Complex networks III

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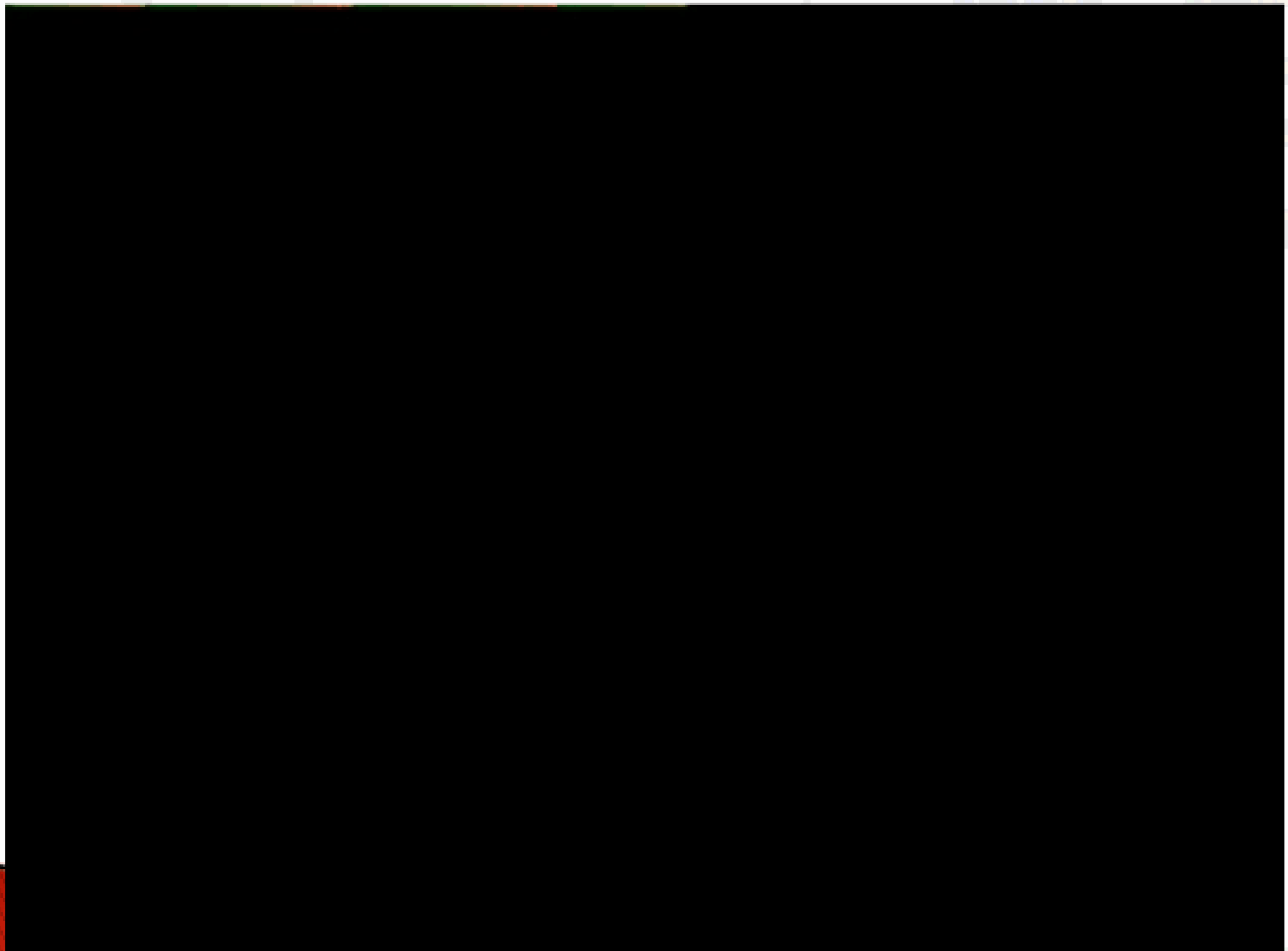
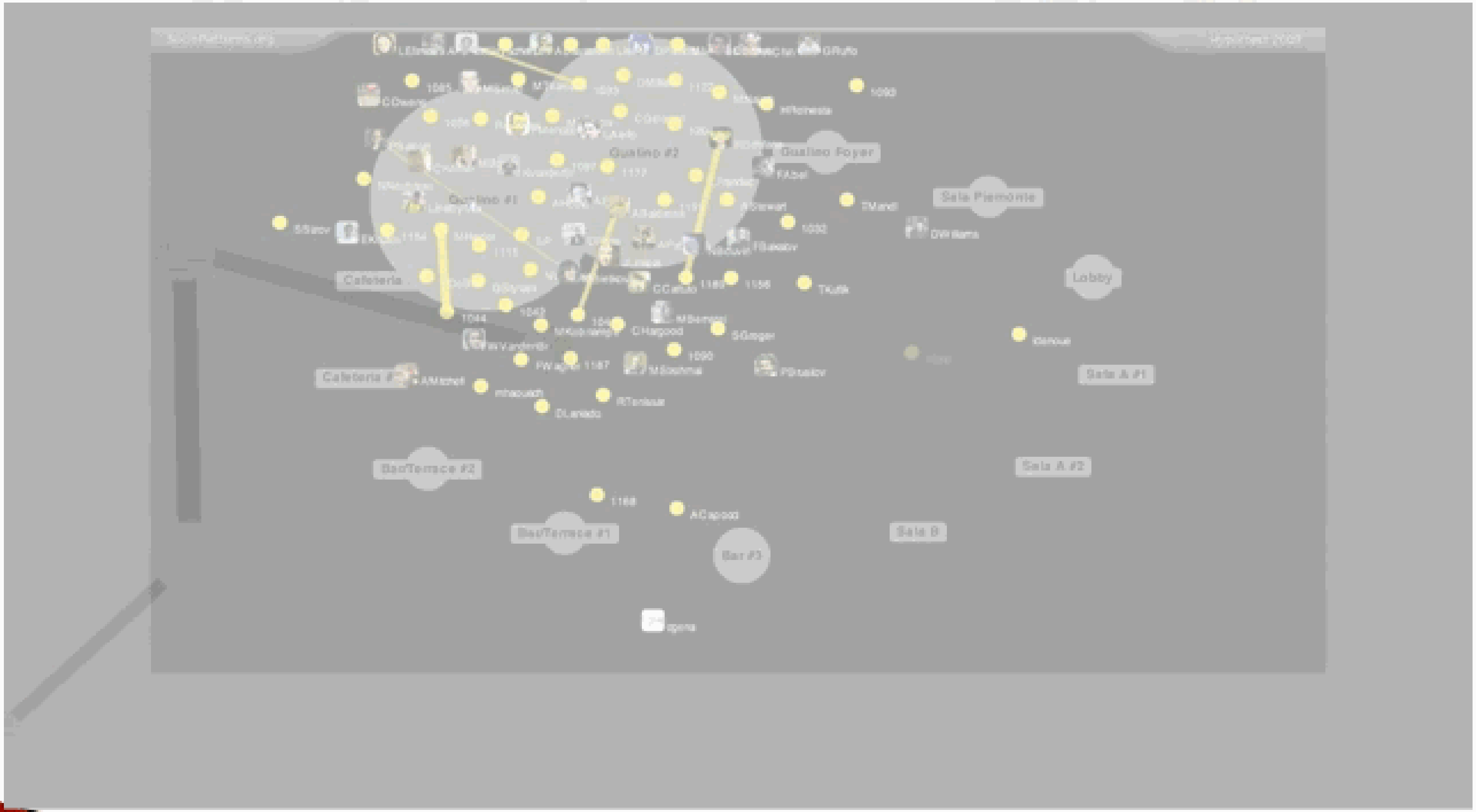
DYNAMICS ON/OF COMPLEX NETWORKS

Something’s going on

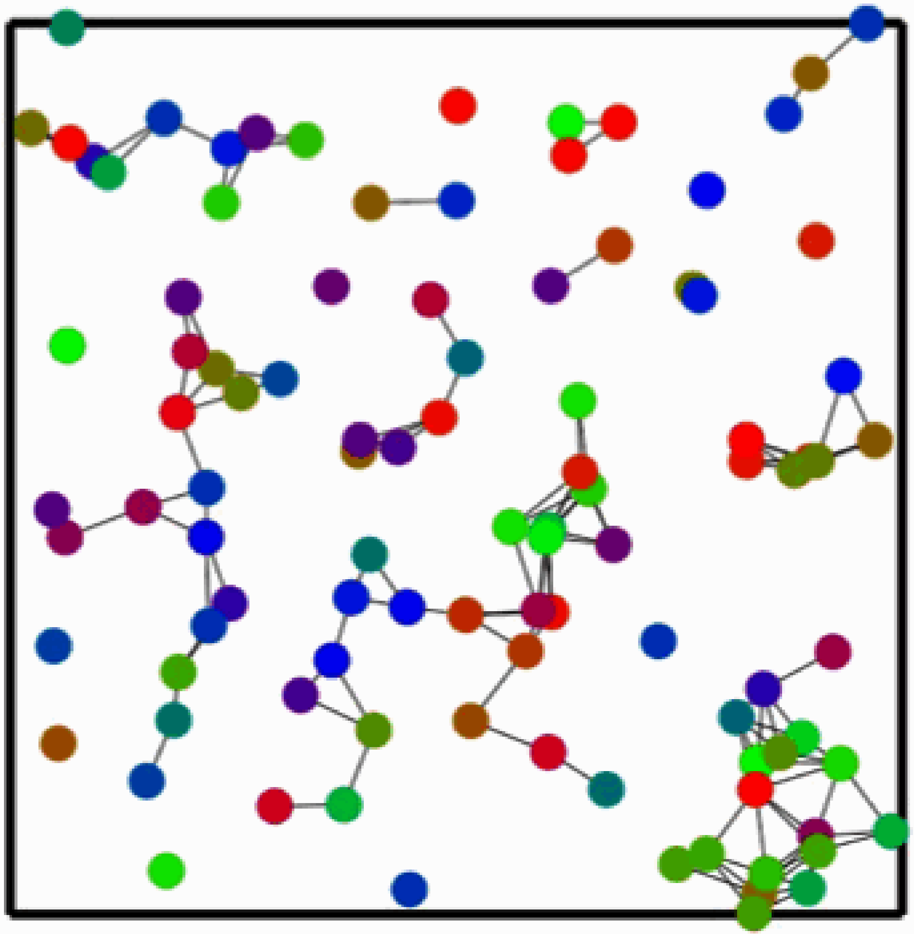
Dynamics on networks



Dynamics of networks



Dynamics on dynamic networks



Outline

- General issues
- Synchronization

- Network (link) dynamics:
  - global goal
  - local goal
- Flow in complex networks:
  - ideas
  - innovations
  - computer viruses
  - problems
  - people

Global vs local optimization

- Design: the goal is to optimize global quantity (distance, clustering, density, ...)
- Evolution: decision taken at node level

