_1 + R_2 + R_3) + \frac{1}{2}(R_1^2 + R_2^2 + R_3^2) + R_1R_2 + R_2R_3 + R_1R_3$$

- Equations of motion

$$x(t) = R_1(t) + R_2(t) + R_3(t) = \frac{1}{\left(\frac{1}{x_0} - 1\right)e^{-2t} + 1}$$

After $t = O(1) \Rightarrow x(t) \approx 1$ (**asymptotic dynamics**)

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- Eliminating for instance R_1 :

$$\left. \begin{aligned} \dot{R}_2 &= 2\delta \frac{\delta H}{\delta R_3} \\ \dot{R}_3 &= -2\delta \frac{\delta H}{\delta R_2} \end{aligned} \right\} \begin{array}{l} \text{Hamiltonian dynamics} \\ H(R_2, R_3) = R_2 R_3 (1 - R_2 - R_3) \equiv E \text{ ('energy')} \end{array}$$

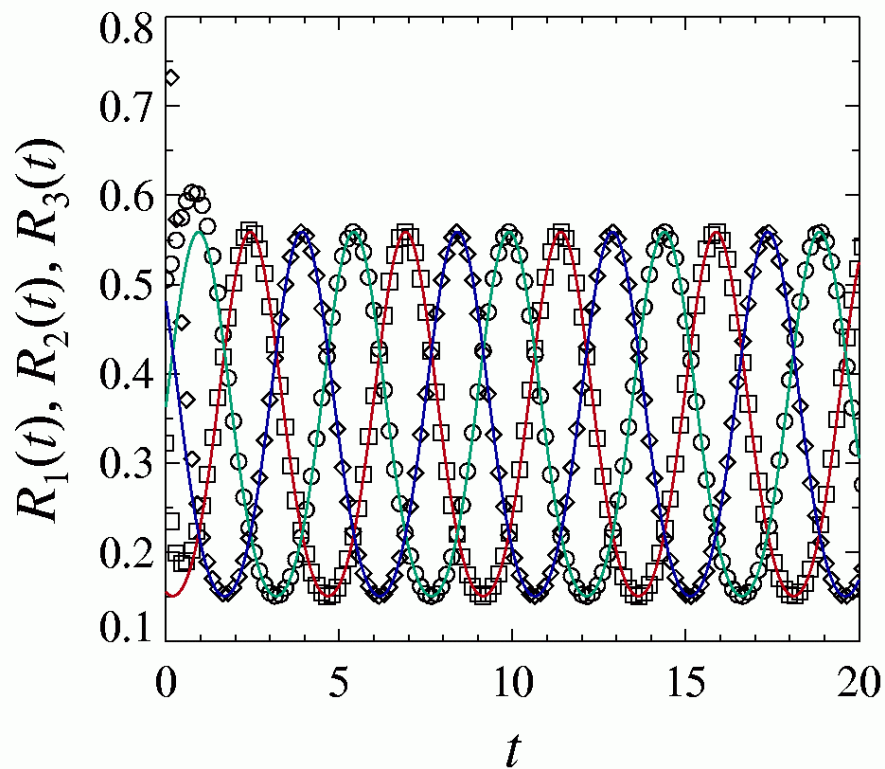
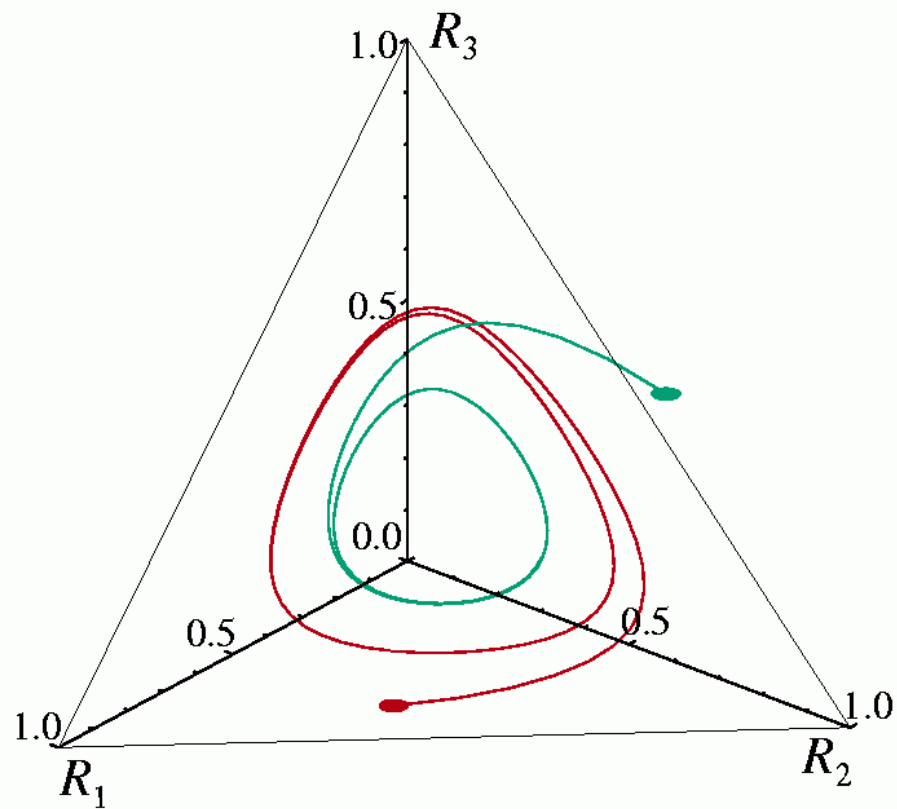
$$E = \frac{R_1(0)R_2(0)R_3(0)}{[R_1(0) + R_2(0) + R_3(0)]^3} \quad (\text{only depends on initial conditions})$$

- Explicit solutions for the period of the orbits and the amplitudes

$$T(E) = \frac{2}{\delta\sqrt{b(a-c)}} K(q) = \boxed{-\frac{3}{2\delta} \log E} \times (1 + O(E))$$

