Biased-voter model - how persuasive a small group can be?

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Abstract

We study a model of confident voters dynamics with bias, i.e. where a fraction of voters prefer a fixed opinion. We consider the model defined on both biased-independent and biased-dependent topologies. In the former case, we obtain analytical results for an all-to-all and an ER random network topologies which we confirm through numerical simulations. In particular, we find that the consensus time scales logarithmically with effective bias, i.e. the value of bias multiplied by the number of biased nodes. In the case of the biased-dependent topology, we consider as the defining parameter of the topology of the network the ratio of the density of connections among only biased nodes (B) and among only unbiased nodes (U). Based on this, we present two models through which this ratio can be varied and through simulations we identify the effect this has on the consensus time. We find that while varying the average degree among B-U voters (everything else constant) has no effect on the consensus time, when the biased voters form a well-organized minority, the time to reach consensus is reduced significantly.

Biased-independent topology

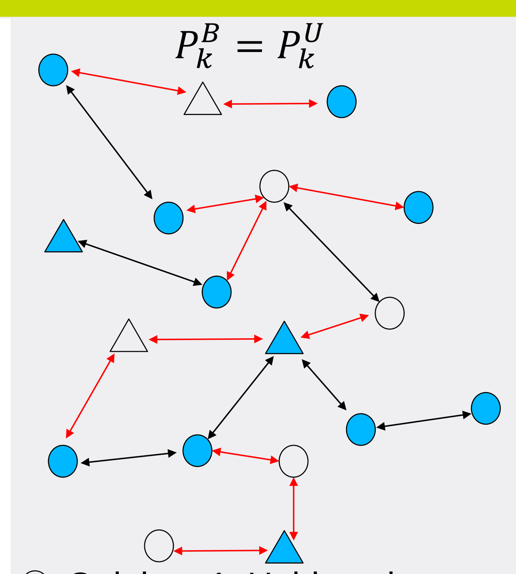
General dynamics

A. Preassign node type (fraction γ of them are biased)
B. Model dynamics:
1. Choose i^{th} voter ~ $\boldsymbol{U(i)}$ & j^{th} neighbor ~$\boldsymbol{U(j)}$, $\boldsymbol{U}$: uniform dist.
2. Update i^{th} voter's opinion s_i according to

$$P\left(s_i \longrightarrow -s_i | s_i, s_j, \boldsymbol{W(i)}\right) = \begin{cases} \frac{1}{2} & s_i \neq s_j \\ \frac{1-v}{2} & 1 = s_i \neq s_j = -1 \\ \frac{1+v}{2} & -1 = s_i \neq s_j = 1 \end{cases}$$

$\boldsymbol{W(i) = U}$: $-1 \longrightarrow 1$, $1 \longrightarrow -1$

$\boldsymbol{W(i) = B}$: $-1 \longrightarrow 1$ (bold arrow), $1 \longrightarrow -1$



○ Opinion -1, Unbiased voter
● Opinion 1 , Unbiased voter
△ Opinion -1, Biased voter
▲ Opinion 1 , Biased voter

Quantities of interest

Spin magnetization:
$\sigma \coloneqq \sum_k P_k \sigma_k$
where $\sigma_k \coloneqq \frac{n_k}{N_k}$

Degree-weighted spin magnetization:
$\sigma_L \coloneqq \frac{1}{\mu} \sum_k k P_k \sigma_k$
where $\mu \coloneqq \sum_k k P_k$

Active link (↔) density:
$$\rho = \frac{1}{2L} \sum_{i,j=1}^{N} (1 - A_{ij}\sigma_i\sigma_j)$$

