Coupling between two quantum dots through a superconducting island

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Jožef Stefan Institute
Ljubljana, Slovenia

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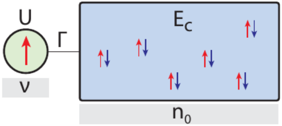
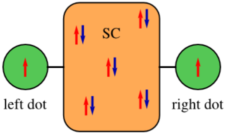
Motivation

Double quantum dot + superconducting island

- Potential qubit realization by using two dots coupled to a superconducting island SC (subgap states)



Source: PRB 104, L241409 (2021)



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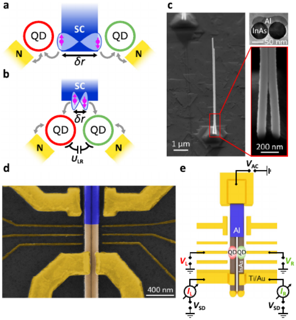

Source: npj Quantum Materials, 7, 88 (2022)

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- Potential qubit realization by using two dots coupled to a superconducting island SC (subgap states)
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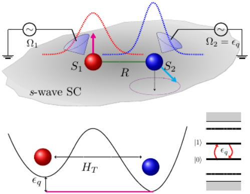
- Interaction between magnetic impurities embedded in superconducting materials

PRL 113, 087202 (2014)

PRX Quantum 2, 040347 (2021) $\Rightarrow$



Interaction between magnetic impurities

- RKKY interaction:

$$H_{ij} = J(\mathbf{R}_1 - \mathbf{R}_2)\mathbf{S}_1\mathbf{S}_2$$

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  PRL **113**, 087202 (2014)
- Between magnetic impurities located closely to each other, the coupling is described by superexchange

Two quantum dots coupled to SC

- Model:

$$H = H_{SC} + H_{QDs} + H_{hyb}$$

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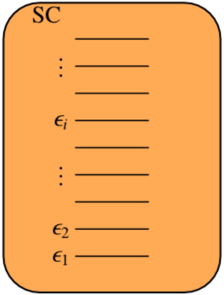
$$H_{SC} = \sum_{\sigma,i=1}^{N} \epsilon_i c^+_{i\sigma} c_{i\sigma} - \frac{g}{N} \sum_{i,j}^{N} c^+_{i\uparrow} c^+_{i\downarrow} c_{j\downarrow} c_{j\uparrow} \qquad \text{(Richardson model)}$$



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$$H_{QDs} = \varepsilon_L \sum_{\sigma} d^+_{L\sigma} d_{L\sigma} + \varepsilon_R \sum_{\sigma} d^+_{R\sigma} d_{R\sigma} + U_L n_{L\uparrow} n_{L\downarrow} + U_R n_{R\uparrow} n_{R\downarrow}$$

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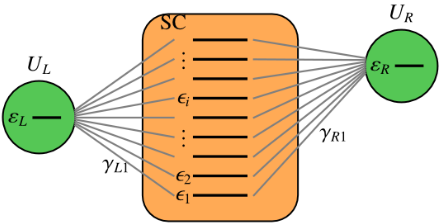
$$H_{hyb} = v_L \sum_{\sigma i} \left(\gamma_{Li}^{\sigma} c_{i\sigma}^{+} d_{L\sigma} + h.c.\right) + v_R \sum_{\sigma i} \left(\gamma_{Ri}^{\sigma} c_{i\sigma}^{+} d_{R\sigma} + h.c.\right)$$



- Strength of couplings: v_L and v_R
- Distribution of couplings: γ_{Li} and γ_{Ri} which fulfill $\sum_i |\gamma_{Li}|^2 = \sum_i |\gamma_{Ri}|^2 = 1$

Two quantum dots coupled to SC

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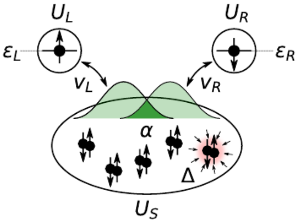
- Hybridizations may also overlap: $\alpha = \sum_i \gamma^*_{Li}\gamma_{Ri}$ $\alpha \in [0;1]$