$E \rightarrow 0$

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$$\mu = 0, \delta = 1.3$$

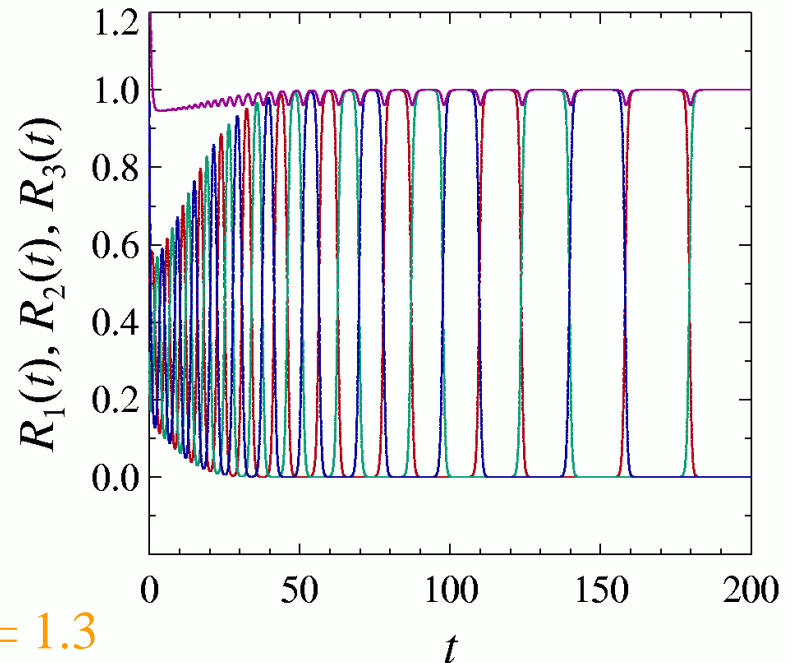
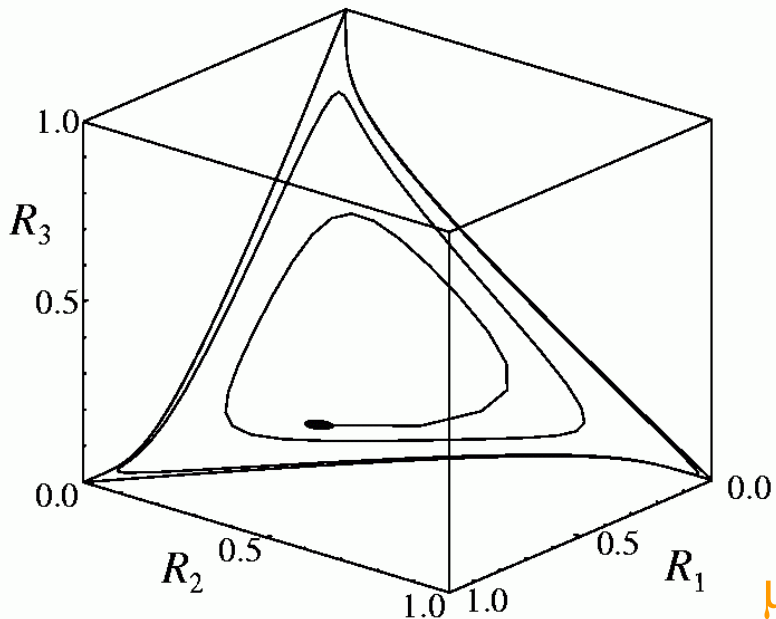
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The case $\mu \gtrsim 0$

• **Time-dependent energy**

$$E(t) = \frac{R_1 R_2 R_3}{(R_1 + R_2 + R_3)^3} \rightarrow \frac{dE}{dt} = -4\mu f(t)E, \quad f(t) > 0$$

• **Motion occurs near the plane $R_1 + R_2 + R_3 = 1$ [after a transient time of order 1]**



$\mu = 0.1, \delta = 1.3$

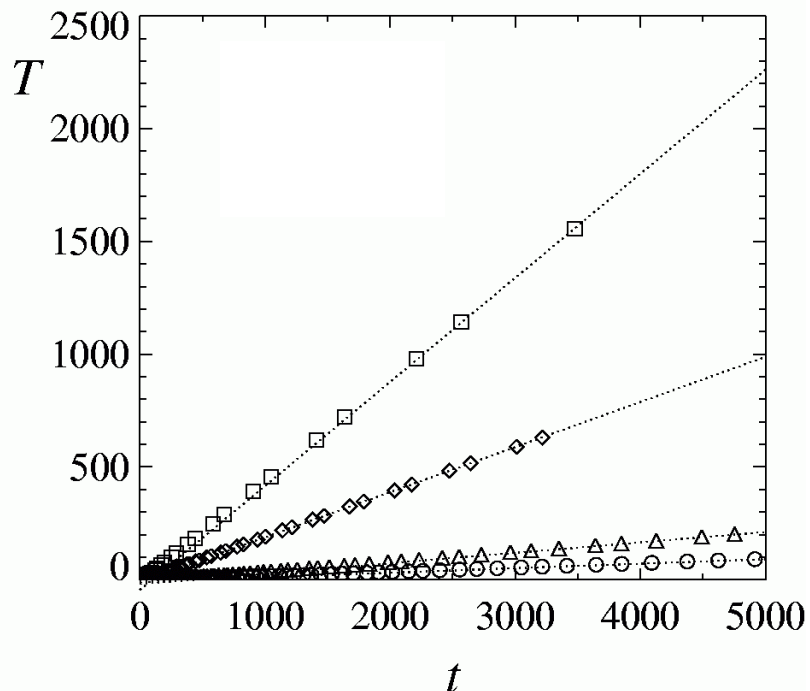
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- **Period** of the orbits is function of **time**

$$E(t) = E(0) \exp(-4\mu \int_{t_0}^t f(t') dt') \approx E(t_0) e^{-4\mu(t-t_0)}$$

$$\boxed{T = T(E(t))} \xrightarrow[\text{long times (small energies)}]{} -\frac{3}{2\delta} \log E(t) = T_0 + \frac{6\mu}{\delta} t$$

- $\triangle \delta=1.3, \mu=0.01$
- $\circ \delta=3, \mu=0.01$
- $\square \delta=1.3, \mu=0.1$
- $\diamond \delta=3, \mu=0.1$



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Busse-Heikes Model with Noise

- **Period increasing** with time is unphysical.

Solution: addition of **fluctuations**

$$\dot{A}_1 = A_1 [1 - |A_1|^2 - (1 + \mu + \delta)|A_2|^2 - (1 + \mu - \delta)|A_3|^2] + \xi_1(t)$$

$$\dot{A}_2 = A_2 [1 - |A_2|^2 - (1 + \mu + \delta)|A_3|^2 - (1 + \mu - \delta)|A_1|^2] + \xi_2(t)$$

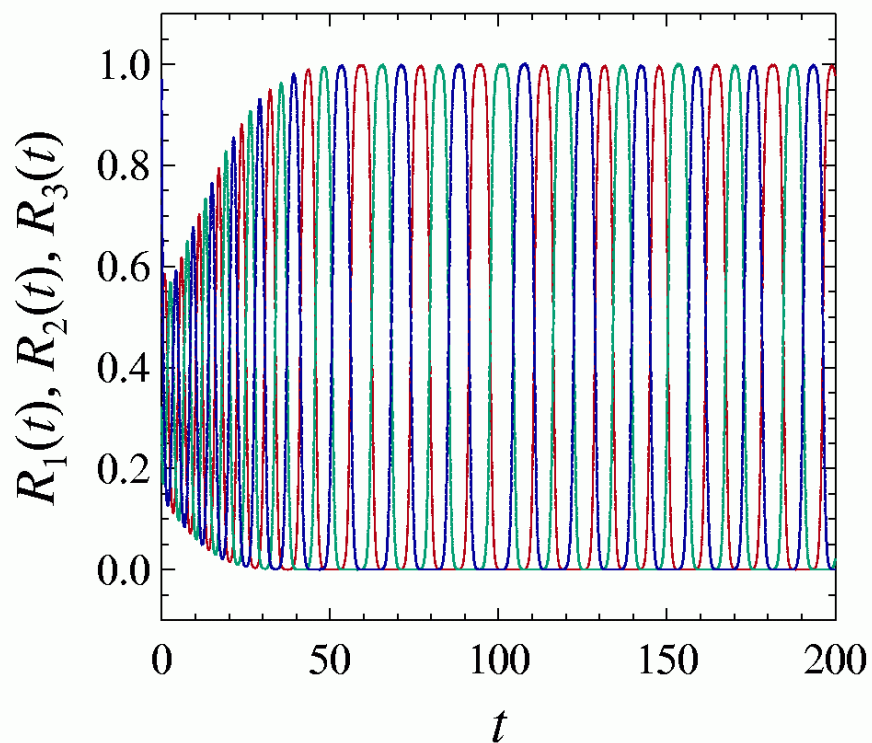
$$\dot{A}_3 = A_3 [1 - |A_3|^2 - (1 + \mu + \delta)|A_1|^2 - (1 + \mu - \delta)|A_2|^2] + \xi_3(t)$$

