EUROPHYSICS LETTERS 15 January 2003

Europhys. Lett., **61** (2), pp. 162–167 (2003)

System size coherence resonance in coupled FitzHugh-Nagumo models

R. Toral[1,2], C. R. Mirasso[1] and J. D. Gunton[2,3]

[1] *Departament de Física, Universitat de les Illes Balears E-07071 Palma de Mallorca, Spain*

[2] *Instituto Mediterráneo de Estudios Avanzados (IMEDEA), CSIC-UIB E-07071 Palma de Mallorca, Spain*

[3] *Department of Physics, Lehigh University - Bethlehem, PA 18015, USA* (*)

(received 26 July 2002; accepted in final form 4 November 2002)

PACS. 05.40.Ca – Noise.
PACS. 05.45.-a – Nonlinear dynamics and nonlinear dynamical systems.
PACS. 05.45.Xt – Synchronization; coupled oscillators.

Abstract. – We show the existence of a system size coherence resonance effect for coupled excitable systems. Namely, we demonstrate numerically that the regularity in the signal emitted by an ensemble of globally coupled FitzHugh-Nagumo systems, under excitation by independent noise sources, is optimal for a particular value of the number of coupled systems. This resonance is shown through several different dynamical measures: the time correlation function, correlation time and jitter.

Noise-induced resonance is a topic that has attracted a lot of attention in the last years. In particular, it has been unambiguously shown that the response of some systems to an external perturbation can be *enhanced* by the presence of noise (*stochastic resonance* [1–4]). A different effect is that of *coherence resonance* [5] by which an excitable system shows a maximum degree of regularity in the emitted signal in the presence of the right amount of fluctuations (or the related one of stochastic resonance without the need of an external forcing [6, 7]). Coherence resonance has also been studied in dynamical systems close to the onset of a bifurcation [8], as well as in other bistable and oscillatory systems [9, 10]. It has also been analyzed in different neuronal models such as the FitzHugh-Nagumo [11, 12] and Hodgkin-Huxley [13] models. It has been observed experimentally in electronic circuits, either excitable [14, 15] or chaotic [16, 17], and in lasers operating in an excitable regime [18].

In an important recent paper [19], Pikovsky *et al.* have shown that when one considers an ensemble of coupled bistable systems subjected to an external periodic forcing (and in the presence of a constant amount of noise), it turns out that an optimal response is obtained for an appropriate value of the number N of coupled systems. In other words, that there is a resonance with respect to the *number* of coupled elements, rather than to the usual one that involves the noise level. The authors speculate that this system size resonance might be

(*) Permanent address.

relevant to neuronal dynamics, in which the neuronal connections or the coupling strengths between neurons can be tuned in order to achieve maximum sensitivity to external signals. Other independent results [20–22] show that specific models of the Hodgkin-Huxley type predict that the pulses of K^+ and Na^+ concentration along biological cell membranes follow optimally an external periodic signal (stochastic resonance) for a given size of the number of ionic gates implied in the ionic transport. These latter results also show that, in the absence of external stimulus, the periodicity of the pulses is also optimal (coherence resonance) for an appropriate number of gates. At variance with the work in [19], there is no direct coupling between the different gates, but the membrane potential depends on the number of open and close gates, and the probability of having an open gate depends on the membrane potential. The number of open gates, hence, acts as a stochastic variable in the evolution equation for the potential.

In this paper, we extend the previous results by considering an ensemble of explicitly coupled excitable systems, each one under the influence of its own noise with a fixed intensity, but without an external forcing. We show that there is a coherence resonance effect as a function of the number N of coupled systems. More specifically, we show that the excitable systems pulse on average with a regularity which is optimal for a specific value of N.