Assumptions

All-to-all topology

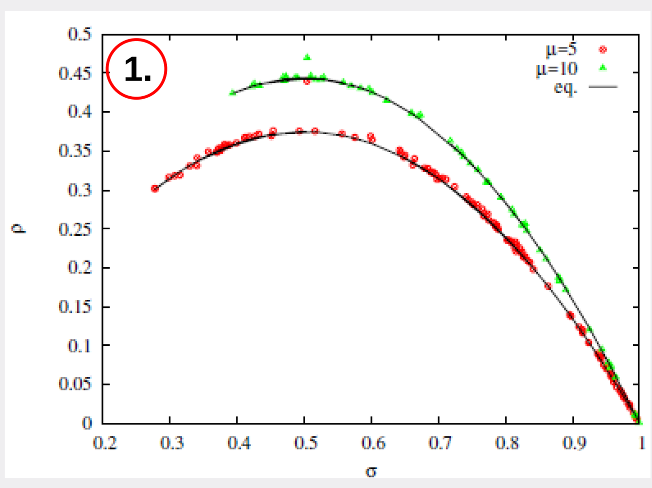
$\mathrm{P}(s_j = +1 | s_i = -1) = \mathrm{P}(s_j = +1) = \sigma \Rightarrow \rho = 2\sigma(1-\sigma)$

ER network

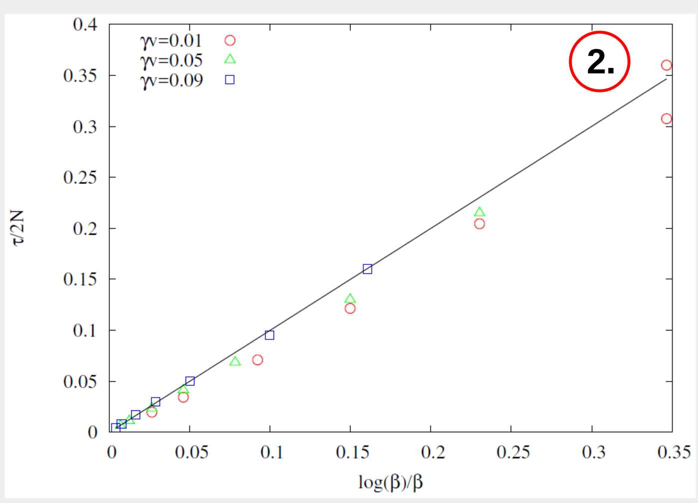
1. Pair approx. $\mathrm{P}(s_j = \mp 1 | s_i = \pm 1)$ indep. of the other links of i^{th} voter ⇒ $\rho^{i,j} \approx \rho$
2. Uncorrelated network
$P(s_j = \mp 1 | s_i = \pm 1, k_j = k) = P(s_j = \mp 1 | s_i = \pm 1) = \rho/2$
3. Adiabatic approx. $\frac{\partial \rho(\sigma(t), t)}{\partial t} = 0$

$$\rho(\sigma) = A_1(\mu,\gamma,v)\sigma(1-\sigma)^{\frac{\mu-1}{\mu}\frac{\gamma v+1}{\gamma v}} \cdot {}_2F_1\left(A_2(\mu,\gamma,v), A_3(\mu,\gamma,v); A_4(\mu,\gamma,v); \sigma\right)$$

All-to-all and ER random network topologies

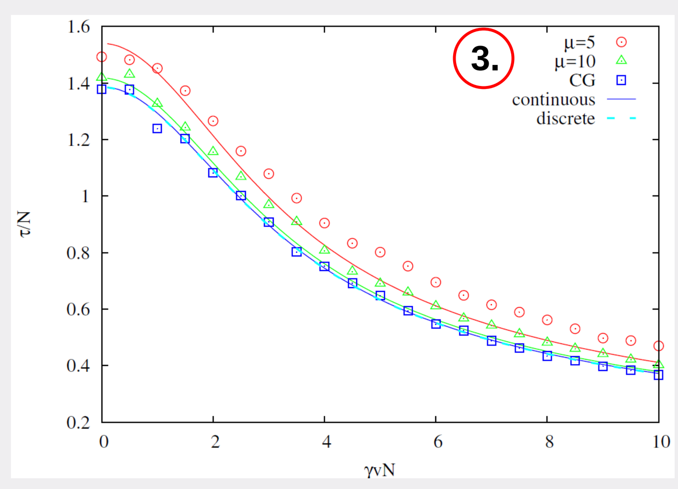


ER network: for $\rho(\sigma)$ computer simulations (symbols) agree with analytical calculation (solid lines).