White noise processes: $\langle \xi_i(t) \xi_j^*(t') \rangle = 2\varepsilon \delta(t-t') \delta_{ij}$

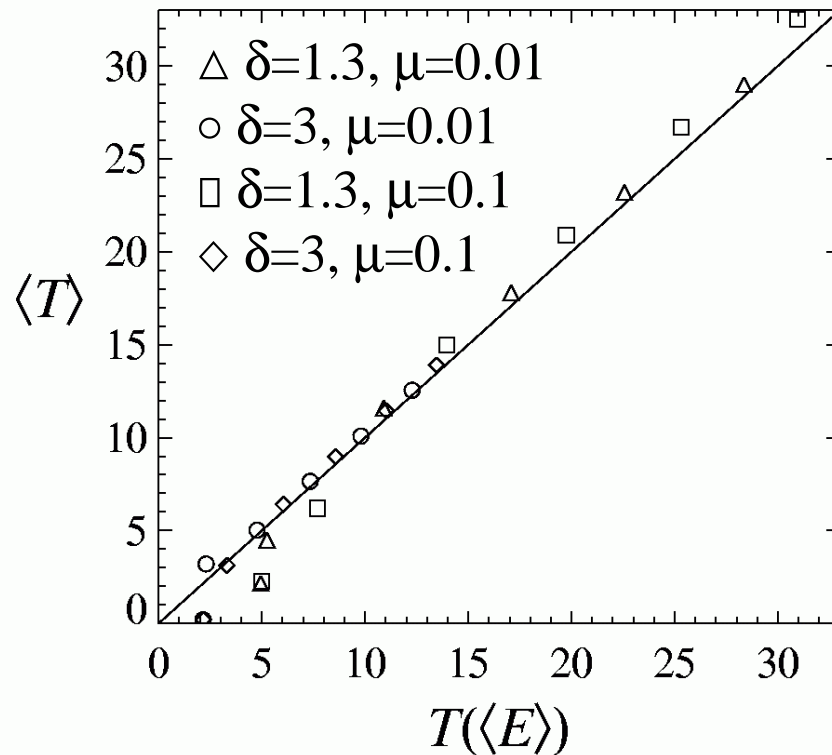
- **Noise prevents** $E(t)$ from decaying to zero.
- **Fluctuating period** is established (but periodic on average)

$$E(t) \rightarrow \langle E \rangle \Rightarrow T(t) \rightarrow \langle T \rangle = T(\langle E \rangle)$$

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$\mu = 0.1, \delta = 1.3, \varepsilon = 10^{-6}$



$\varepsilon = 10^{-2} \text{---} 10^{-6}$

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Average energy

$$P_{\text{st}}(\mu = 0) \propto \exp[-V(\mu = 0) / \varepsilon] \rightarrow P_{\text{st}}(\mu) \propto \exp[-\Phi(\mu) / \varepsilon] \approx \exp[-V(\mu) / \varepsilon]$$

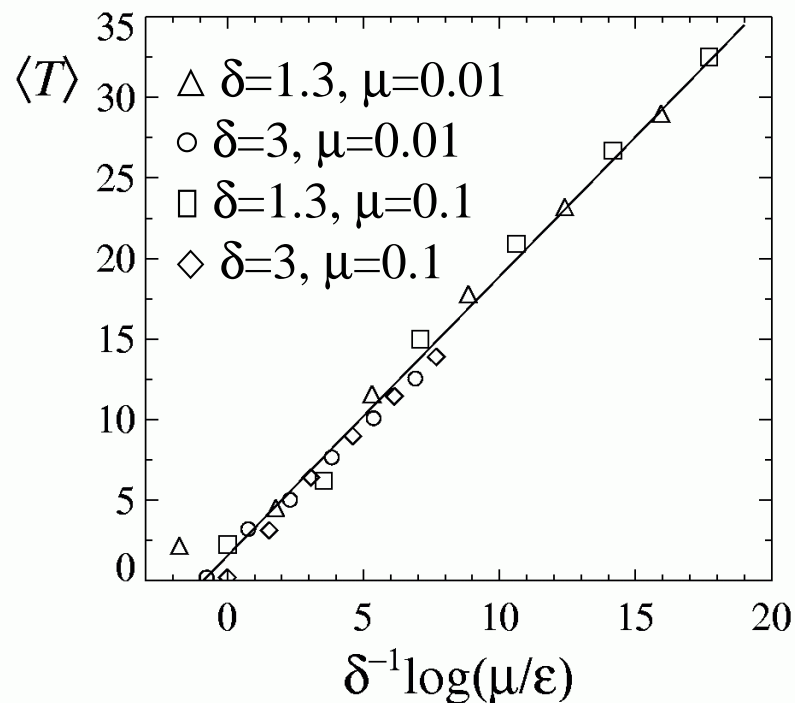
μ small

$$\langle E \rangle \cong \frac{\int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^\infty dR_3 E \exp(-V / \varepsilon)}{\int_0^\infty dR_1 \int_0^\infty dR_2 \int_0^\infty dR_3 \exp(-V / \varepsilon)}$$

Saddle-point type integration:

$$\langle E \rangle \approx (\varepsilon / \mu)^2, \quad \varepsilon \rightarrow 0, \quad \mu \text{ small}$$

$$\Rightarrow \boxed{\langle T \rangle = T(\langle E \rangle) \approx \frac{3}{\delta} \log(\mu / \varepsilon)}$$



$$\varepsilon = 10^{-2} \text{ --- } 10^{-6}$$

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ONE-DIMENSIONAL SYSTEMS

- Three competing nonconserved **real** order parameters with short-range interactions

$$\dot{A}_1 = \partial_{xx} A_1 + A_1 [1 - A_1^2 - (1 + \mu + \delta) A_2^2 - (1 + \mu - \delta) A_3^2]$$

$$\dot{A}_2 = \partial_{xx} A_2 + A_2 [1 - A_2^2 - (1 + \mu + \delta) A_3^2 - (1 + \mu - \delta) A_1^2]$$

$$\dot{A}_3 = \partial_{xx} A_3 + A_3 [1 - A_3^2 - (1 + \mu + \delta) A_1^2 - (1 + \mu - \delta) A_2^2]$$

