Eigenvalues of random matrices with generalised correlations: A path-integral approach

Joseph W. Baron, Thomas Jun Jewell, Christopher Ryder, Tobias Galla

IFISC (CSIC-UIB) Palma de Mallorca – Spain.

josephbaron@ifisc.uib-csic.es

Introduction

- Random matrix theory allows one to find the eigenvalues of a large matrix given only statistical information about its elements. Applications include opinion dynamics, neural networks, complex ecosystems and spin glasses.
- We generalise previous results by allowing for correlations between any pair of elements in the matrix.
- To this end, we introduce a new formalism which maps the random matrix problem onto a disordered dynamical system.
- We use a path-integral approach to treat the new correlations as a perturbation to old results, vastly simplifying the problem.

Random matrix and eigenvalue spectrum

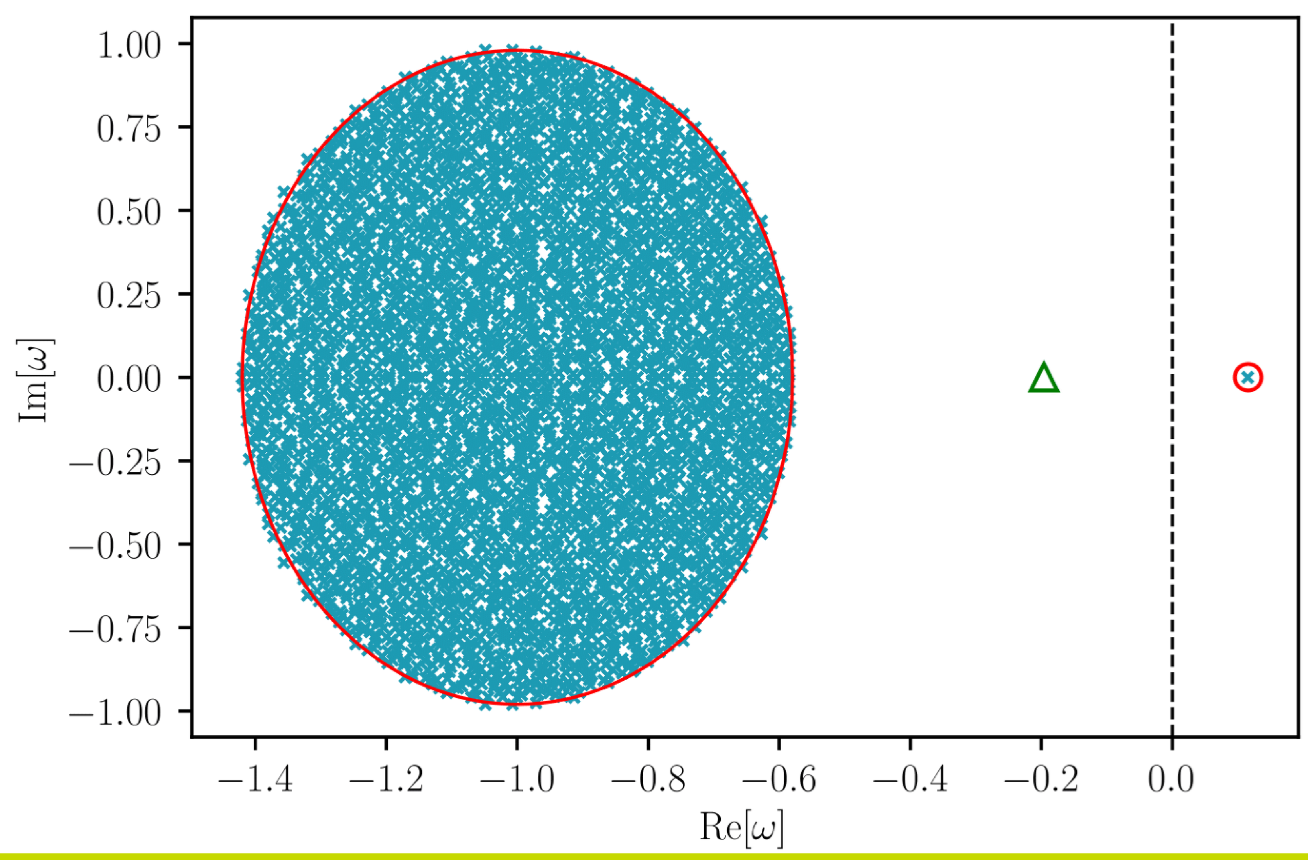
■ We consider matrices with elements drawn from a joint distribution with the following statistics ($M_{ii} = -1$ and none of i, j, k are equal)

$$\mathrm{E}(M_{ij}) = \frac{\mu}{N}, \mathrm{Var}(M_{ij}) = \frac{\sigma^2}{N}, \mathrm{Corr}(M_{ij}, M_{ji}) = \Gamma, \mathrm{Corr}(M_{ij}, M_{ki}) = \frac{\gamma}{N}$$

$$\mathrm{Corr}(M_{ij}, M_{ik}) = \frac{r}{N}, \mathrm{Corr}(M_{ji}, M_{ki}) = \frac{c}{N},$$

■ These are the most general statistics that preserve symmetry between different positions in the matrix. New correlations are given by r, c and γ.