Motivated by the biological applications to neuronal dynamics suggested in [19], we consider an ensemble of generic FitzHugh-Nagumo systems, each one described by the activator, x_i, and inhibitor, y_i, variables, $i = 1, \ldots, N$. The FitzHugh-Nagumo model provides the simplest representation of firing dynamics and has been widely used as a prototypic model for spiking neurons as well as for cardiac cells [23, 24]. Our aim is to consider this model as a prototypical model, without making reference to any specific application, in order to clarify the minimal elements that an excitable model has to have in order to display the phenomenon of system size coherence resonance. This is equivalent to the approach used in ref. [19], dealing with system size stochastic resonance, with focus on generic (φ^4, Ising) models for phase transitions, rather than on a specific application. The dynamical FitzHugh-Nagumo equations modified to account for the global coupling are as follows:

$$\epsilon \dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{K}{N}\sum_{j=1}^{N}\left(x_j - x_i\right), \quad (1)$$

$$\dot{y}_i = x_i + a + D\xi_i(t), \quad (2)$$

where independent noises of intensity D have been added to the slow variables y_i as in ref. [5]. The $\xi_i(t)$ are white noises with Gaussian distribution of zero mean and correlations $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$. The difference in the time scales of x_i and y_i is measured by ϵ, a small number. The systems are globally coupled by a gap-junctional form, as indicated by the last term of eq. (1), where K is the coupling strength. The assumption of global coupling is not strictly necessary, but it will allow us to justify a theoretical approach based on a mean-field–type approximation. Similar globally coupled models have been used previously to study array-enhanced stochastic resonance in the coupled FitzHugh-Nagumo equation [25].

We consider the excitable regime, $a > 1$. In the absence of coupling, $K = 0$, each FitzHugh-Nagumo system emits pulses. The pulses are trajectories that exit the basin of attraction of the stable fixed point and are triggered by the influence of the noise term. For zero coupling, coherence resonance exists when the regularity of the time between pulses is optimal for a certain value of the noise intensity D [5]. This behavior is the same for all the systems, although the response of each one is uncorrelated with any other.

Let us now study the collective response of the coupled system and compare it with the individual responses. For the collective response, we introduce the average values of the

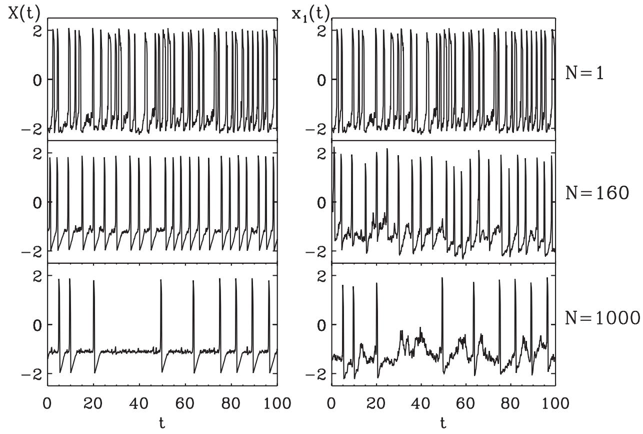

Fig. 1 – Time series for the averaged variable $X(t)$ (left panel), and for the individual variable $x_1(t)$ (right panel) of the set of coupled FitzHugh-Nagumo systems, as obtained from a numerical integration of eqs. (1)-(2), for different values of the number of coupled elements: $N = 1$ (top), $N = 160$ (middle) and $N = 1000$ (bottom). Observe that the largest regularity is obtained for the intermediate value of N. The equations have been integrated numerically using a stochastic Runge-Kutta method (known as the Heun method [26]) with a time step $h = 10^{-4}$ and setting the following parameters: $a = 1.1$, $\epsilon = 0.01$, $K = 2$, $D = 0.7$.

activator and inhibitor variables as

$$X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t), \qquad Y(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t). \tag{3}$$

By following the approach by Desai and Zwanzig [27] (see also ref. [19]), it is possible to reach an approximate effective equation for these average values of the form

$$\epsilon\dot{X} = F(X, K) - Y, \tag{4}$$

$$\dot{Y} = X + a + \frac{D}{\sqrt{N}}\xi(t), \tag{5}$$

where $\xi(t)$ is a white-noise source. Although the exact form of the function $F(X, K)$, which depends on the global variable X as well on the coupling strength K, and the analysis of the approximations assumed in the derivation will be presented elsewhere, we need only to remark here that in the (exact) equation for $Y(t)$ the noise intensity appears rescaled as $D/\sqrt{N}$. Therefore, this approximation suggests that the optimal effective noise intensity for the appearance of coherence resonance can be achieved by varying the number of coupled elements N, as in the case of stochastic resonance for the bistable system considered in [19]. To go beyond this approximation, we numerically integrate the equations of motion (1) and (2).