Source: arXiv:2303.14410

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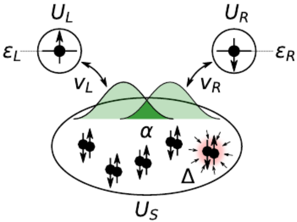

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- We will focus on *close* QDs, $\alpha = 1$.
- Main question: What is the ground state depending on $\varepsilon_{L/R}$, $v_{L/R}$ and U? What spin configuration?

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- In the single QD - SC system, flat-band limit provided a good description
- Ground state of the superconductor with fixed number of Cooper-pairs M

$$|\Psi_M^N\rangle \sim \left(\sum_i c_{i\uparrow}^{+} c_{i\downarrow}^{+} \right)^M |0\rangle$$

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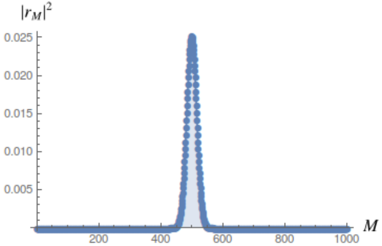
- Diagonalization with Bogoliubov (finite chemical potential)
- Ground state is a mixture of $|\Psi^N_M\rangle$ with different values of M

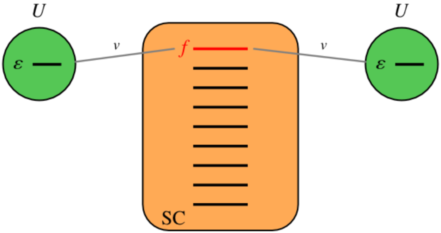
$$|\Psi_{BCS}\rangle = \sum_M r_M |\Psi^N_M\rangle$$

r_M describes a binomial distribution



Complete overlap, $\alpha = 1$, analytical results

- Fix the number of Cooper pairs, M, and assume the limit $U \to \infty$ and the thermodynamic limit
- Symmetric coupling v and $\varepsilon < 0$
- Quantum numbers:
  - total number of particles (even/odd)
  - total spin ($S = 0$ singlet, $S = 1/2$ doublet, $S = 1$ triplet)
  - total S_z
  - parity (symmetric/antisymmetric)



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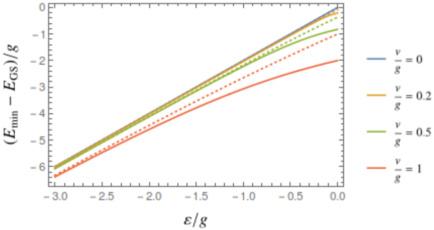
- Even number of particles (singlet/symmetric or triplet/antisymmetric)



———	singlet	$\frac{1}{\sqrt{2}}\left(\lvert\uparrow_L, \downarrow_R\rangle - \lvert\downarrow_L, \uparrow_R\rangle\right) \otimes \lvert\Psi_M^N\rangle_{SC}$
- - - -	triplet	$\lvert\uparrow_L, \uparrow_R\rangle \otimes \lvert\Psi_M^N\rangle_{SC}$

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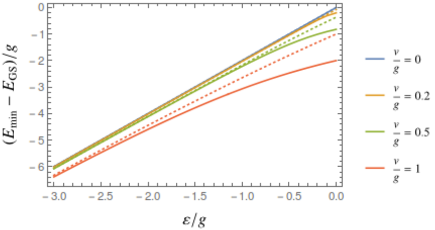

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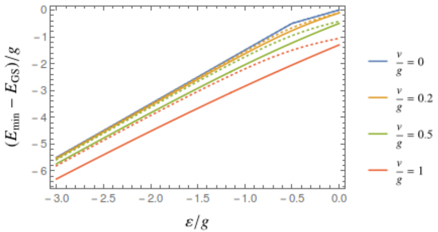
- The difference in minimal energies: $E_T - E_S = \frac{2v^4}{\lvert\varepsilon\rvert^3}$