Evolution

- Bornholdt & Rohlf: Global criticality from local dynamics
  - Network of interconnected binary elements
  - The dynamics reaches an attractor
  - Change the connectivity of a node according to its behavior during the attractor
- Gleiser & Zanette: rearrangements of links
  - Synchronization
  - Change of connectivity pattern to adjust to neighbors
  - Final state: community structure

Optimization

- Global goal:
  - Distance: related to minimal cost in transportation
  - Number of connections: costly connections
  - A combination of parameters
- Initial configuration: random graph
- Change connections
- Accept if there is an improvement
- Stars vs trees

Synchronization
A. Arenas, A.D.-G., C.J. Pérez-Vicente

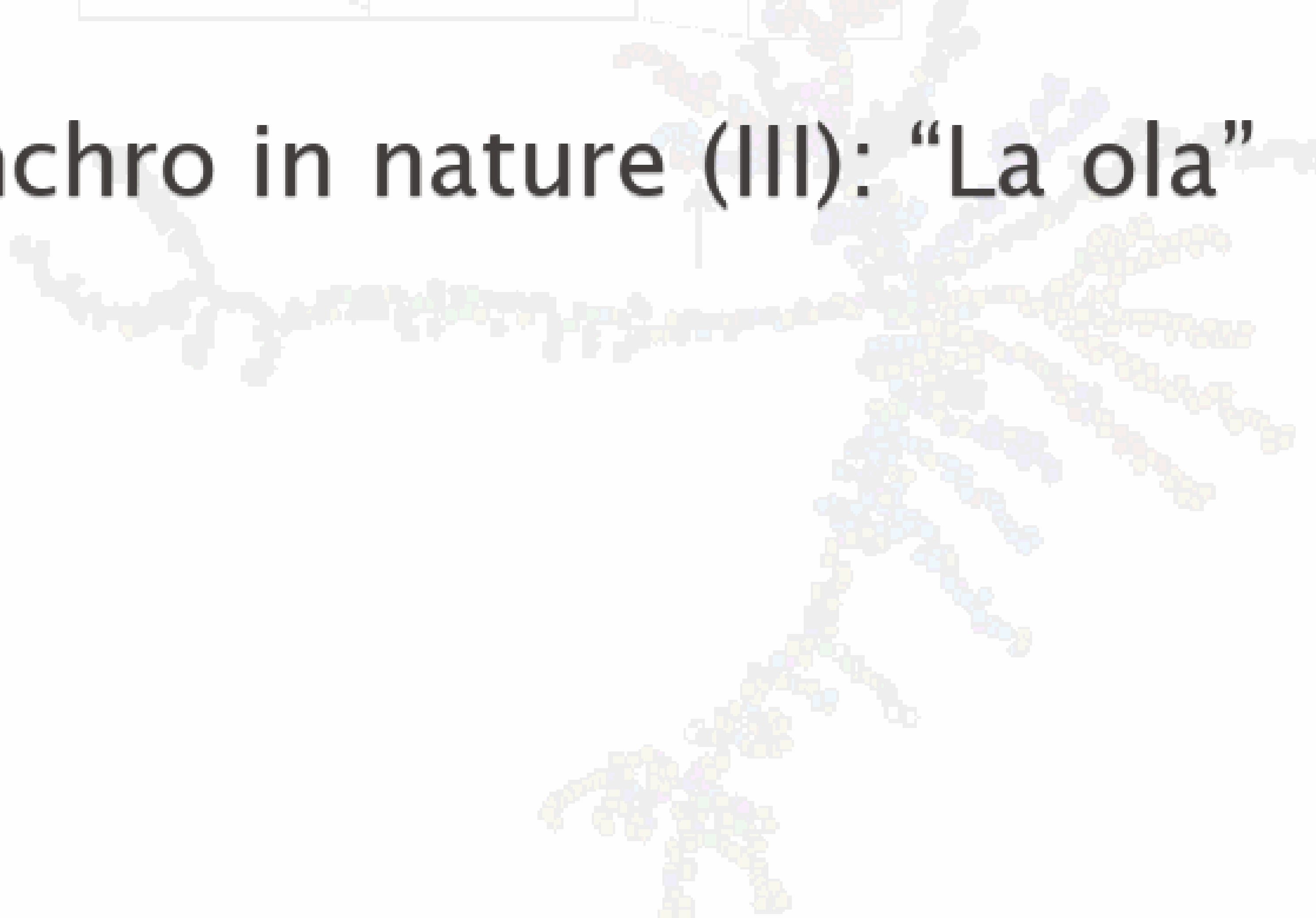
- Community identification: maximization of modularity -> very best partition
- Communities can be hierarchically organized
- Synchronization: dynamics at all scales from the innermost to the global scale

Synchro in nature (I): hands clapping

- Sound file (play)

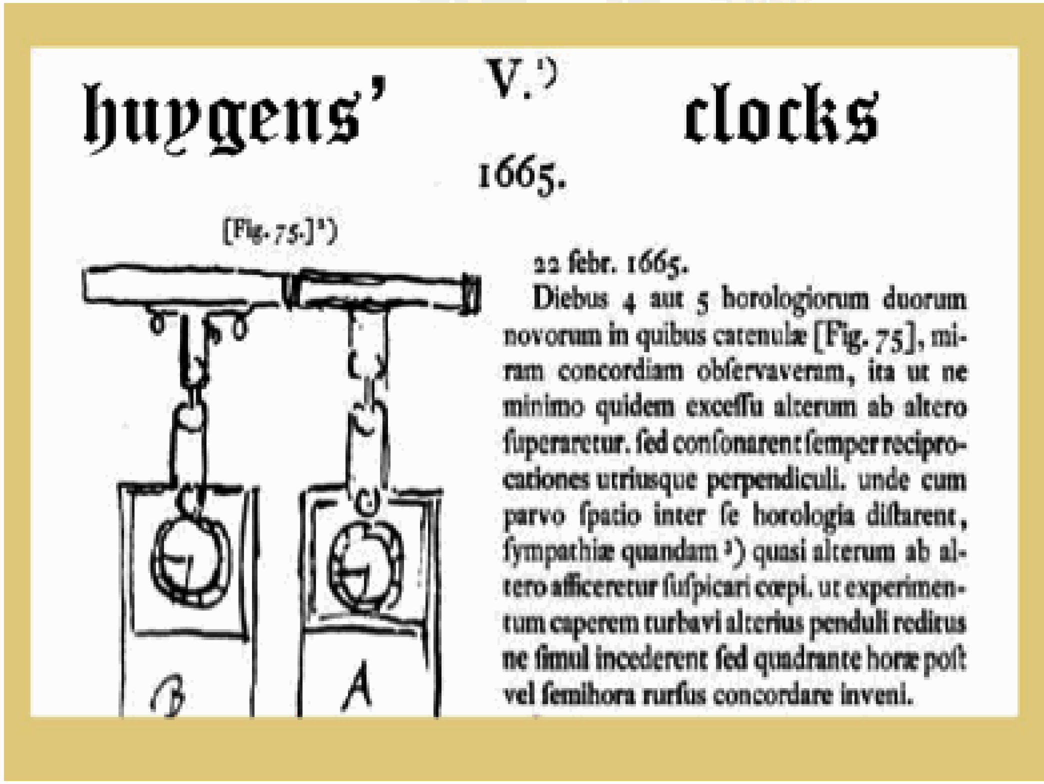
Synchro in nature (II): flashing fireflies

Synchro in nature (III): “La ola”

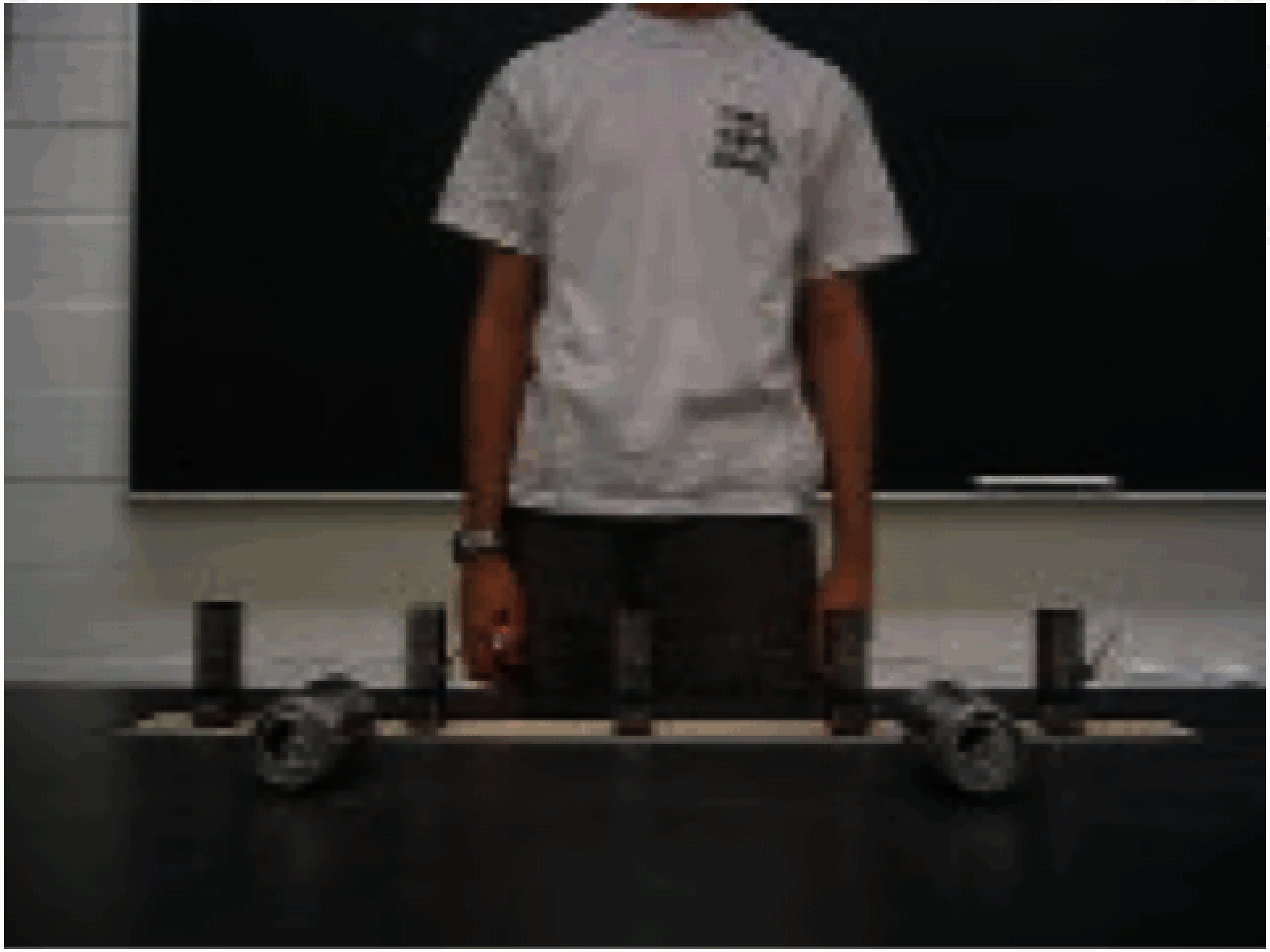


Synchro in nature (IV)

- Hearts beats
- Cricket chirps
- Laser
- Superconductivity
- Menstrual synchrony in women living together
- ……..