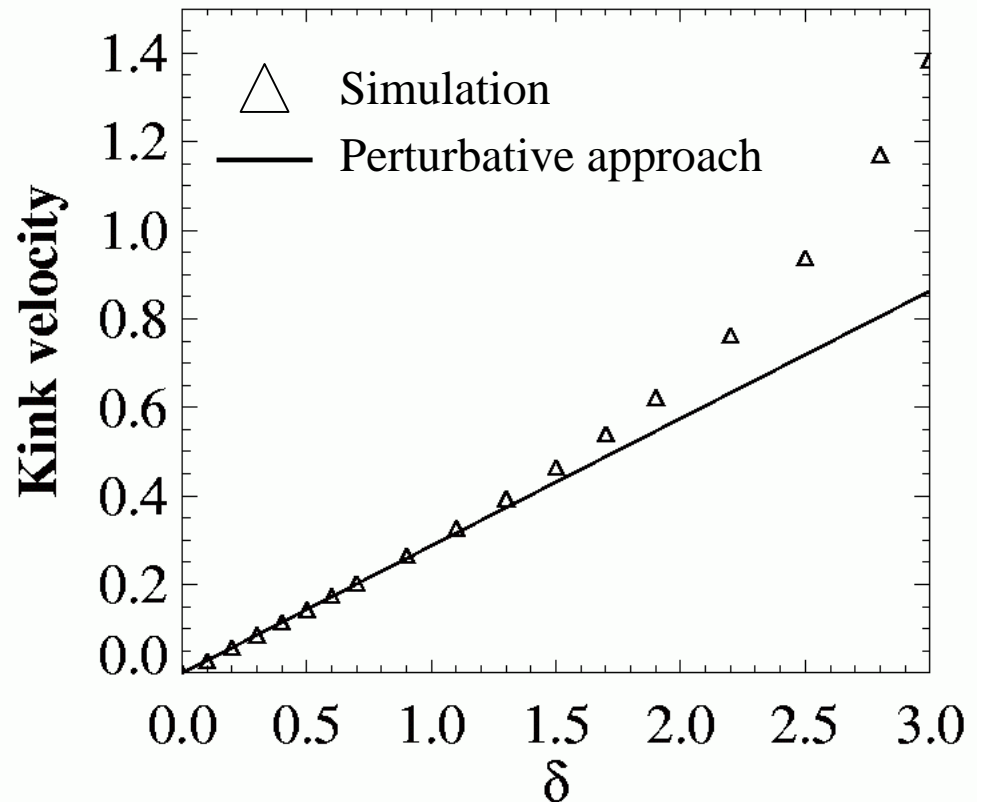
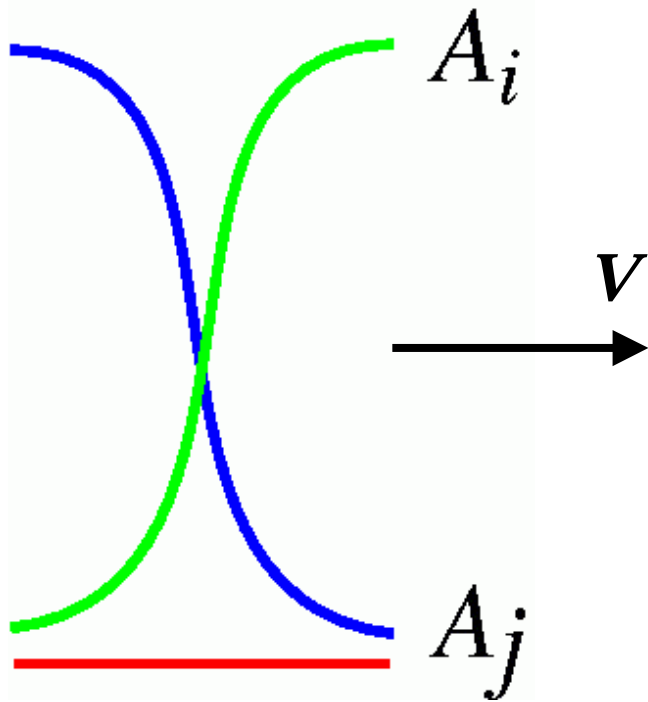
- Prothotypical **nonpotential** problem to study **domain growth** and **dynamical scaling**
- We focus on the region **below** the KL instability: coexistence of three stable homogeneous states under **nonpotential** dynamics ($\delta \neq 0$)

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Dynamics of an Isolated Kink

- Isolated kink moves due to **nonpotential** effects

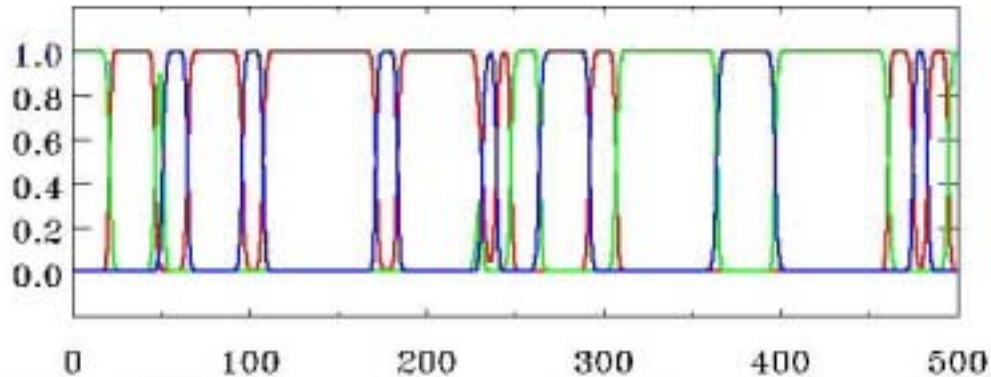
$$v(\mu, \delta) = h(\mu) \delta + O(\delta^2)$$



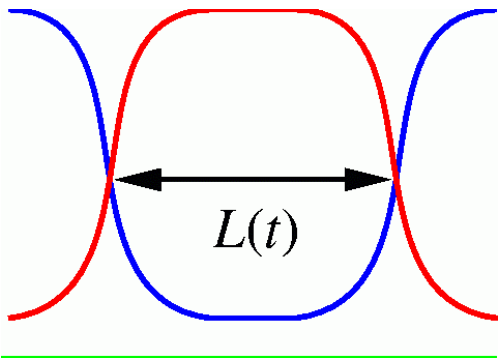
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Multikink Configurations