Complete overlap, $\alpha = 1$, analytical results

- Odd number of particles, doublet (symmetric or antisymmetric)



———	antisymmetric	$\frac{1}{\sqrt{6}}\left(2\vert\uparrow,\uparrow\rangle \otimes \vert\downarrow,\Psi^{N-1}_{M-1}\rangle - (\vert\uparrow,\downarrow\rangle + \vert\downarrow,\uparrow\rangle) \otimes \vert\uparrow,\Psi^{N-1}_{M-1}\rangle\right)$
- - - -	symmetric	$\frac{1}{\sqrt{2}}(\vert\uparrow,\downarrow\rangle - \vert\downarrow,\uparrow\rangle) \otimes \vert\uparrow,\Psi^{N-1}_{M-1}\rangle$

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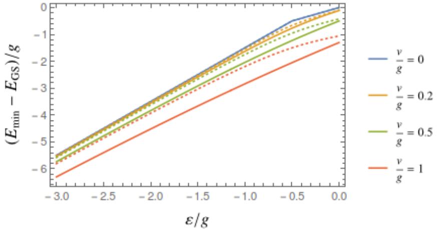

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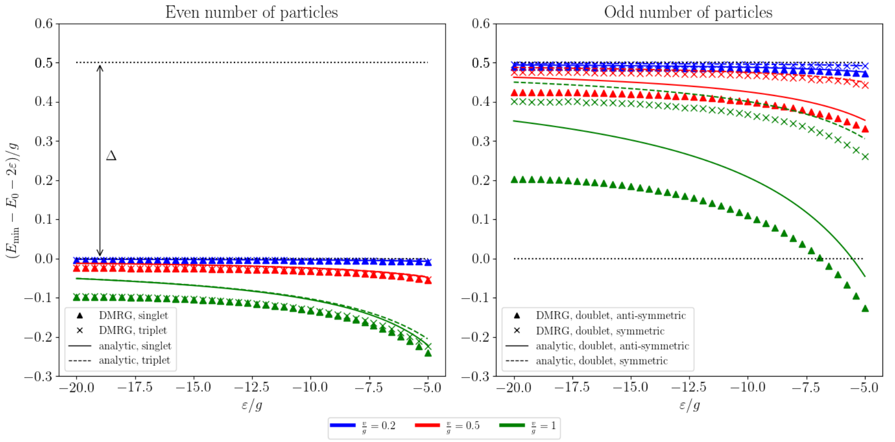
- The difference in minimal energies: $E_{D,sym} - E_{D,asym} = \frac{2v^2}{|\varepsilon|}$

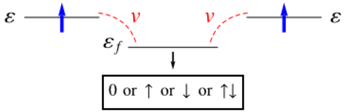
Comparison with DMRG

- Density Matrix Renormalization Group simulation with $U/g = 40$ and $N = 80$



Superexchange through a single level

- Two dots coupled to a single level



- Hamiltonian

$$H = \varepsilon \sum_{\sigma\delta=L,R} d^+_{\delta\sigma} d_{\delta\sigma} + U \sum_{\delta=L,R} n_{\delta\uparrow} n_{\delta\downarrow} + v \sum_{\sigma\delta=L,R} \left(d^+_{\delta\sigma} f_\sigma + h.c.\right) + \varepsilon_f \sum_\sigma f^+_\sigma f_\sigma$$

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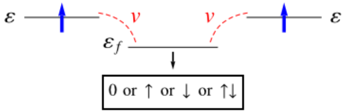