The left panel of fig. 1 shows the time trace for the variable $X(t)$, while the right panel of the same figure shows the time trace for the variable $x_i(t)$ of one of the elements chosen randomly, for three different values of the number of coupled elements (see the caption of the figure for details of the parameters). Notice that for $N = 160$, the regularity of the amplitude of the emitted pulses is better than that corresponding to larger or smaller values of N. This is a clear signature of coherence resonance. Moreover, it can be seen that the regularity in the averaged variable $X(t)$ is better than in one of the individual elements, showing that the coupling allows for a smoothness of the trace. It is worth noting that the peaks in the

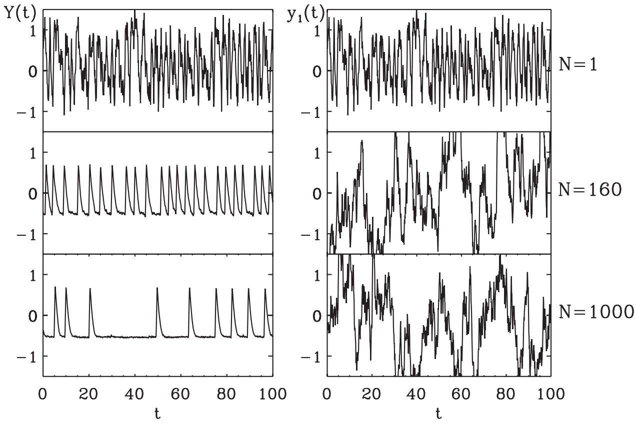

Fig. 2 – Time series for the averaged variable $Y(t)$ (left panel), and the individual variable $y_1(t)$ (right panel) of the set of FitzHugh-Nagumo systems, eqs. (1)-(2). Similarly as in fig. 1, observe that again the largest regularity for the averaged Y variable is obtained for the intermediate value of N. In this case, however, there is no obvious increase in the regularity of the y_i individual variables.

collective variable $X(t)$ and $x_i(t)$ are very well synchronized in time, indicating that the individual systems are pulsing synchronously in time. In fig. 2 (left panel) we plot the time trace for the slow variable $Y(t)$, as well as a time trace for a single one $y_i(t)$ (right panel). At variance with the fast variable X, it turns out that the averaged $Y(t)$ shows a very nice regular behavior for an intermediate number of elements, while the individual traces $y_i(t)$ do not.

We have computed two indicators commonly used to quantify this effect [5]. First, we have computed the time correlation function $C_X(t)$ of the averaged X variable, defined as

$$C_X(t) = \frac{\langle \delta X(t')\delta X(t+t')\rangle}{\langle \delta X(t')^2\rangle}, \qquad \delta X(t) = X(t) - \langle X(t')\rangle \tag{6}$$

and similarly for the correlation function $C_Y(t)$ for the averaged Y variable. Here the averages $\langle\,\rangle$ are with respect to the time t', after a small transient has been neglected. Figure 3 shows this correlation function for both the X and Y variables. It can be seen that the correlations extend further in time for an intermediate value, neither very large nor very small, of the number of coupled systems N. To obtain a quantitative indicator of this effect, we define the characteristic correlation times τ_X and τ_Y for each variable as

$$\tau_{X,Y} = \int_0^\infty |C_{X,Y}(t)|\,\mathrm{d}t. \tag{7}$$

In practice, the upper limit of the integral is replaced by a value $t_{\max}$ such that the correlation function can be considered as decayed to its asymptotic value $C_{X,Y} = 0$ ($t_{\max} = 50$ for the data shown in fig. 3). We have plotted these two correlation times in the left panel of fig. 4. Both times reach a maximum at approximately the same value $N \approx 160$, indicating that, for the set of parameters chosen, the maximum extent of the time correlation occurs for this number of coupled excitable systems.