Synchro in the lab



Synchronization dynamics

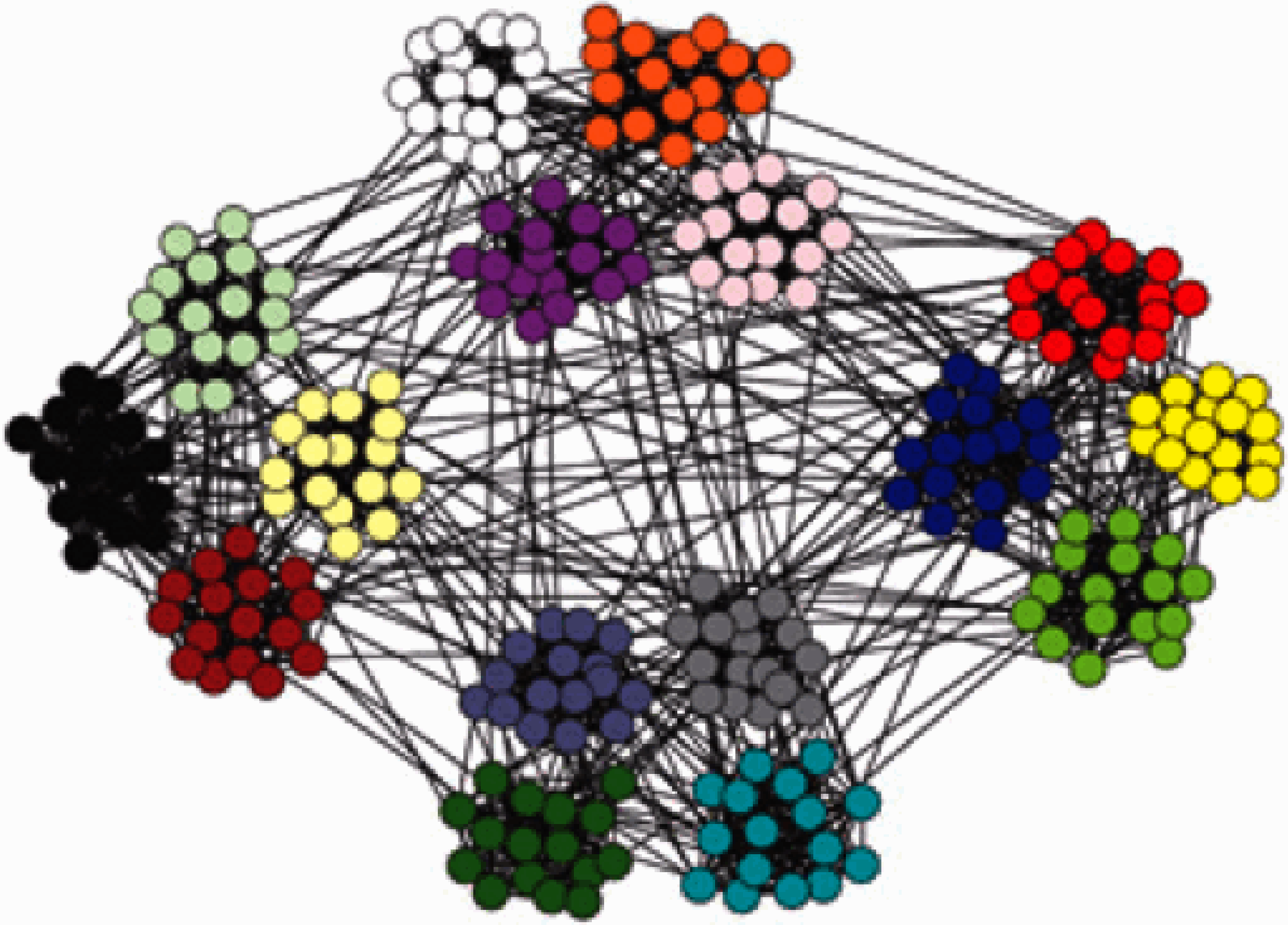
- Synchronization of Kuramoto oscillators

$$\frac{d\theta_i}{dt} = \omega_i + \sigma \sum_j A_{ij} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N$$

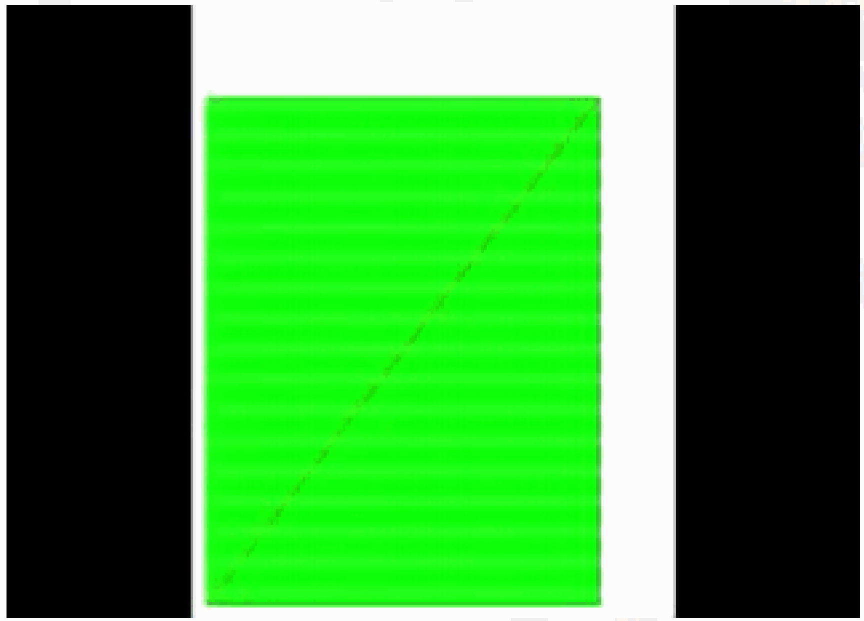
Kuramoto (the Applet)

$$\frac{d\theta_i}{dt} = \sigma \sum_j A_{ij} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N$$

Hierarchical structure of communities



Time evolution of the correlation matrix



$$\rho_{ij}(t) = < cos(\theta_i(t) - \theta_j(t)) >$$

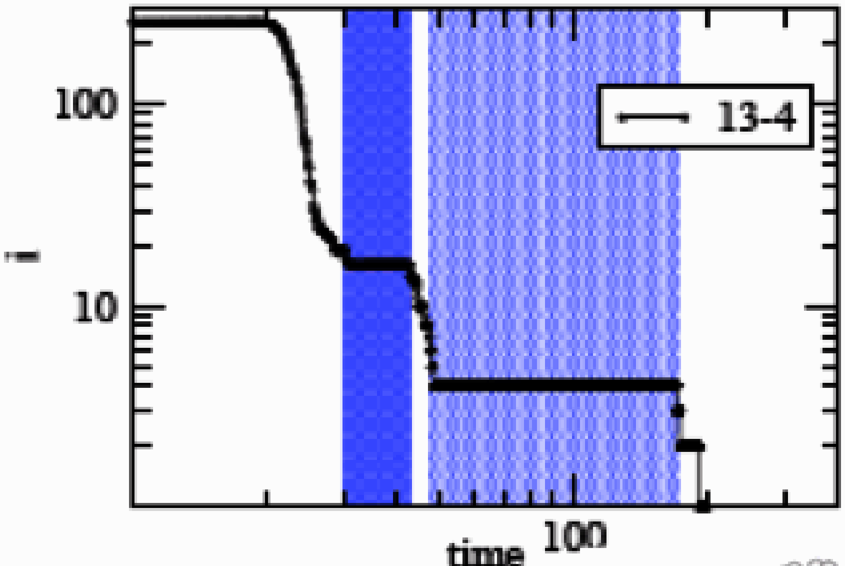
New graphical representations

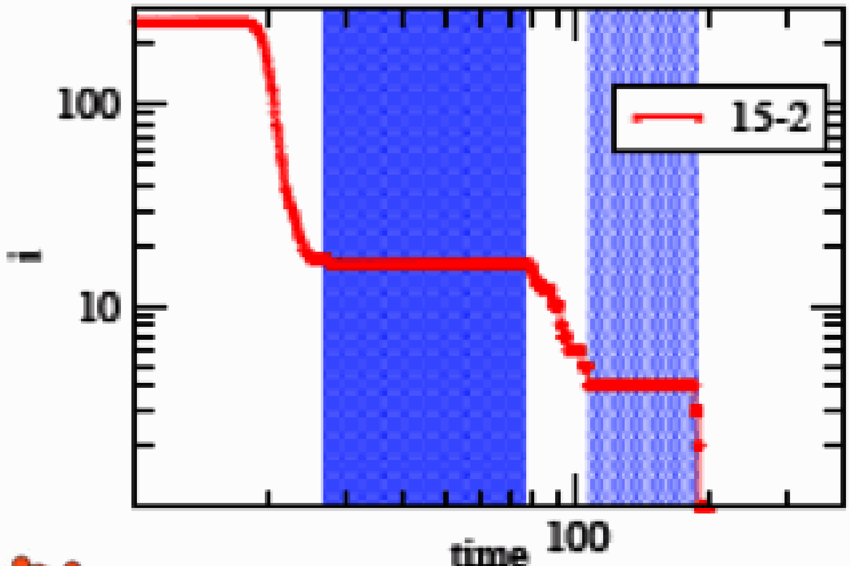
- Two nodes are connected if they are “synchronized”
- Dynamic connectivity matrix

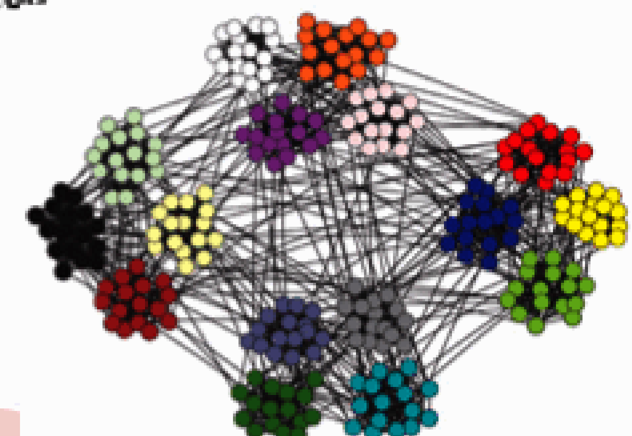
$$\mathcal{D}_t(T)_{ij} = \begin{cases} 1 & \text{if } \rho_{ij}(t) > T \\ 0 & \text{if } \rho_{ij}(t) < T \end{cases}$$

- Fixed time – moving threshold
- Fixed threshold – time evolution of the network
- Structure at different time scales