• **Kink motion** → **annihilation** → **domain growth**



• **Front motion due to: attractive interaction forces + nonpotential effects**



$$\partial_t L(t) = \pm 2v(\mu, \delta) - \gamma \exp(-\mu^{1/2} L(t))$$

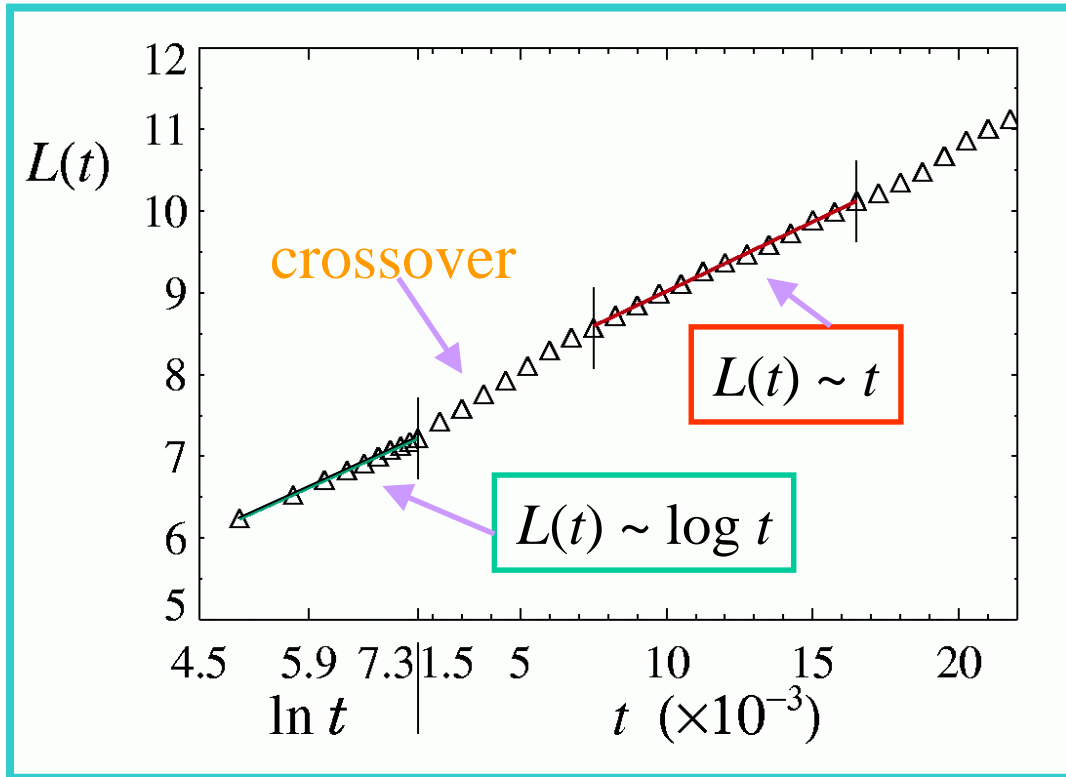
$$L(t) \sim t$$

$$L(t) \sim \log t$$

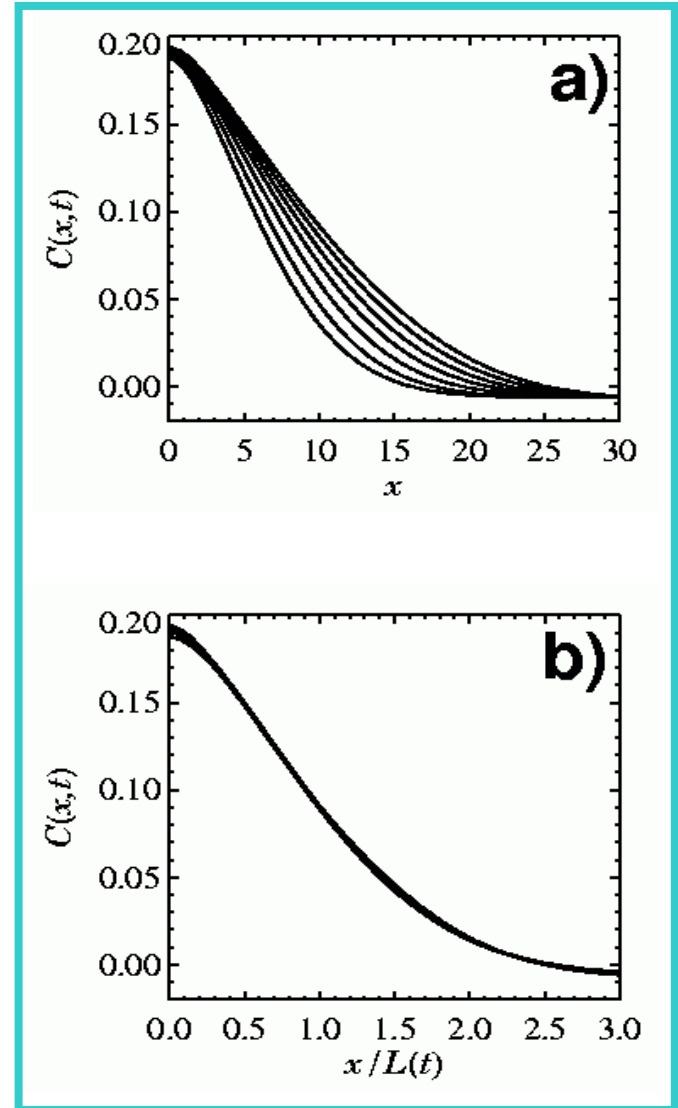
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Domain Growth and Dynamical Scaling

$$L(t) \sim \log t + \text{crossover} + t$$



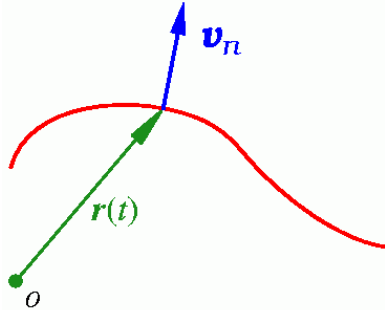
$$\eta = 3.5, \delta = 10^{-3}$$



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TWO-DIMENSIONAL SYSTEMS

- We take **real** variables, **isotropic** diffusion terms and we focus on the region **below** the KL instability point
- Normal front **velocity**



$$V_n(\mathbf{r}, t; \mu, \delta) = -\kappa(\mathbf{r}, t) + v_p(\mu, \delta)$$

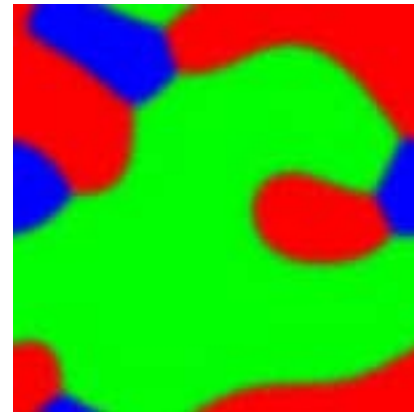
$$v_p(\mu, \delta) \sim \delta + O(\delta^2)$$