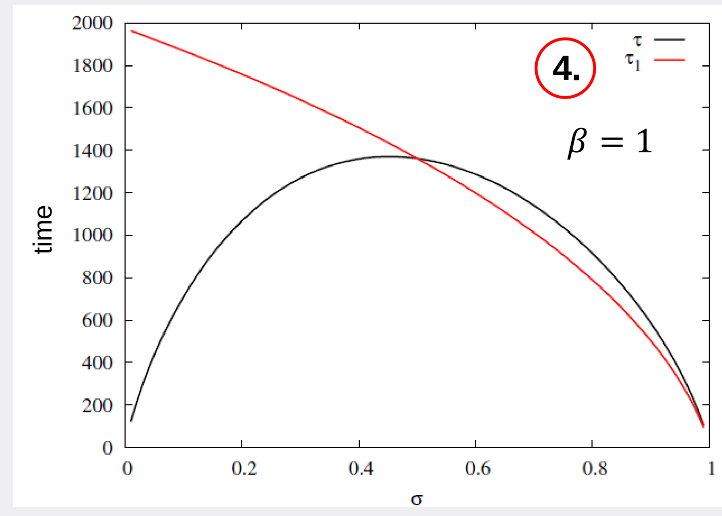


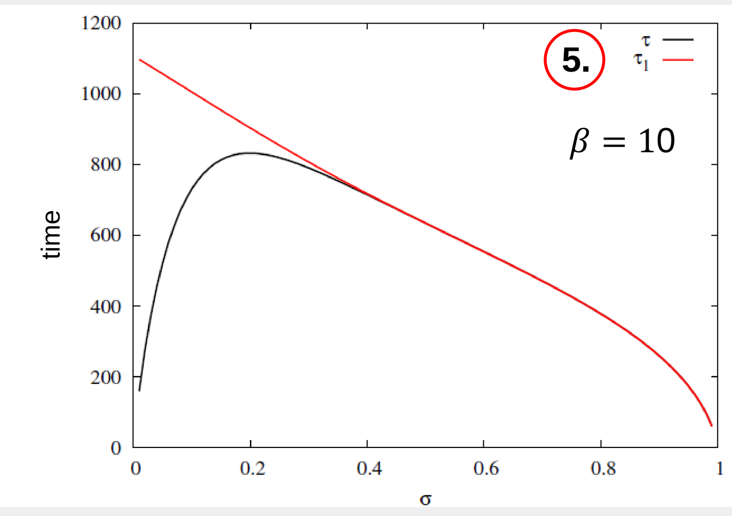
Absorbing state τ VS effective bias $\beta = 2\gamma v N$ at the asymptotic limit for an all-to-all network:

$$\lim_{\beta \to \infty} \tau(\sigma_0 = 1/2; \beta) \to \frac{\ln(N)}{\gamma v}$$



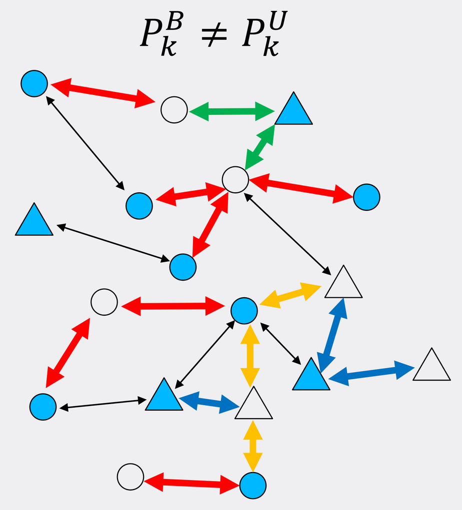
- *Absorbing state τ VS effective bias $\gamma v N$.*
- *$\tau \uparrow$ as average degree $\mu \downarrow$*
- *τ ER > τ all-to-all*





- *Time to reach consensus τ (black line) and time to reach consensus in preferred state, τ_1 (red line)*
- *τ and τ_1 coincide for $\sigma = \frac{1}{2}$ in the all-to-all case.*
- *For large β and $\sigma(t=0)$, τ and τ_1 coincide as the prob. to reach consensus at preferred state is almost 1.*