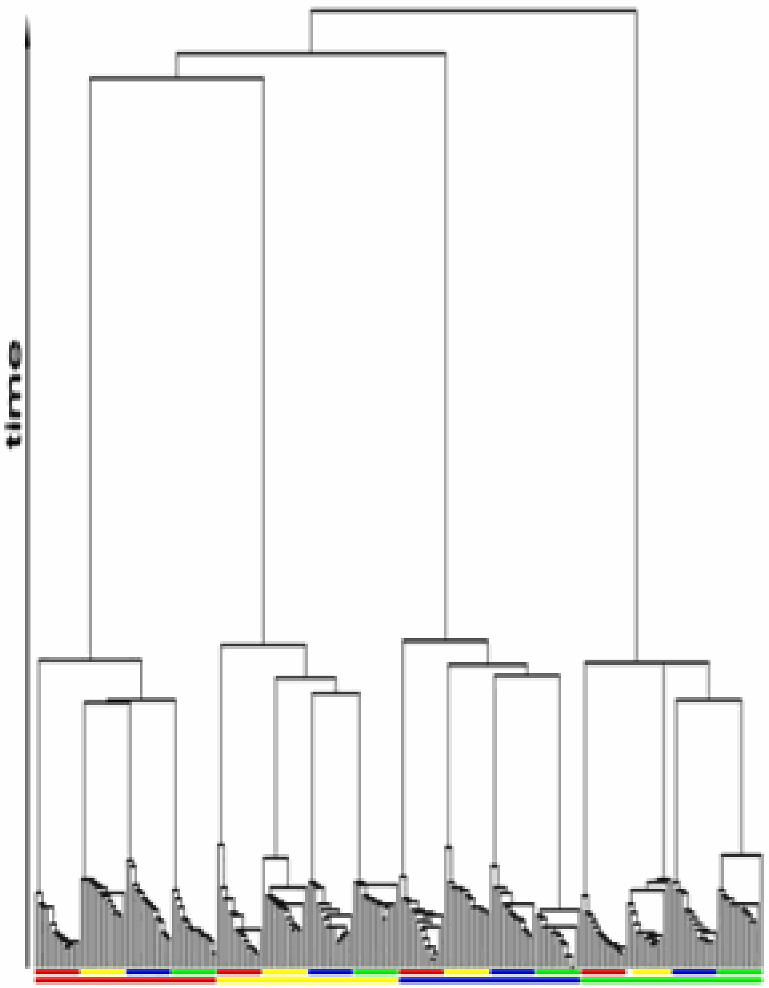
Number of connected components

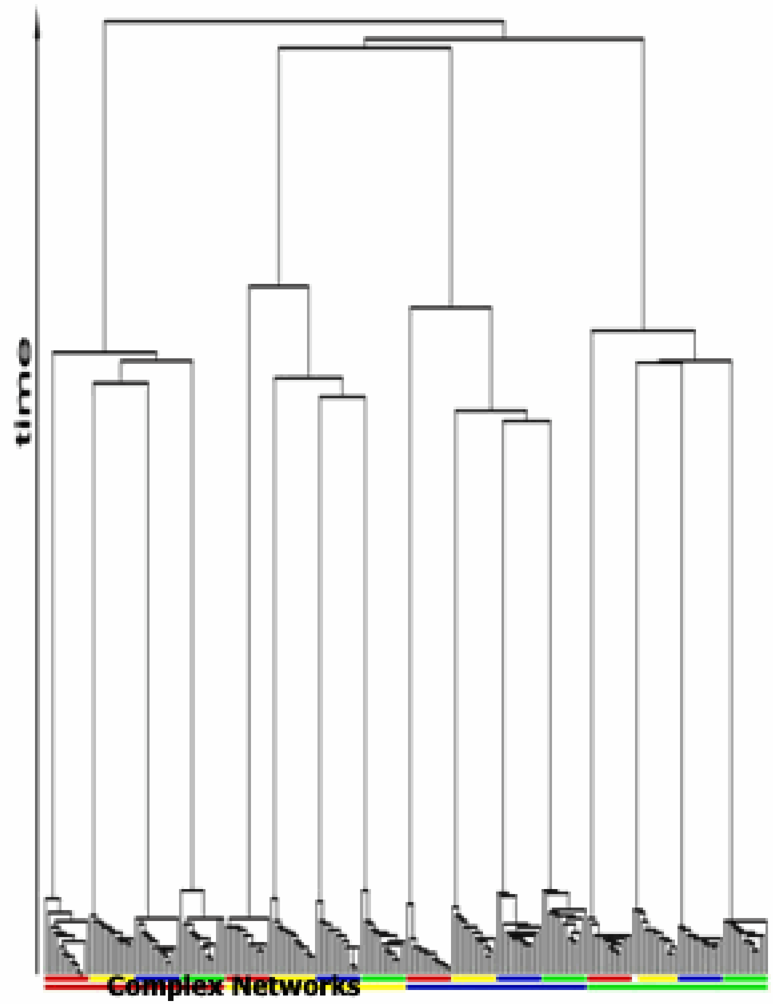




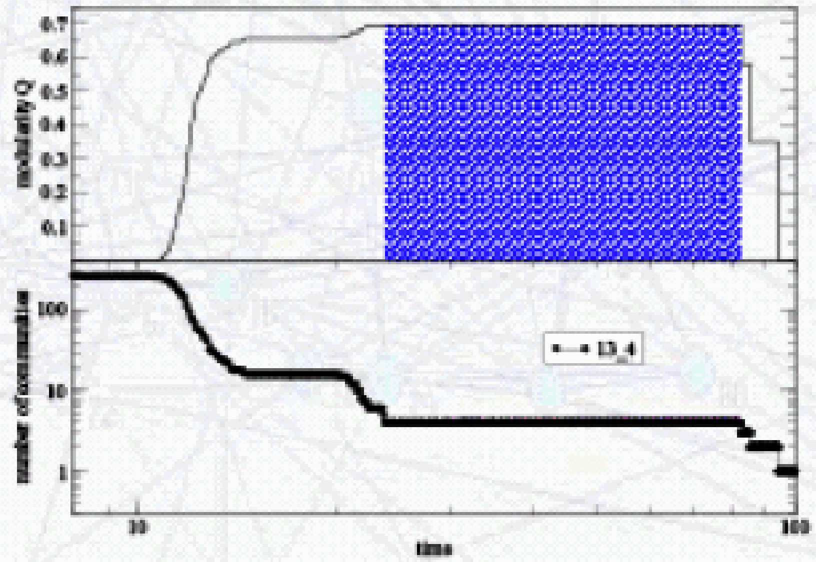


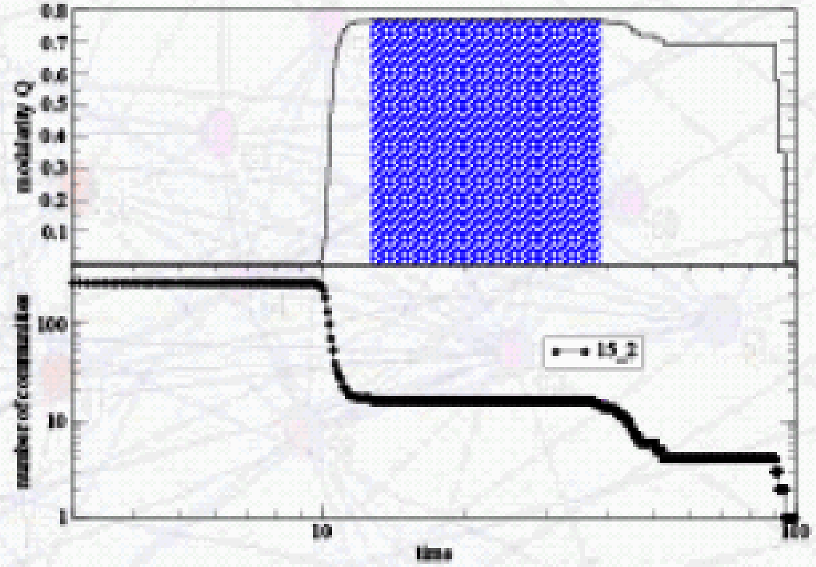
Dendogram of connections





Number of communities and modularity



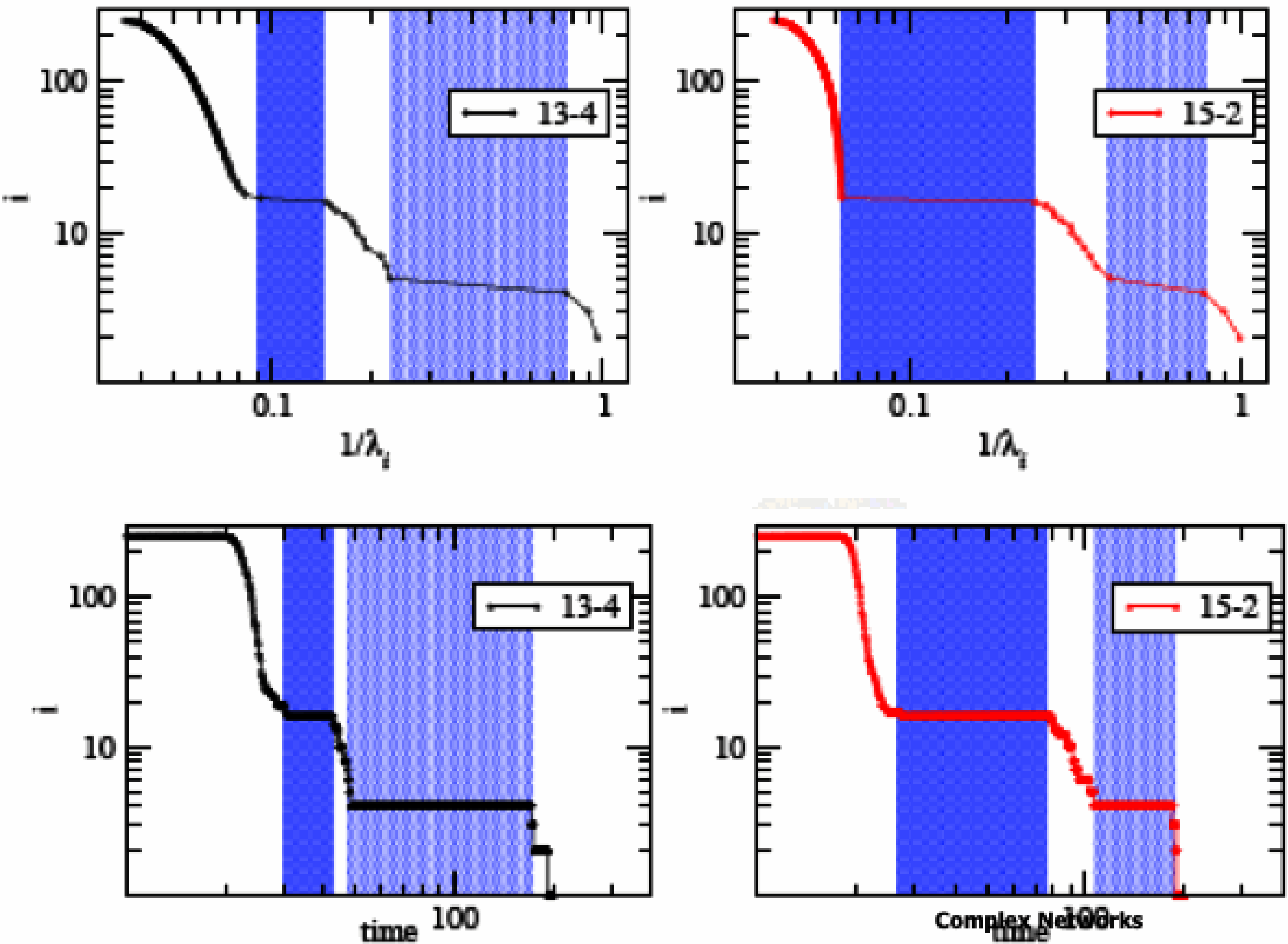


Spectral properties

- Spectrum of the Laplacian matrix
- We order the eigenvalues

  $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

Spectral versus dynamics



Other (small) networks

- See the URL:
  http://www.ffn.ub.es/albert/synchro.html

Spectral properties

- Laplacian matrix
- Adjacency (connectivity) matrix

Laplacian matrix

- Diffusion equation. Laplacian operator

$$\frac{dn(x,t)}{dt} = \nabla^2 n(x,t)$$

- Discrete Laplacian operator (1d)