- **Absence of coarsening** for system sizes large enough



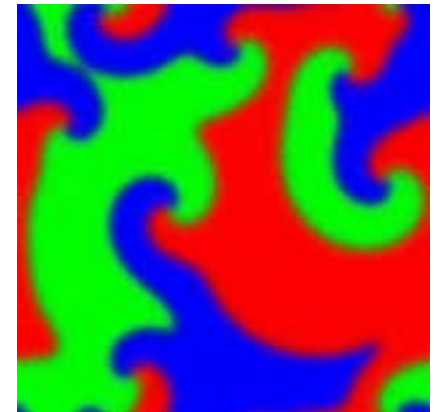
nonpotential dynamics
+
formation of vertex points

POTENTIAL



$$\mu = 3, \delta = 0$$

NONPOTENTIAL



$$\mu = 3, \delta = 2$$

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Spiral dynamics

Rotation angular velocity

$$\omega(\mu, \delta) \propto v_p^{1/2} \kappa_0^{3/2}$$

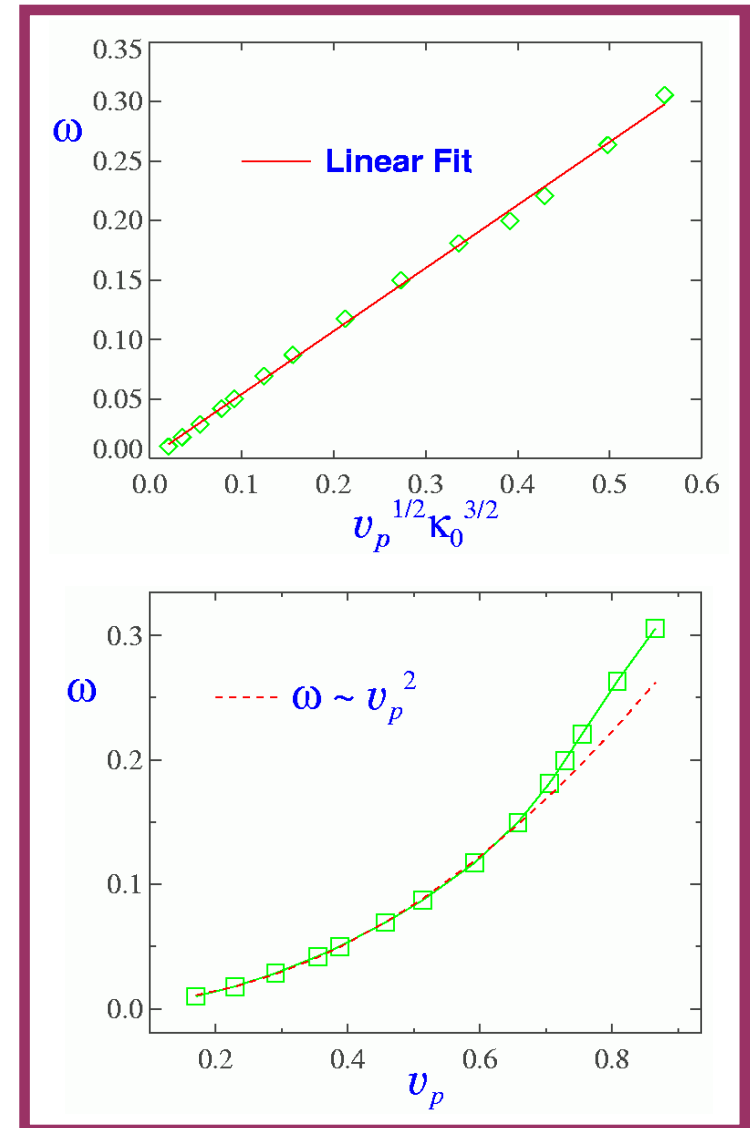
v_p small



$$\kappa_0 \sim v_p(\delta)$$

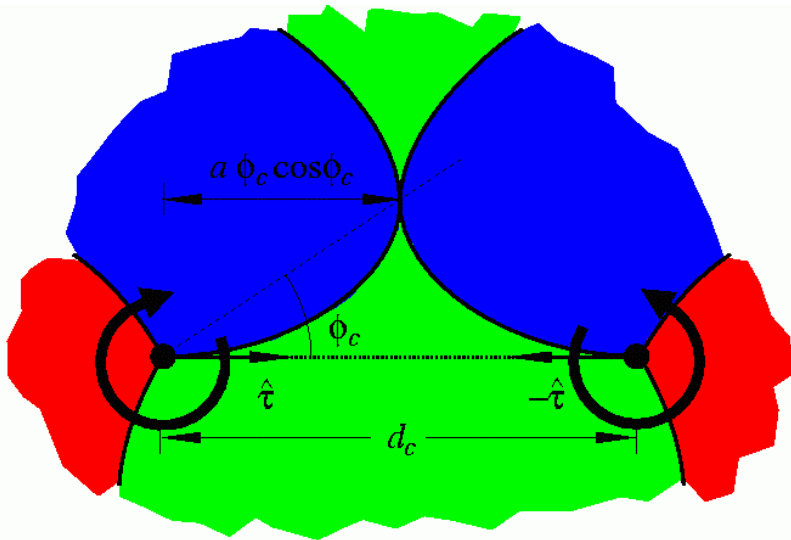


$$\omega \sim v_p(\delta)^2 \sim \delta^2$$

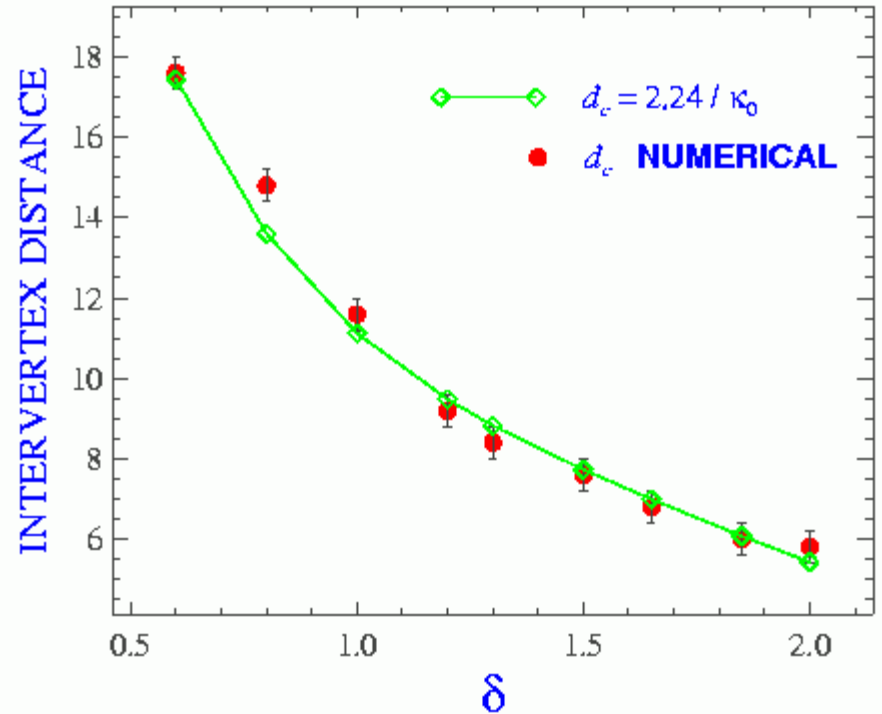


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Critical distance for vertex annihilation



$$d_c \cong 2.24/\kappa_0 \sim \delta^{-1}$$





Consequence: coarsening will occur for system sizes $S \lesssim d_c$

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Vertex Motion

- For **long times** vertices **diffuse** through the system
- Vertex dynamics affected by the **boundary conditions**