Another common indicator for the regularity of the emitted pulses can be obtained by the *jitter* of the time between pulses [5]. A pulse in the $X(t)$ variable is defined when $X(t)$ exceeds a certain threshold value X_0 (taken arbitrarily as $X_0 = 0.3$, although other values yield similar results). The jitter R_X is defined as the root mean square of the time T_X between

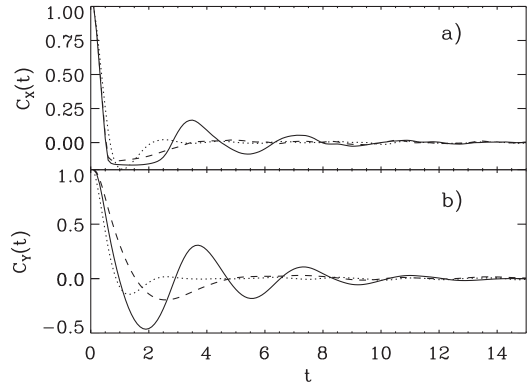

Fig. 3 – Correlation functions $C_X(t)$ and $C_Y(t)$ of the averaged variables $X(t)$ and $Y(t)$, respectively, for the cases of $N = 1$ (dotted line), $N = 160$ (solid line) and $N = 1000$ (dashed line). Notice that, in agreement with the qualitative results derived from figs. 1 and 2, the slower decay of the correlations corresponds to the intermediate values of the system size N. Same parameters as in fig. 1.

two consecutive pulses normalized to its mean value:

$$R_X = \frac{\sigma[T_X]}{\langle T_X \rangle} \tag{8}$$

and an equivalent definition holds for the jitter R_Y of the Y variable. The smaller the value of $R_{X,Y}$, the larger the regularity of the pulses (a value of $R_{X,Y} = 0$ indicates a perfectly periodic signal). It is shown in the right panel of fig. 4 that indeed the jitter in both variables have a well-defined minimum at a value of $N \approx 80$, again showing the existence of the system size resonance. When comparing with the results of the correlation time, it is not uncommon that the two indicators (the correlation time τ and the jitter R) have their optimal values at different values of the system parameters [5, 16].

In summary, we have shown that an ensemble of globally coupled FitzHugh-Nagumo excitable systems subjected to independent noises pulse on average with a regularity that is maximum for a given value of the number N of coupled systems. An approximate calculation indicates that the collective variable $Y(t)$ is subjected to a noise of effective intensity $D/\sqrt{N}$.

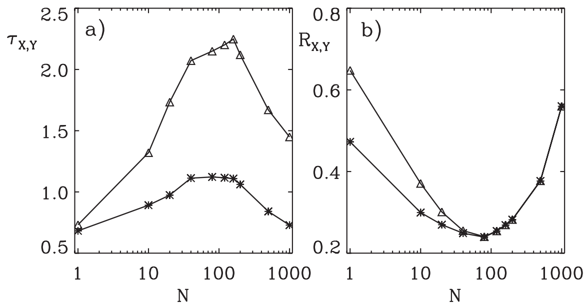

Fig. 4 – Panel (a) plots the correlation times τ_X and τ_Y as obtained by integration of the absolute value of the respective correlation functions. Clear maxima (maximum extent of the correlations) can be observed around $N = 160$. Panel (b) plots the jitter of the time between consecutive pulses of the collective variables $X(t)$ (asterisks) and $Y(t)$ (triangles). Clear minima (optimal regularity in the emitted pulses) can be observed around $N = 80$ in both cases.

Therefore, even in the presence of a large amount of noise (D large), it is possible to couple the right number of systems in order to optimize the periodicity of the emitted pulses $X(t)$. Notice that, since the individual variables $x_i(t)$ follow the same pattern as the collective one $X(t)$ (see fig. 1), the periodicity is optimal in the individual trajectories $x_i(t)$ as well. Given that the FitzHugh-Nagumo system has been used previously to model some biological systems, we believe that our results, in the same lines than those of ref. [20, 21], can be relevant when analyzing the collective response of such systems in a noisy environment, and can help to explain the observed size of some ensembles of excitable cells in living organisms.