$$\frac{dn_i(t)}{dt} = (n_{i+1}(t) - n_i(t)) - (n_i(t) - n_{i-1}(t))$$

$$\frac{dn_i(t)}{dt} = n_{i+1}(t) + n_{i-1}(t) - 2 \cdot n_i(t)$$

- Discrete Laplacian operator (2d)

$$\frac{dn_{i,j}(t)}{dt} = n_{i+1,j}(t)+n_{i-1,j}(t)+n_{i,j+1}(t)+n_{i,j-1}(t)-4\cdot n_{i,j}(t)$$

- In general

$$\frac{dn_i(t)}{dt} = \sum_i^N a_{i,j} n_j(t) - k_i \cdot n_i(t) = -L_{ij} n_j(t)$$

- Where we have introduced the Laplacian matrix

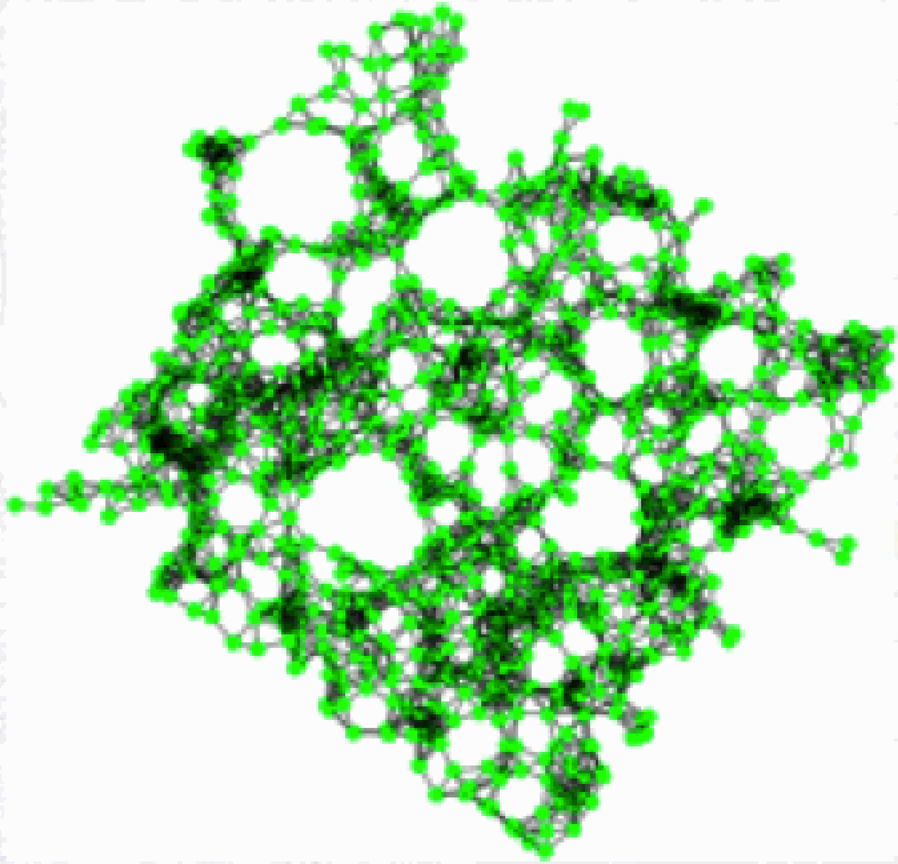
$$L_{ij} = k_i \delta_{ij} - a_{ij}$$

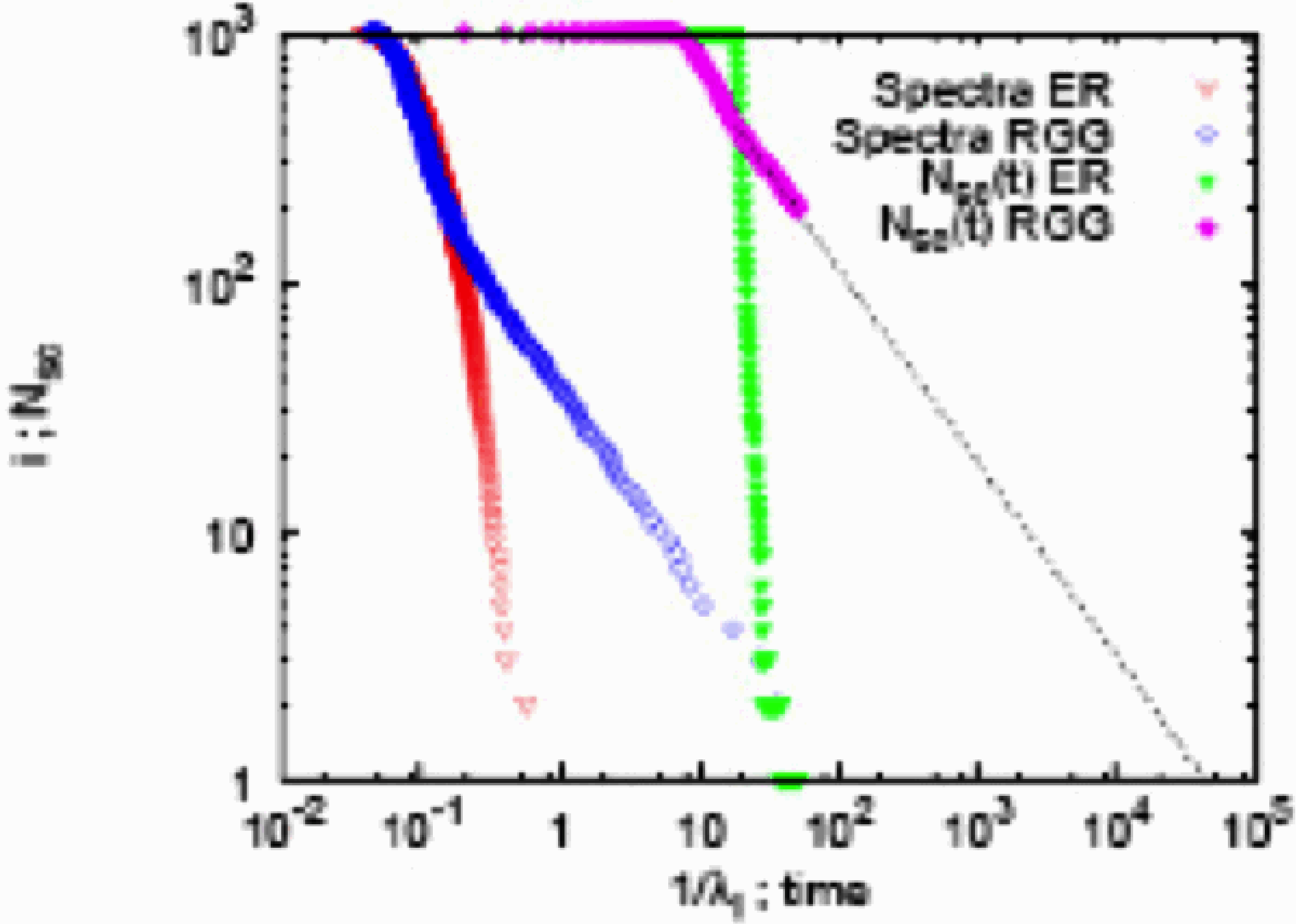
Properties of Laplacian matrix

- Important on dynamical properties
- Discrete spectrum: eigenvalues and eigenvectors
- Ordered $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$
- Number of 0 eigenvalues is equal to the number of (dis)connected components
- λ_2 is related to the time the system needs to be synchronized. Intuitively, when it is zero there are at least two disconnected components and the system will never synchronize

Networks without community structure

Random Geographic Graph (RGG)



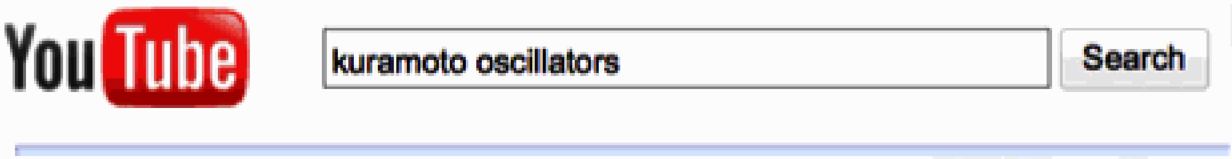


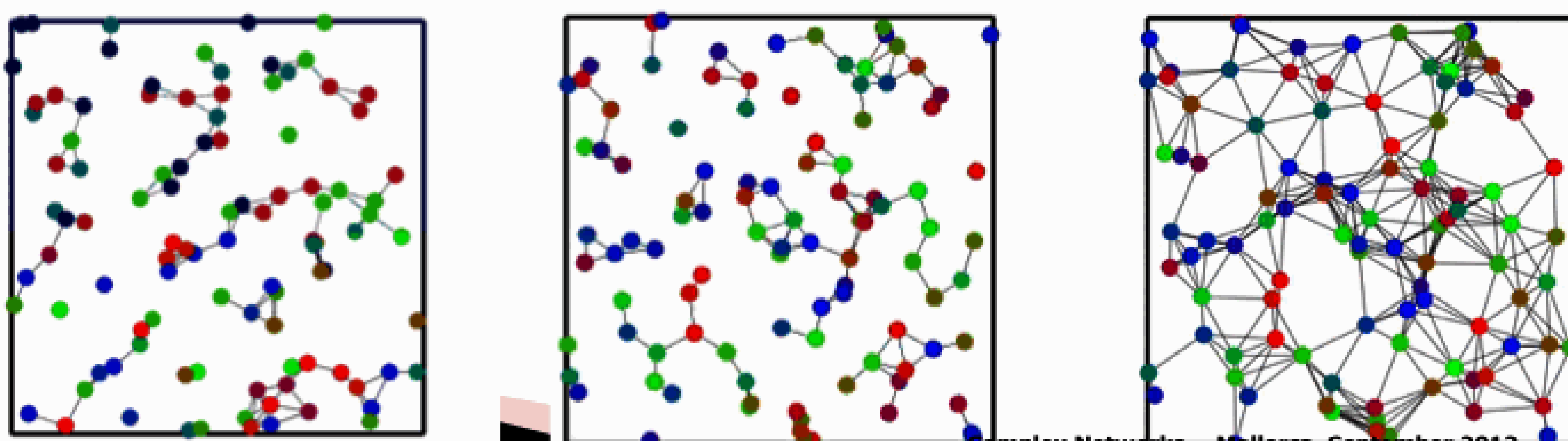
Spectra ER
Spectra RGG
$N_{ss}(t)$ ER
$N_{ss}(t)$ RGG
i ; N_{ss}
$1/\lambda_i$; time
10^3
10^2
10
1
10^{-2}
10^{-1}
1
10
10^2
10^3
10^4
10^5