• **Periodic BC:**

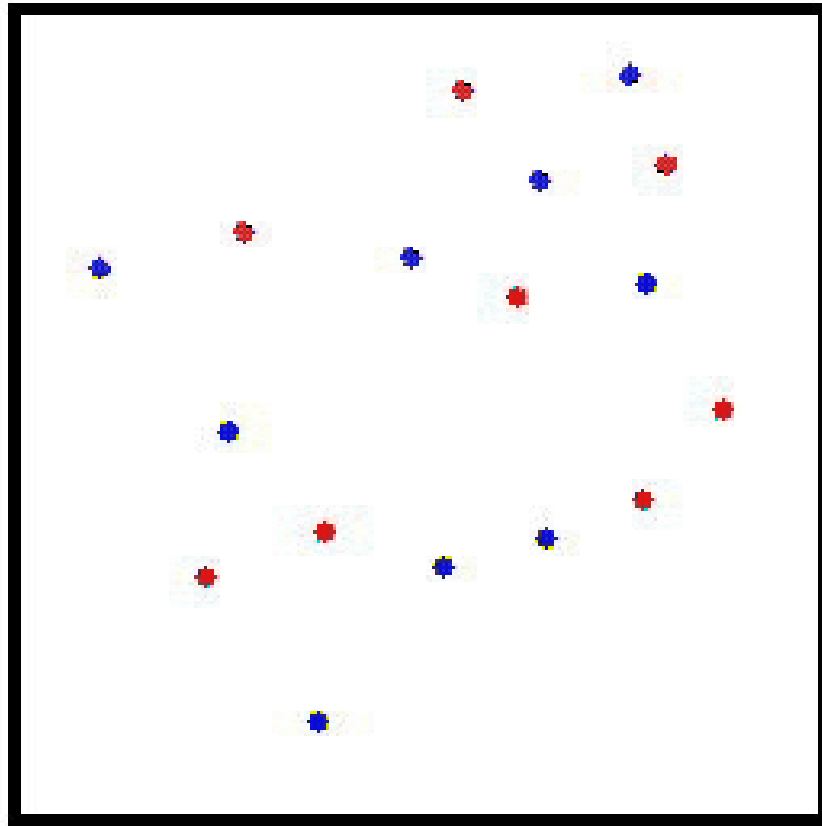
- **Even** number of vertices: half  and half 
- Vertices **disappear by pairs** of opposite sense of rotation
- **Correlated** motions are observed

• **Null BC:**

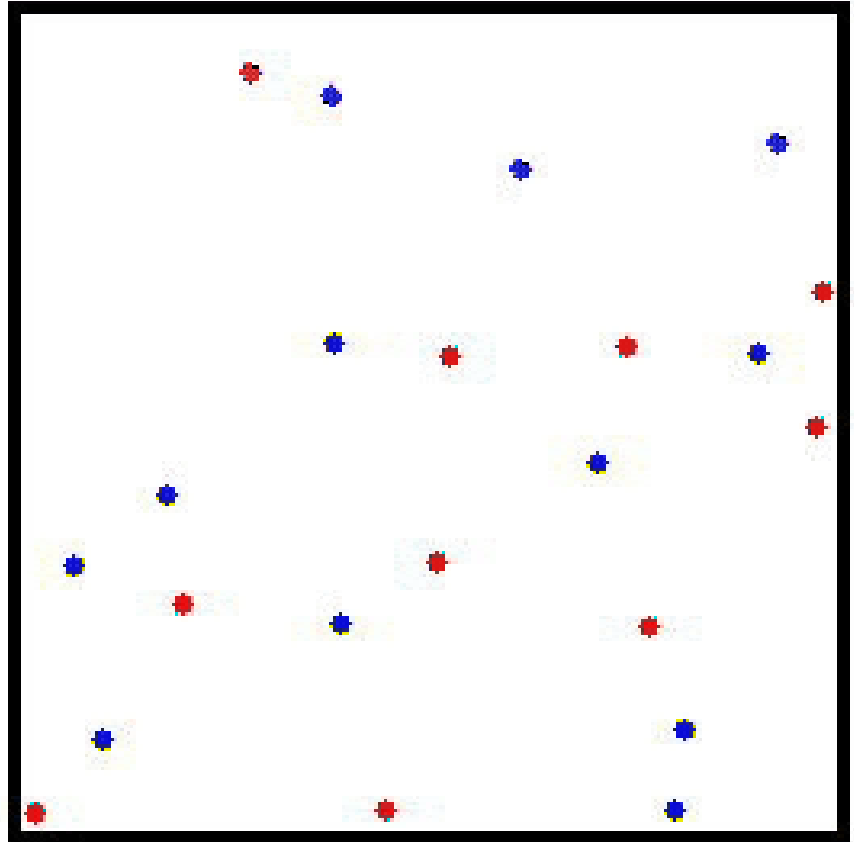
- **No restrictions** about the number of vertices
- Vertices may **disappear through the edges** of the system
- **No correlated** motions observed