■ Most of the eigenvalues are confined to an ellipse in the complex plane (also true when $r = c = \gamma = 0$) except one outlier, which we attempt to find



Corresponding dynamical system

■ Letting $z_{ij} = M_{ij} + \delta_{ij} - \frac{\mu}{N}$, consider the following set of coupled ODEs

$$\dot{x}_i = -x_i + \sum_{j \neq i} z_{ij} x_j + h_i(t)$$

■ The response functions $R_{ij}(t - t') = \frac{\delta x_i(t)}{\delta h_j(t')}$ yield the outlier eigenvalue via

$$1 - \frac{\mu}{N} \sum_{ij} \hat{R}_{ij}(1 + \lambda_{outlier}) = 0$$

where $\hat{R}_{ij}(\mathrm{u})$ is the Laplace transform of the response function. If we can find R_{ij}, we can find $\lambda_{outlier}$.

Path-integral and series expansion

■ The response functions can be written as a path-integral expression

$$\langle R_{ij}(t - t') \rangle = -i \left\langle x_i(t) \hat{x}_j(t') \right\rangle_{S_0 + S_{int}}$$

$$\langle \cdots \rangle_{S_0 + S_{int}} = \int D[\{x(t)\}, \{\hat{x}(t)\}] [\cdots] e^{S_0 + S_{int}}$$

■ The action has two contributions:
S_0, which involves terms proportional to Γ (correlations between z_{ij} and z_{ji})
S_{int}, which involves terms proportional to γ, c and r (correlations between non-transpose pairs of elements)

■ One can expand the exponential above to obtain a series in S_{int}

■ Performing the averages over paths, one obtains an expression for $R_{ij}(t - t')$ in terms of the response functions of the system with $\gamma = c = r = 0$, $R^0_{ij}(t - t')$, which are known

Final expression for the outlier

$$\lambda_{outlier} = -1 + \mu + \frac{\mu}{2}\left(1 + \frac{\Gamma}{\gamma}\right)\left(\sqrt{1 + \frac{4\gamma\sigma^2}{\mu^2}} - 1\right)$$

■ We note that in-row (r) and in-column correlations (c) do not contribute

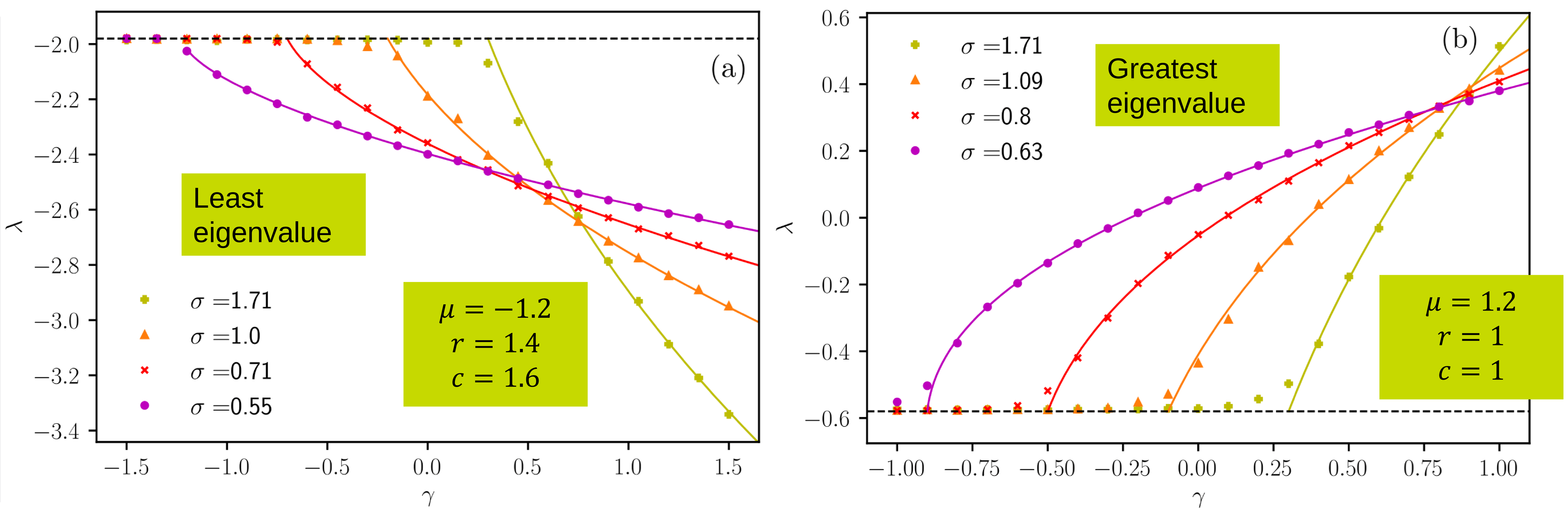
■ For $|\mu| \leq (1 - \gamma)\sigma$, the outlier is absorbed into the ellipse and there is no outlier eigenvalue

Verification of the result

■ Edges of the ellipse at $\lambda = -1 \pm (1 + \Gamma)\sigma$ are shown as a dotted line

■ For each value of σ in this figure, we keep $(1 + \Gamma)\sigma = \mathrm{const.}$

■The presence of correlations proportional to γ can significantly affect the position of the outlier eigenvalue and, therefore, the stability of the system associated with the matrix M



Conclusions

■ We found the eigenvalue spectrum of an ensemble of random matrices with generalised correlations

■Correlations between elements M_{ij} and M_{ki} affected the outlier enough to have impact on the stability of the system for which M was the Jacobian

■To take into account previously neglected correlations, we employed a path intergal approach, which allowed us to treat them as a perturbation to previous results