Mobility creates complex time-dependent networks

- Agents that move create contact networks on its way
- Interact with nearest neighbors:
  - range of interaction
  - pre-define a number of interacting

Movies





Synchronization in complex nets

- **Synchronization in complex network:**
  Interplay between topology and dynamics

  A. Arenas, A.D.-G., J. Kurths, Y. Moreno, C. Zhou, Phys. Rep. 469, 93 (2008)
- **Spectral analysis:**
- **In most of the previous studies, network topology is fixed**

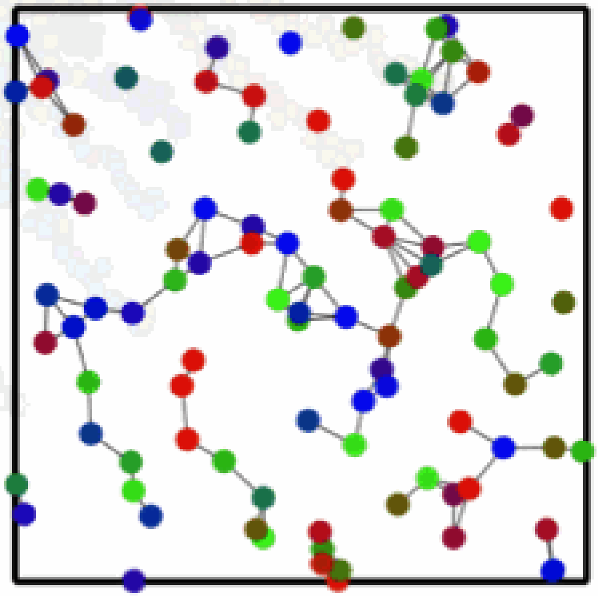
What happens if topology changes in time?
Is spectral approach possible?

Kuramoto Model

N. Fujiwara, J. Kurths, A.D-G, PRE (2011)

$$\varphi_i(t+\tau_P) = \varphi_i(t) + \sum_{j=1}^{N} \sigma(d_{ij}) \sin(\varphi_j(t) - \varphi_i(t))$$

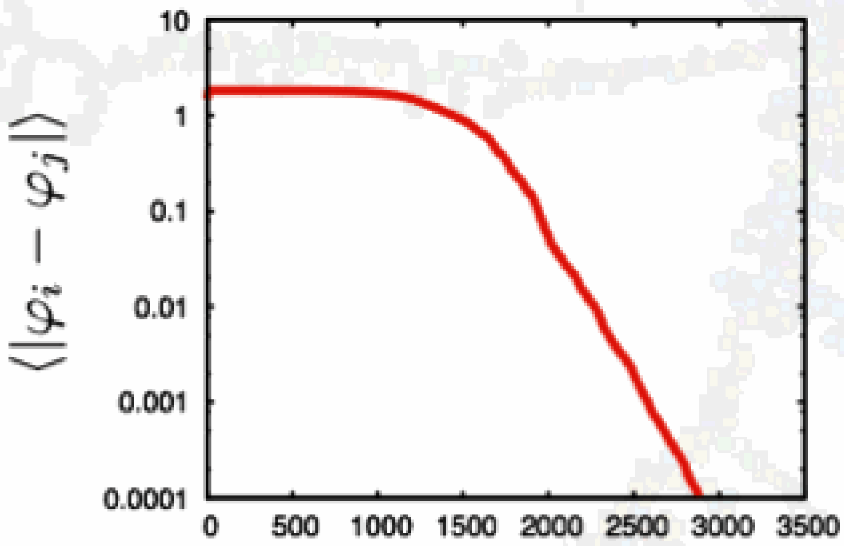
$$\sigma(d_{ij}) = \begin{cases} \sigma & (d_{ij} < d) \\ 0 & (d_{ij} > d) \end{cases}$$



Applet

- Java applet simulation
- http://complex.ffn.ub.es/~albert/mobile/Kuramoto.html

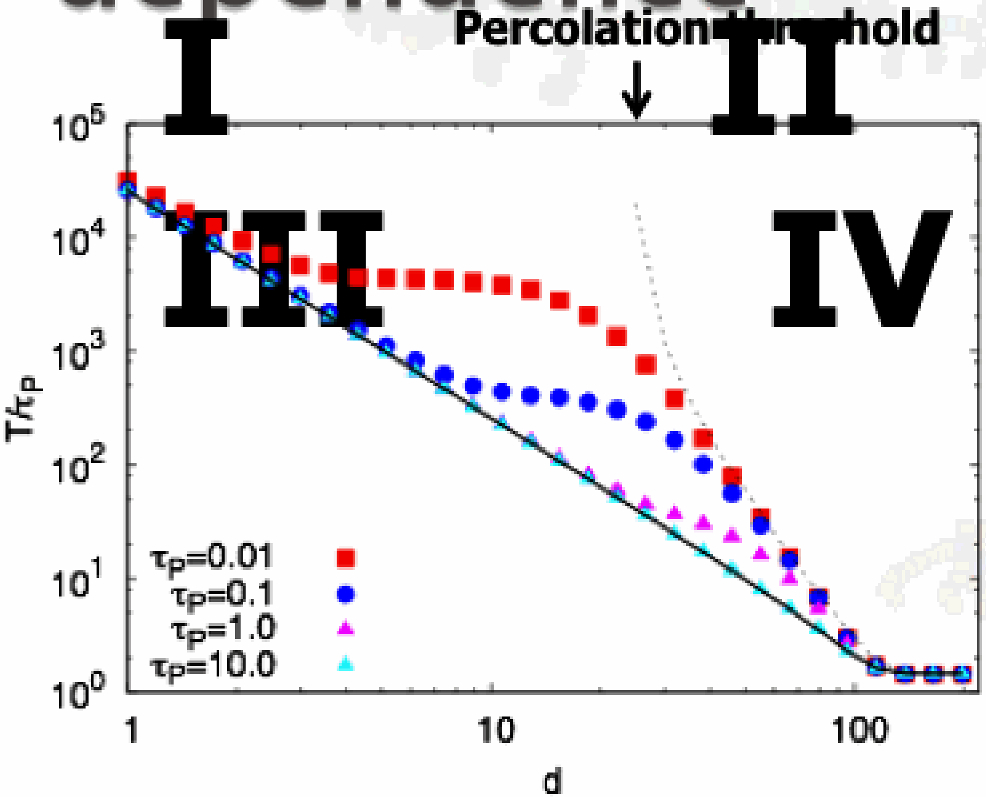
Time evolution



time

T/τ_P

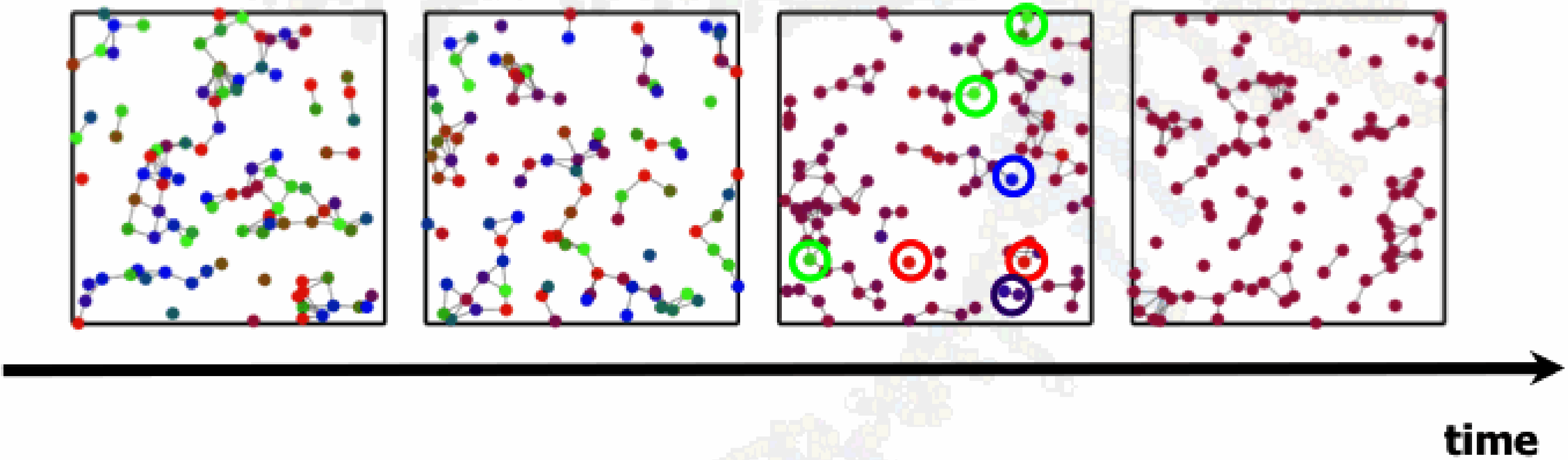
d (interaction range) dependence



$N = 100,\ L = 200,\ v = 10,\ \tau_M = 1.0,\ \sigma = 0.005$

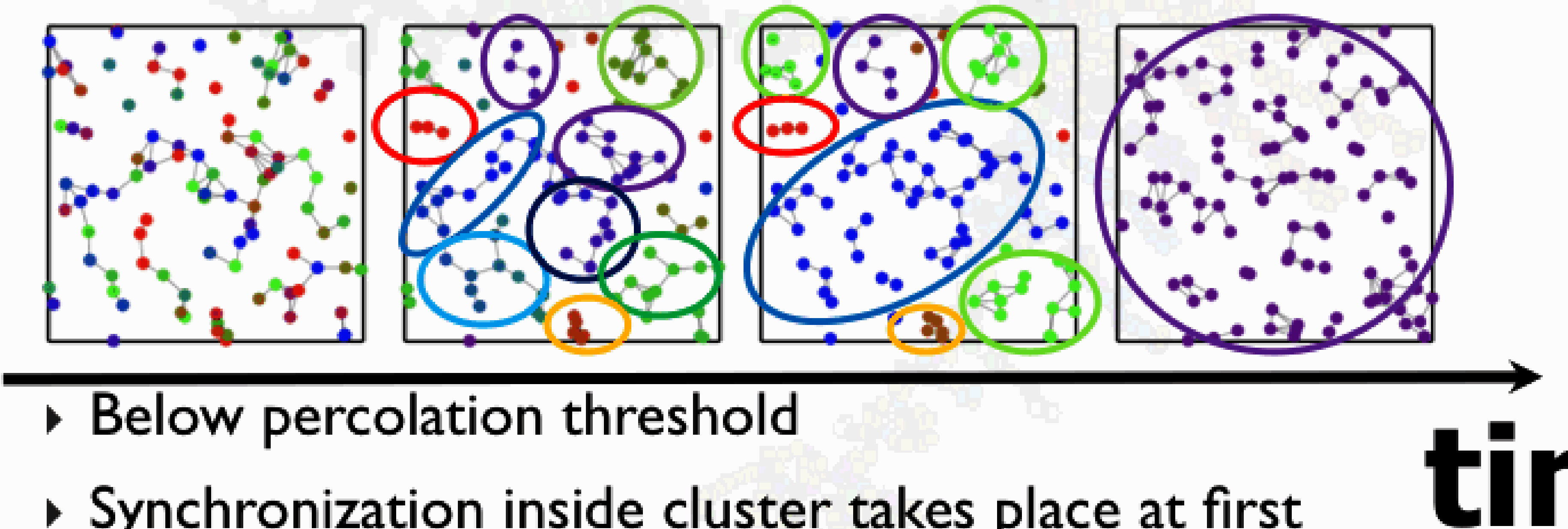
- I: fast switching
- II: multi cluster
  - local synchronization
  - slow topology change
- III: single cluster
  - local synchronization
- IV: complete graph