Biased-dependent topology



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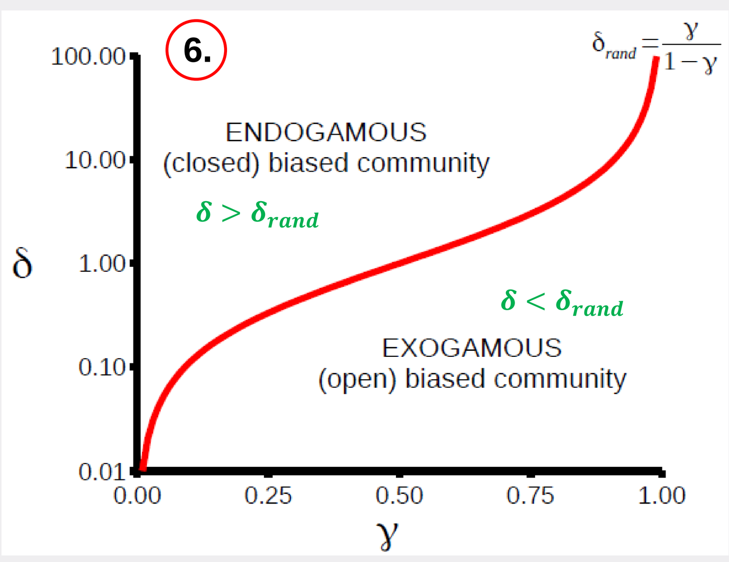
Biased -1 to Biased 1
Biased -1 to Unbiased 1
Unbiased -1 to Biased 1
Unbiased -1 to Unbiased 1

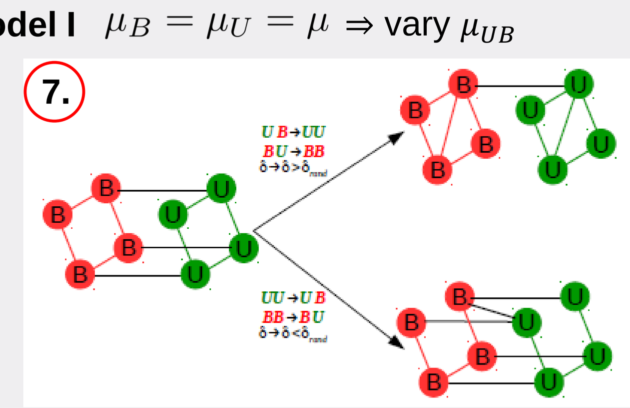


$L = L_{BB} + L_{UU} + L_{UB} = \frac{1}{2}\mu N$

$\mu = \gamma\mu_B + (1-\gamma)\mu_U$	μ_B	Biased network av. deg.
$\mu_B = \mu_{BB} + \mu_{BU}$	μ_U	Unbiased network av. deg.
$\mu_U = \mu_{UB} + \mu_{UU}$	μ_{BB}	Biased ONLY av. deg.
$\mu_{BU} N_B = \mu_{UB} N_U$	μ_{UU}	Unbiased ONLY av. deg.
$\boxed{\delta = \frac{\mu_{BB}}{\mu_{UU}}}$	μ_{UB}	Unbiased-Biased av. deg.