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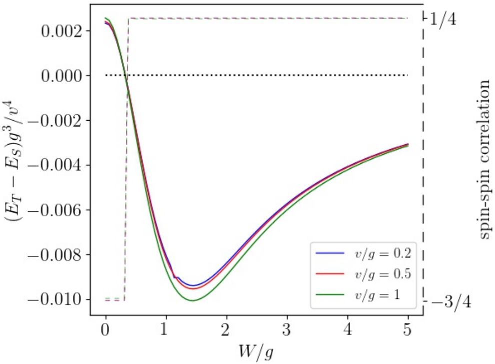
- The occupation of the mediating level determines whether ferromagnetic or antiferromagnetic alignment is favoured
- Doubly occupied or empty intermediate level favours antiferromagnetic alignment
- Singly occupied intermediate level favours antiferromagnetic alignment

Effects of finite bandwidth

- DMRG results:
  - even number of particles: transition from singlet to triplet
  - odd number of particles: no transition, always ferromagnetic alignment



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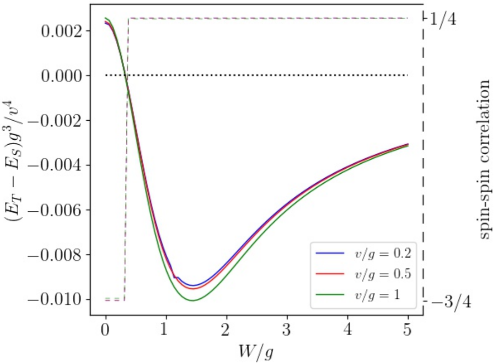

- Kinetic energy enhances the probability of single occupancy on the distinguished level

Summary

- In the flatband limit, QD-SC-QD system features
  - singlet (antiferromagnetic alignment) for even number of particles
  - parity-antisymmetric doublet (ferromagnetic alignment) for odd number of particles
- The features can be understood through the occupancy of the distinguished level of the superconductor
- Finite bandwidth plays an important role as opposed to single QD - SC

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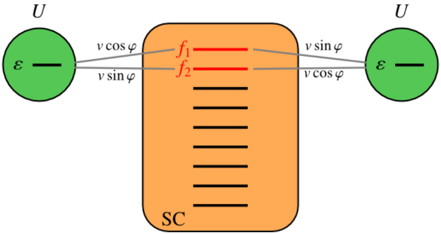
Thank you for your attention!

Löwdin orthogonalization

- Two quantum dots coupled to the SC in a symmetric way: $\varepsilon_L = \varepsilon_R = \varepsilon$, $v_L = v_R = v$ and $U_L = U_R = U$
- In the flatband limit, the SC levels are identical

$\Rightarrow$ freedom to choose basis of SC levels $\quad f_{i\uparrow} = \sum_{j=1}^{N} U_{ij} c_{i\uparrow} \qquad f_{i\downarrow} = \sum_{j=1}^{N} U_{ij}^{*} c_{i\downarrow}$

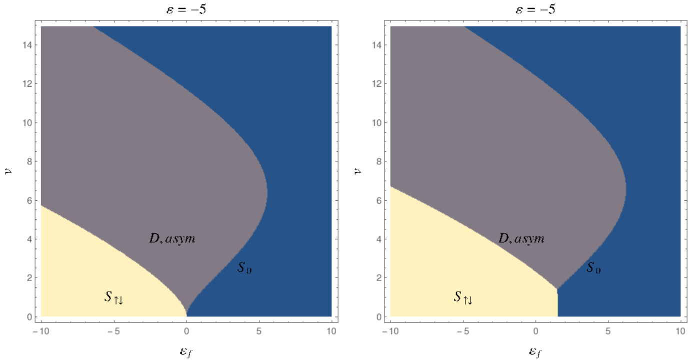
- We choose a basis in which the dots are coupled to two SC level only



$$\alpha = \sin(2\varphi)$$

Superexchange through a single level

- Phase diagram for the ground state for $U = 10$ and $\varepsilon = -5$



$g = 0$ $\qquad$ $g = 1$