Region I:
global synchronization



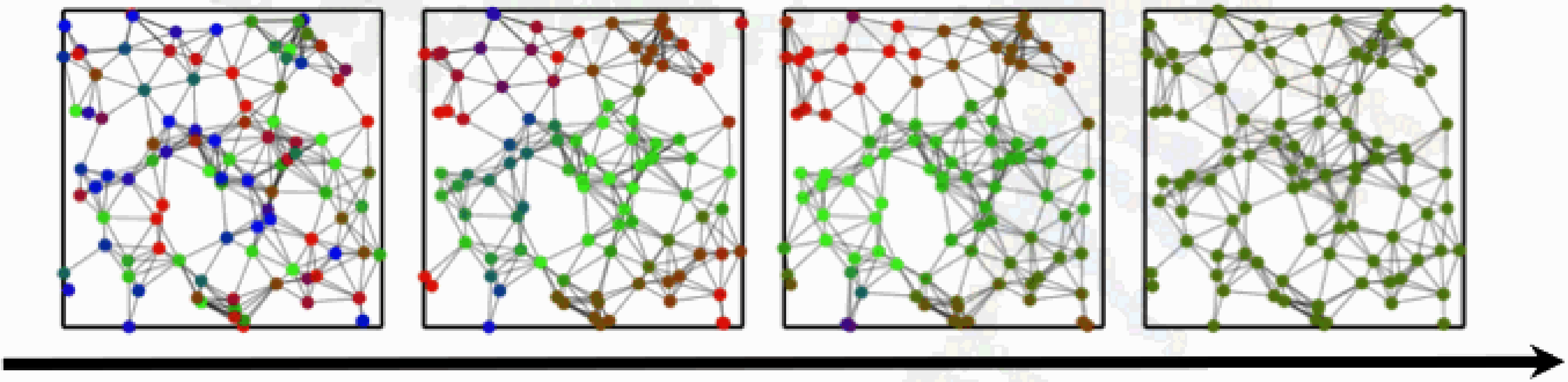
- Global synchronization
- Non-synchronized oscillators are isolated
- Fast switching approximation describes the synchronization dynamics
- Effect of agent dynamics disappear

Region II:
local synchronization multiple cluster



- Below percolation threshold
- Synchronization inside cluster takes place at first
- Global synchronization is achieved due to the motion of agents
- Characteristic time is dominated by cluster interaction (agent motion)
- Slower than fast switching approximation

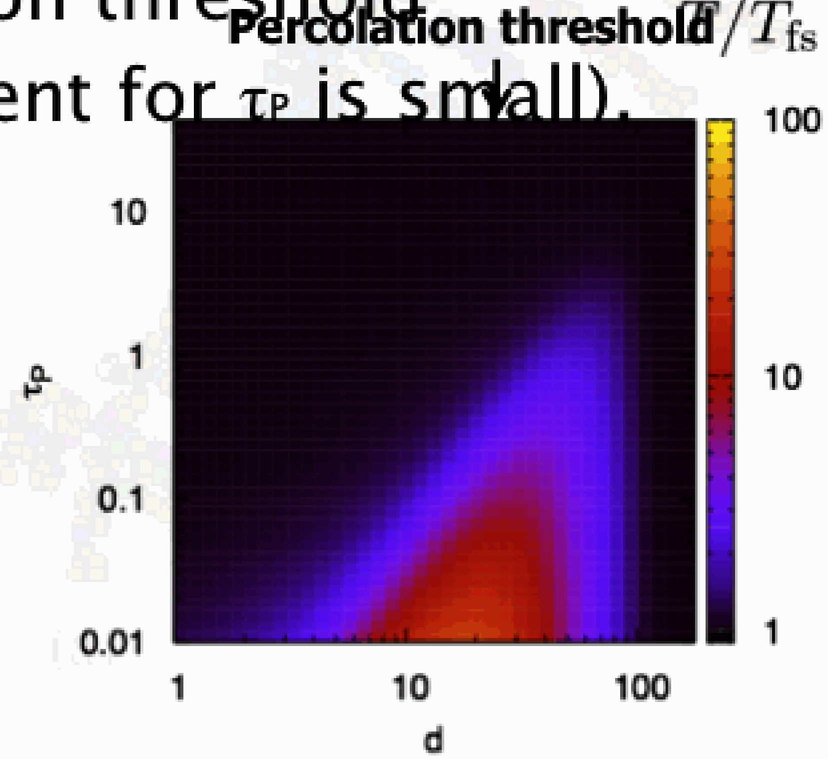
Region III: single cluster small jump



- Above percolation threshold, whole network is connected
- Local synchronization takes place at first
- Second smallest eigenvalues of instantaneous Laplacian
- matrices λ_2 (t)dominate the synchronization dynamics
- $T \sim 1/(\sigma<\lambda_2>)$
- Slower than fast switching approximation

Deviation from fast switching approximation in d-taup plane

- T deviates from T_{fs} for d value near percolation threshold
- Smaller τ_P (jump of agent for τ_P is small), larger T/T_{fs}



Dynamic transition: local to global synchronization

- Number of steps for a cluster to internally synchronize

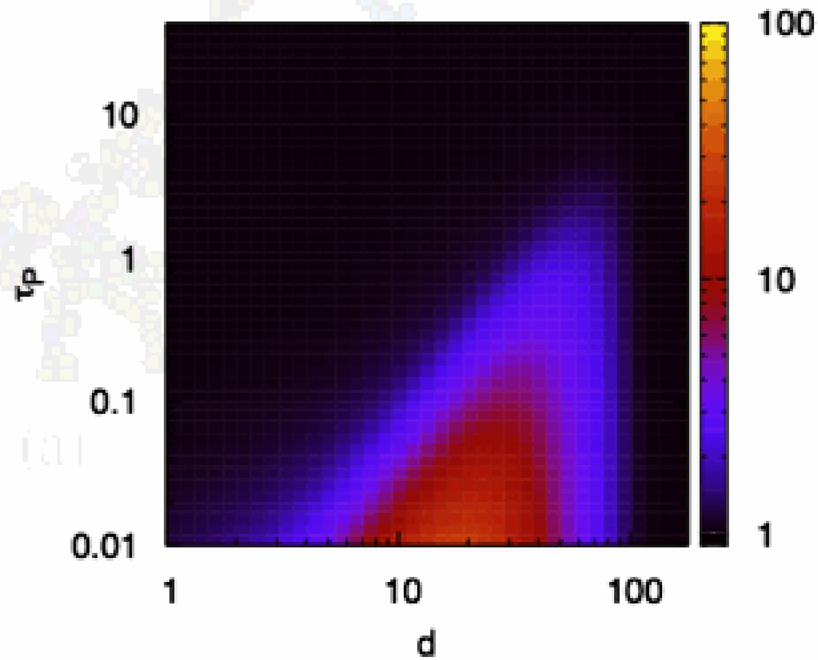
$$n_s = \frac{1}{\sigma \lambda \xi(d)},$$

- **Number of steps for an agent to leave a cluster**

$$n_m = \frac{\xi^2(d)}{v^2 \tau_M \tau_P}.$$

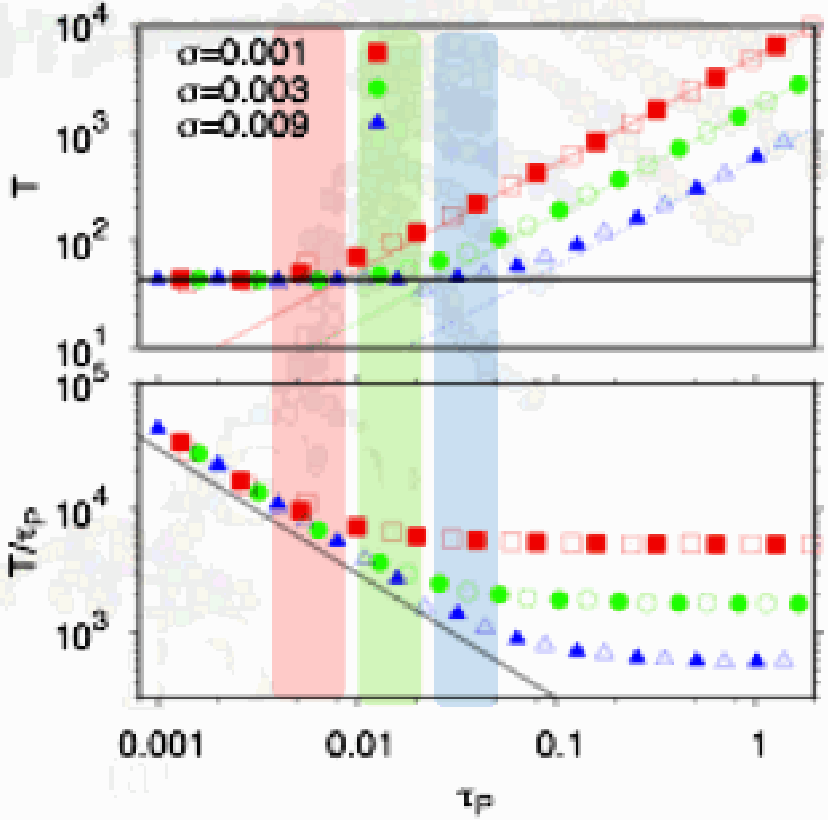
Transition

$$\eta = \frac{n_m}{n_s} = \frac{\sigma f(d)}{v^2 \tau_M \tau_P}.$$



Dependence of characteristic time on signal interval (below percolation)

- For large τ_P, T is large (slow synchronization achievement)
- For small τ_P, more number of signals are required to synchronization (low efficiency)



which both fast synchronization and

Matrix product for linearized equation

- When the phase difference is small, the linearized equation describes the synchronization dynamics

$$\varphi_i(t+\tau_P) = \varphi_i(t) - \sigma \sum_{j=1}^{N} L_{ij}(t)\varphi_j(t),$$