Fix γ, μ

Model I $\mu_B = \mu_U = \mu \Rightarrow$ vary μ_{UB}

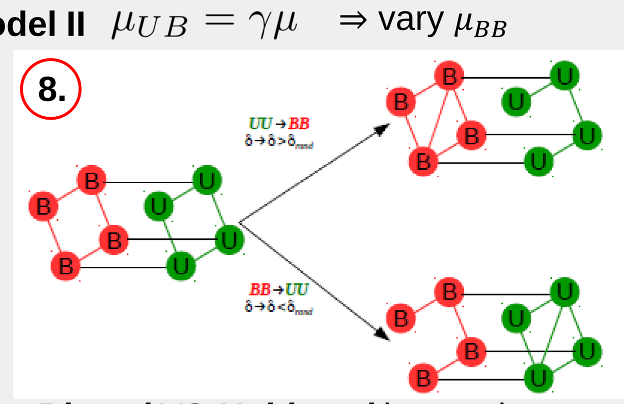


$\mu_{UB} \downarrow$ $\mu_{BB} \uparrow$ $\mu_{UU} \uparrow$

$\mu_{UB} \uparrow$ $\mu_{BB} \downarrow$ $\mu_{UU} \downarrow$

Intergroup VS ***Intra***group interactions

Model II $\mu_{UB} = \gamma\mu \Rightarrow$ vary μ_{BB}



$\mu_{BB} \uparrow$ $\mu_{UU} \downarrow$

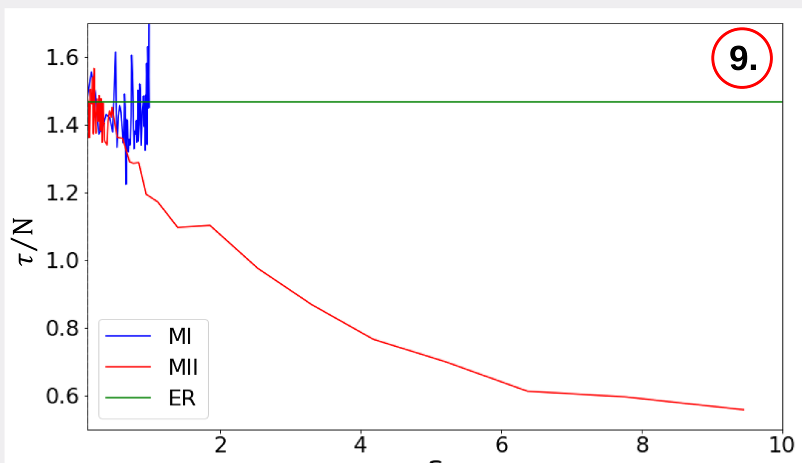
$\mu_{BB} \downarrow$ $\mu_{UU} \uparrow$

Biased VS ***Unbiased*** interactions

$$\frac{d\left(\rho_{X^s Y^{-s}}\right)/2}{dt} = \sum_{\{W,\hat{s}\},k,k_{W\overline{W}}} p_W \sigma_{k,W}^{\hat{s}} P^W_{(k,k_{W\overline{W}})} \left.\frac{\Delta\rho_{\{W,\hat{s}\},\{\widehat{W},-\hat{s}\}}}{\Delta t}\right|_{(k,k_{W\overline{W}})}$$

$W, \widehat{W} \in \{X,Y\}$ s.t. W the i^{th} voter's state and $\widehat{W}$ the state of his neighbor & $\overline{W} \neq W$

$P^W_{(k,k_{W\overline{W}})} = P_k^W P(k_{W\overline{W}} | k)$



- *Time to reach consensus, τ for the biased-independent ER topology (green line, analytical result) and the biased-dependent topologies using model I (blue line) and model II (red line).*
- *Model II: $\tau \downarrow$ as $\delta \uparrow$*

Conclusions

- Devised a model of biased voters as heterogeneity to a group of unbiased voters with confidence.
- Obtained analytical results for special cases of various observables for the biased-independent topologies scenarios.

$$P_1(\sigma_0 \approx 1/2) = \frac{1}{1 + e^{-\tilde{\beta}/2}} \qquad \tilde{\beta} = \begin{cases} \beta = 2\gamma v N & \text{all-to-all} \\ \beta_\mu = 2\gamma v \tilde{N} & \text{ER network } \tilde{N} = \frac{\mu}{\mu+1} \end{cases} \qquad \lim \tau(\sigma_0 \approx 1/2; \beta) \propto \begin{cases} N & \tilde{\beta} \to 0 \\ \log N/\gamma v & \tilde{\beta} \to \infty \end{cases}$$

Consensus time scales logarithmically with effective bias $\gamma v N$.

- Considered biased dependent topologies. 2 models to study exogamous to endogamous transition for biased communities.

Main result: The better connected the biased community is, the faster consensus is reached.

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