* * *

This work is supported by the Ministerio de Ciencia y Tecnología (Spain) and FEDER, projects BFM2001-0341-C02-01 and BFM2000-1108, and NSF grant DMR9813409. JDG acknowledges IBERDROLA for financial support.

REFERENCES

[1] BENZI R., SUTERA A. and VULPIANI A., *J. Phys. A*, **14** (1981) 453.
[2] NICOLIS C. and NICOLIS G., *Tellus*, **33** (1981) 225.
[3] MOSS F., BULSARA A. and SHLESINGER M. F. (Editors), *Proceedings of the NATO Advanced Research Workshop: Stochastic Resonance in Physics and Biology*, *J. Stat. Phys.*, **70** (1993).
[4] GAMMAITONI L., HÄNGGI P., JUNG P. and MARCHESONI F., *Rev. Mod. Phys.*, **70** (1998) 223.
[5] PIKOVSKY A. S. and KURTHS J., *Phys. Rev. Lett.*, **78** (1997) 775.
[6] GANG H., DITZINGER T., NING C. Z. and HAKEN H., *Phys. Rev. Lett.*, **71** (1993) 807.
[7] RAPPEL W. and STROGATZ S., *Phys. Rev. E*, **50** (1994) 3249.
[8] NEIMAN A., SAPARIN P. and STONE L., *Phys. Rev. E*, **56** (1997) 270.
[9] LINDNER B. and SCHIMANSKY-GEIER L., *Phys. Rev. E*, **61** (2000) 6103.
[10] PRADINES J. R., OSIPOV G. V. and COLLINS J. J., *Phys. Rev. E*, **60** (1999) 6407.
[11] LINDNER B. and SCHIMANSKY-GEIER L., *Phys. Rev. E*, **60** (1999) 7270.
[12] RIPOLL MASSANÉS S. and PÉREZ VICENTE C. J., *Phys. Rev. E*, **59** (1999) 4490.
[13] LEE S. G., NEIMAN A. and KIM S., *Phys. Rev. E*, **57** (1998) 3292.
[14] POSTNOV D. E., HAN S. K., YIM T. G. and SOSNOVTSEVA O. V., *Phys. Rev. E*, **59** (1999) R3791.
[15] HAN S. K., YIM T. G., POSTNOV D. E. and SOSNOVTSEVA O. V., *Phys. Rev. Lett.*, **83** (1999) 1771.
[16] PALENZUELA C., TORAL R., MIRASSO C., CALVO O. and GUNTON J., *Europhys. Lett.*, **56** (2001) 347.
[17] CALVO O., MIRASSO C. and TORAL R., *Electron. Lett.*, **37** (2001) 1062.
[18] GIACOMELLI G., GIUDICI M., BALLE S. and TREDICCE J. R., *Phys. Rev. Lett.*, **84** (2000) 3298.
[19] PIKOVSKY A., ZAIKIN A. and DE LA CASA M. A., *Phys. Rev. Lett.*, **88** (2002) 050601.
[20] JUNG P. and SHUAI J. W., *Europhys. Lett.*, **56** (2001) 29.
[21] SCHMID G., GOYCHUK I. and HÄNGGI P., *Europhys. Lett.*, **56** (2001) 22.
[22] SHUAI J. W. and JUNG P., *Phys. Rev. Lett.*, **88** (2002) 068102.
[23] KOCH C., *Biophysics of Computation* (Oxford University Press, New York) 1999.
[24] GLASS L., HUNTER P. and MCCULLOCH A. (Editors), *Theory of Heart* (Springer-Verlag, New York) 1991.
[25] KANAMARU T., HORITA T. and OKABE Y., *Phys. Rev. E*, **64** (2001) 031908.
[26] SAN MIGUEL M. and TORAL R., in *Instabilities and Nonequilibrium Structures VI*, edited by TIRAPEGUI E., MARTÍNEZ J. and TIEMANN R. (Kluwer Academic Publishers) 2000.
[27] DESAI R. C. and ZWANZIG R., *J. Stat. Phys.*, **19** (1978) 1.