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Null BC



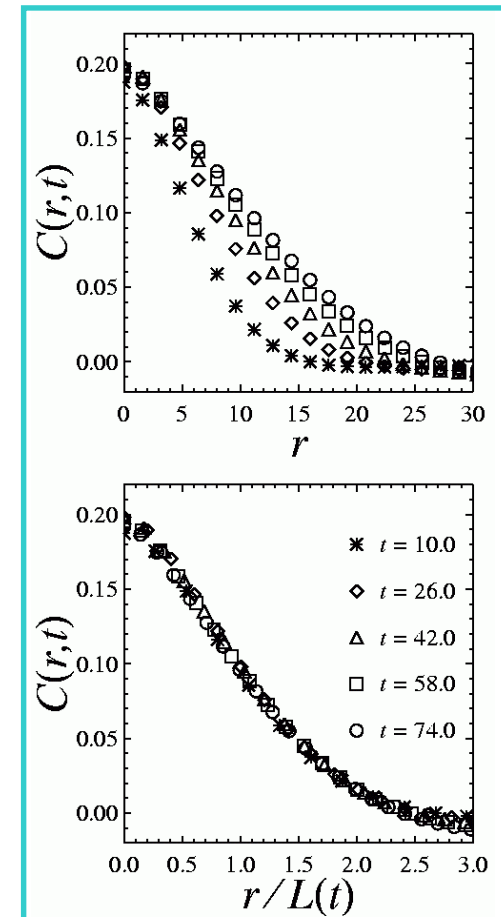
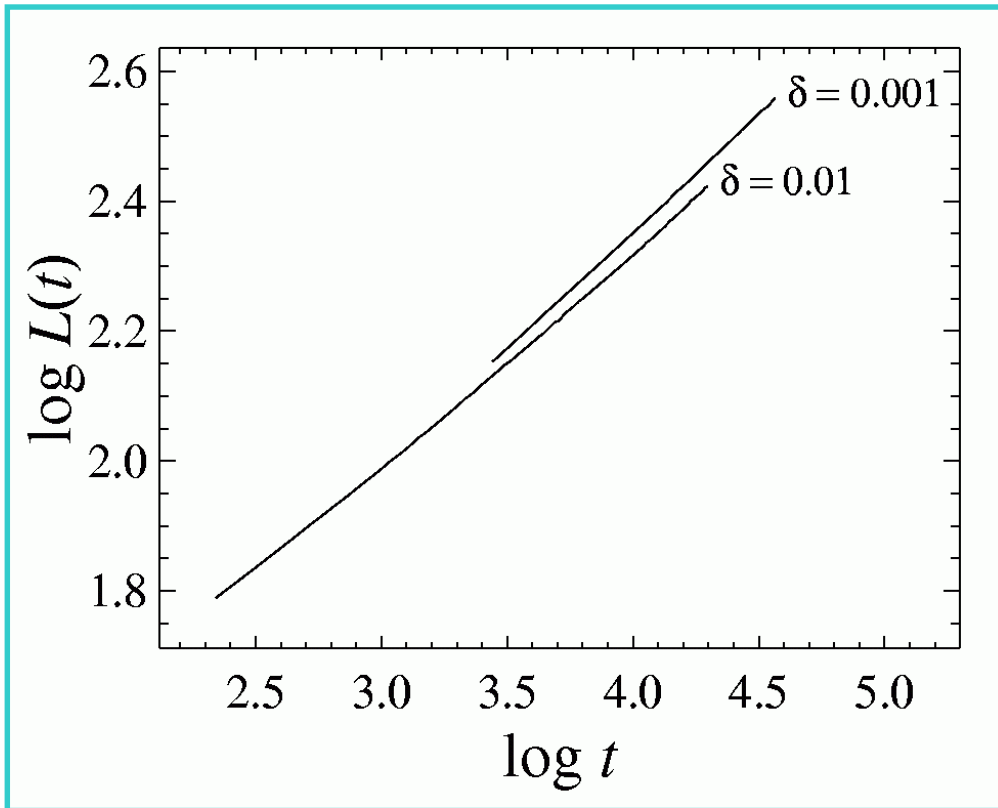
Periodic BC



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Domain Growth and Dynamical Scaling

- $\delta = 0$ (potential limit) $\Rightarrow L(t) \sim t^{1/2}$ + dynamical scaling (3 fields)
- $\delta \neq 0$ (nonpotential limit) $\Rightarrow L(t) \sim t^{1/2}$ + dynamical scaling (2 fields)



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Spatial-dependent Terms

- Alternative explanation for **period stabilization**
- Two kinds of **differential operators**

$$\mathcal{L}_j, j=1,2,3 \begin{cases} \nabla^2 & \text{ISOTROPIC} \\ (\hat{\mathbf{e}}_j \cdot \nabla)^2 & \text{ANISOTROPIC } (\leftarrow \text{NWS, GOS terms}) \end{cases}$$

- We focus on the region **beyond** the KL instability point
- Dynamics depends on the **type** of spatial derivatives and on the size of μ

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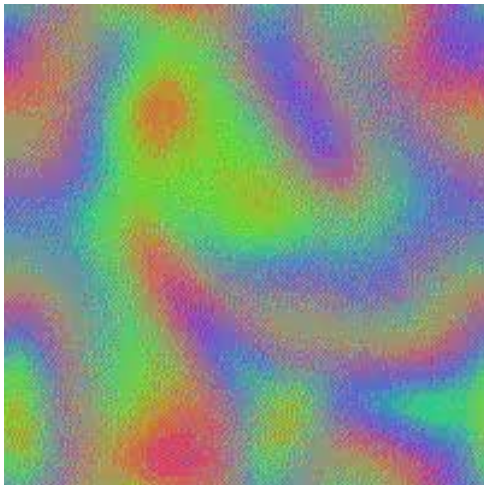
• μ small

• Similar morphology of domains

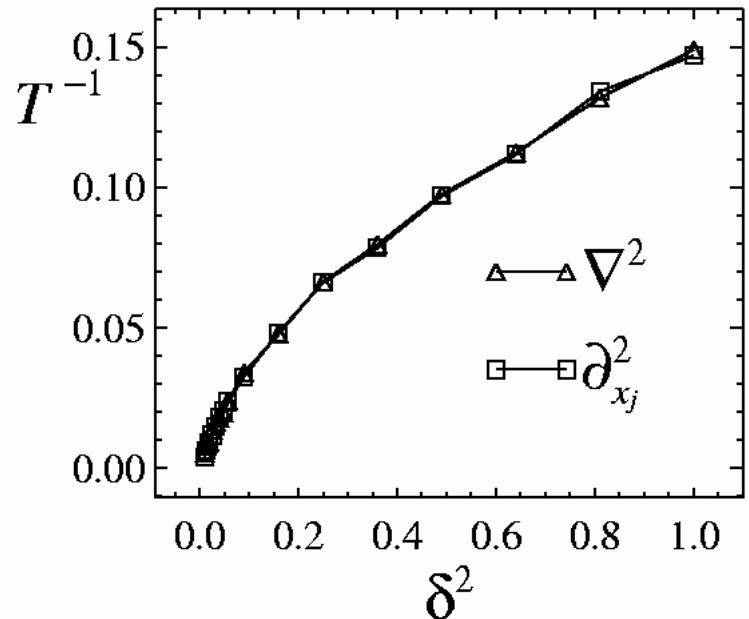
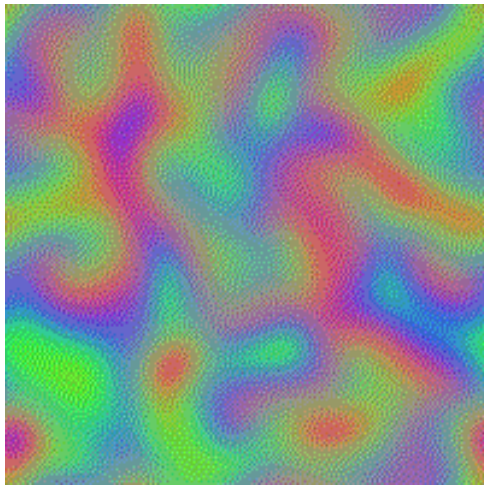
• Alternating period dominated by the KL instability

• Intrinsic KL period stabilizes to a statistically constant value with both kinds of terms

$$\nabla^2$$



$$(\hat{e}_j \cdot \nabla)^2$$



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• μ large

• Different morphology of domains

• Intrinsic KL period $\left\{ \begin{array}{l} \nabla^2: \text{diverges with time} \\ (\hat{\mathbf{e}}_j \cdot \nabla)^2: \text{saturates to a constant value} \end{array} \right.$

• Different alternating periods

∇^2

$(\hat{\mathbf{e}}_j \cdot \nabla)^2$