In our case **Laplacian matrix depends on time**

- consider the transformation of the normal modes (eigenmode of L)

$$\varphi_j(t) = \sum_{k=1}^{N} U_{jk}(t)\theta_k(t), \quad \sum_{k=1}^{N} L_{jk}(t)\theta_k(t) = \lambda_j(t)\theta_j(t)$$

- we get the time evolution of the normal modes as

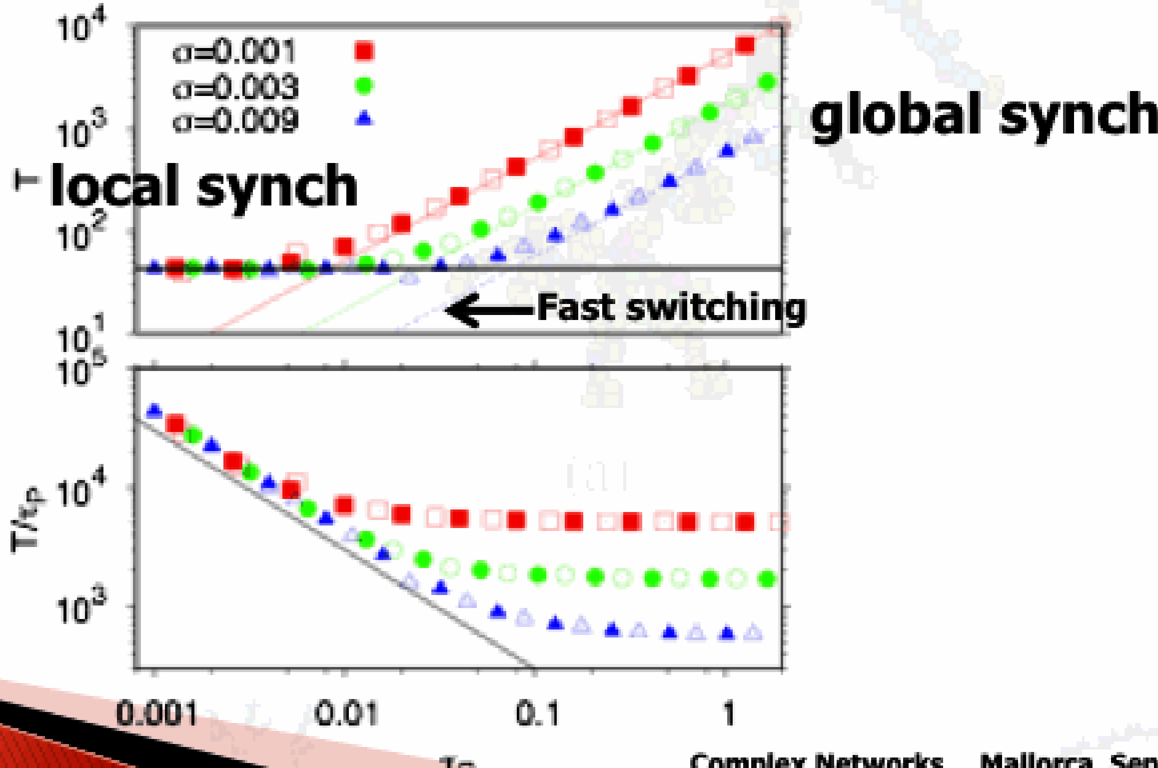
$$\theta_l(t+\tau_P) = \sum_{i,k} U^T_{li}(t+\tau_P)U_{ik}(t)\left[1-\sigma\lambda_k(t)\right]\theta_k(t)$$

$$\equiv \sum_k O_{lk}(t)[1-\sigma\lambda_k(t)]\theta_k(t)$$

agent mobility **oscillator dynamics**

Matrix product for linearized equation

- Finally, we get $\theta_{k_n}(t+n\tau_P) = \prod_{q=0}^{n-1}\left[\sum_{k_q=1}^{N} O_{k_{q+1}k_q}(1-\sigma\lambda_{k_q})\right]\theta_{k_0}(t)$
- Compare empirical T with second smallest eigenvalue of the product of matrices (independent way), and they coincide for any value of the parameters even when fast switching approximation does not work



Derivation of fast switching approximation

$$\theta_{k_n}(t+n\tau_P) = \prod_{q=0}^{n-1}\left[\sum_{k_q=1}^{N} O_{k_{q+1}k_q}(1-\sigma\lambda_{k_q})\right]\theta_{k_0}(t)$$

$$\prod_{q=1}^{n}(1-\sigma\lambda_{l_q}) \approx e^{n\langle\log(1-\sigma\lambda)\rangle}$$

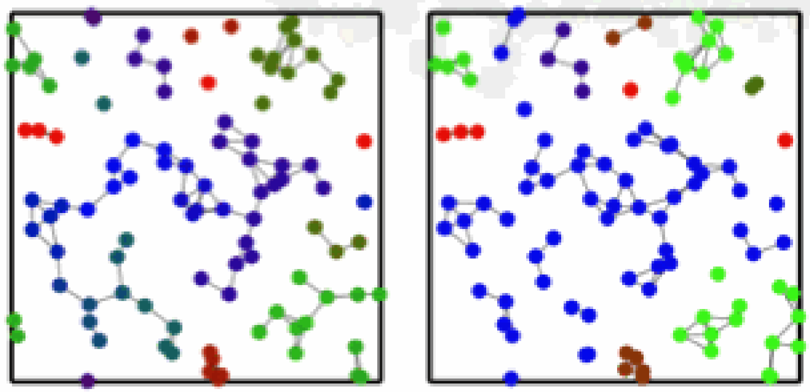
$$T = -\tau_P/\langle\log(1-\sigma\lambda)\rangle$$

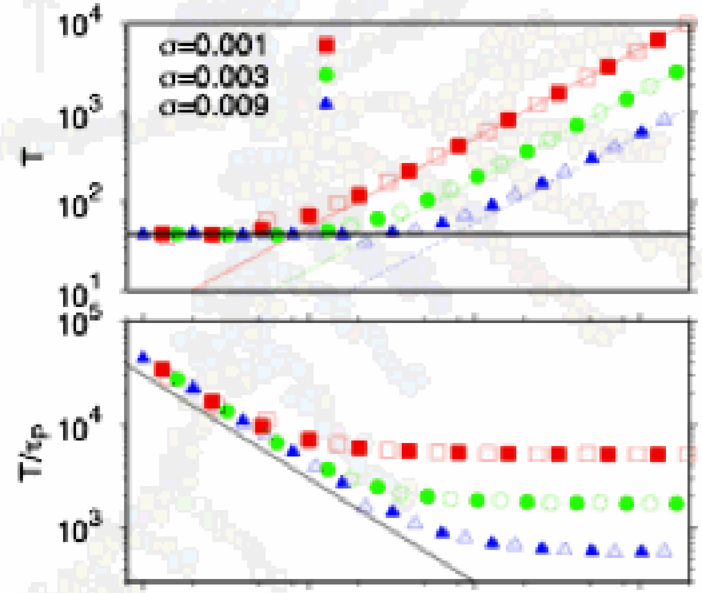
$$\frac{\tau_P}{T} = \sigma\langle\lambda\rangle$$

$$\frac{\tau_P}{T} = (N-1)\sigma\rho$$

$$\rho = \begin{cases} \pi d^2/L^2 & d < \frac{L}{2} \\ L\sqrt{4d^2-L^2} + d^2[\pi - 4\cos^{-1}(\frac{L}{2d})] & \frac{L}{2} < d < \frac{L}{\sqrt{2}} \\ 1 & d > \frac{L}{\sqrt{2}} \end{cases}$$

Multiple cluster local synchronization

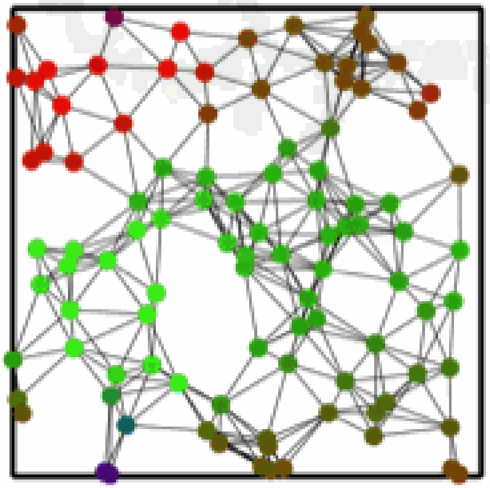


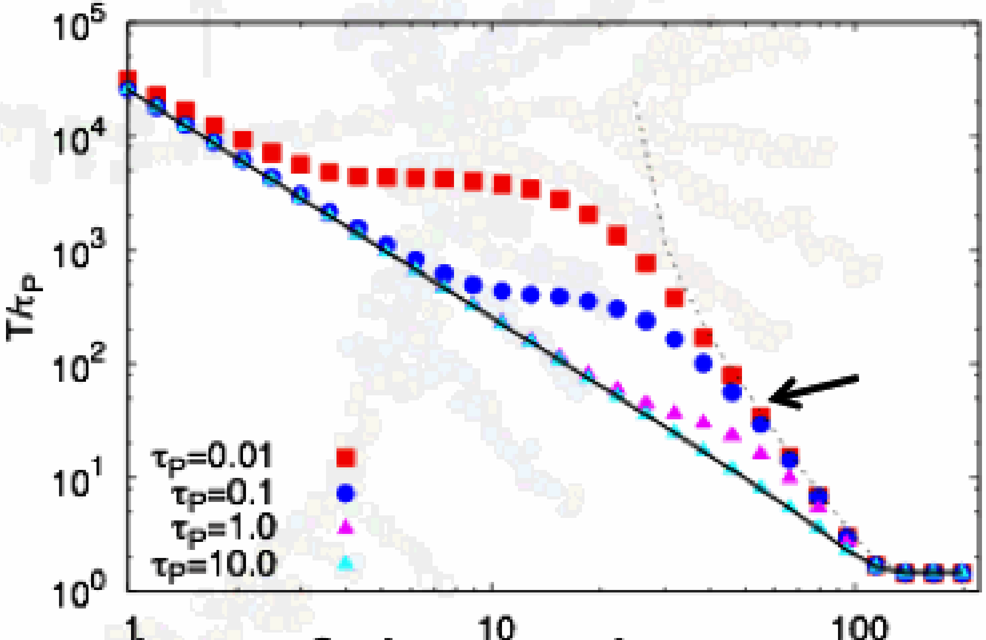


- Non-zero eigenmodes decay before the topology changes. Zero eigenmodes dominates global dynamics
- Matrix product approximation suggests that efficiency scales as

$$\frac{T}{\tau_P} \propto (v^2 \tau_M \tau_P)^{-1}$$

Local synchronization single cluster





- Second smallest eigenvalue of the Laplacian matrix dominates the instantaneous decay rate
- When topology change is slow, dynamics is governed by the average of second smallest eigenvalue (algebraic connectivity)