

Coherence resonance in chaotic systems

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Abstract. – We show that it is possible for chaotic systems to display the main features of coherence resonance. In particular, we show that a Chua model, operating in a chaotic regime and in the presence of noise, can exhibit oscillations whose regularity is optimal for some intermediate value of the noise intensity. We find that the power spectrum of the signal develops a peak at finite frequency at intermediate values of the noise. These are all signatures of coherence resonance. We also experimentally study a Chua circuit and corroborate the above simulation results. Finally, we analyze a simple model composed of two separate limit cycles which still exhibits coherence resonance, and show that its behavior is qualitatively similar to that of the chaotic Chua system.

When a dynamical system is subjected to an external periodic forcing, it is a standard result that synchronization between the system and the forcing can occur under a large variety of conditions. A resonance is defined as the presence of a maximum in the response of the system as a function of some control parameter (for instance, the frequency of the external signal). Although one might naively believe that fluctuations, either in the forcing or in the intrinsic dynamics, will worsen the quality of the synchronization, it is nowadays well established that, in some cases, the response of a nonlinear dynamical system to an external forcing can be *enhanced* by the presence of noise (fluctuations). The prototypical and pioneering example is that of *stochastic resonance* [1, 2] by which a bistable system under the influence of a periodic forcing, and in the presence of fluctuations, shows an optimum response, a resonance, for a given value of the noise intensity. The relevance of this phenomenon has been shown for some physical and biological systems described by a nonlinear dynamics [3–5].

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That noise can have a *constructive* role has been one of the most astonishing discoveries of the last decades in the field of stochastic processes. Besides the above-mentioned stochastic resonance, purely temporal dynamical systems can display phenomena such as noise-induced transitions [6] or noise-induced transport [7]. In spatially extended systems, on the other hand, noise is known to induce a large variety of ordering effects [8]. In all the cases, the common feature is that some sort of *order* appears only in the presence of the right amount of noise.

The possibility of having stochastic resonance without the need of an external forcing has attracted much attention recently [9–11]. In particular, the phenomenon named *coherence resonance* [11] was shown to appear in excitable systems under the influence of fluctuations. An excitable system has a stable fixed point with a finite basin of attraction. When a perturbation is such that the system crosses a threshold value, the return to the fixed point is by executing a large excursion in the configuration space, thus generating a pulse in the time evolution. One of the main features of excitable systems is that the generated pulse is basically independent of the magnitude of the perturbation that induced its firing. Therefore, the duration of the pulse, the excursion time t_e , is a characteristic of the system and not of the perturbation. The total time between pulses, t_p , is composed of two times: the excursion time t_e and the time needed for the activation of the pulse, t_a . If the firing of the pulses is produced by random fluctuations, the activation time t_a is a random variable. According to Kramers formula, for small noise, the mean activation time behaves as $\langle t_a \rangle \sim \exp[A/D^2]$, where A is a constant and D is the noise intensity [12]. The variance of the activation time is $\sigma^2[t_a] \approx \langle t_a \rangle^2$. At the same time, the excursion time t_e depends weakly on D , such that its mean value $\langle t_e \rangle$ can be considered constant and its variance can be estimated as $\sigma^2[t_e] \approx D^2 \langle t_e \rangle$. For small D , we have $\langle t_a \rangle \gg \langle t_e \rangle$ and we can approximate the time between pulses by the activation time $t_p \approx t_a$. The relative fluctuations of the time between pulses, defined as $R = \sigma[t_p]/\langle t_p \rangle$, is in this limit of small noise $R \approx \sigma[t_a]/\langle t_a \rangle \approx 1$. For large noise, the activation time is very small and the system fires a pulse every time it returns from an excursion. The pulse time is dominated by the excursion time and we can approximate $R \approx \sigma[t_e]/\langle t_e \rangle \sim D \langle t_e \rangle^{-1/2}$. If the excursion time is large, and the threshold of excitation is small, it is possible that for intermediate values of D , it is $R(D) < 1$. In this case, and according to the generic behaviors described above ($R(D) \rightarrow 1$ for small D and $R(D) \sim D \langle t_e \rangle^{-1/2}$ for large D) there will be a minimum in the relative fluctuations of the time between pulses. This is the signature of coherence resonance. A similar effect is that of stochastic resonance without external periodic force which can occur in a system near a limit cycle bifurcation point [9, 10].

To summarize, the main feature of a system displaying coherence resonance is that a quasi-periodic signal is generated by a combination of the internal nonlinear dynamics and fluctuations without the need for the presence of an external, deterministic, periodic signal. The periodicity of the pattern is optimal (resonance) for a certain value of the noise intensity. The original studies have been extended to consider other excitable systems such as the FitzHugh-Nagumo model [13, 14], the Hodgkin-Huxley model for neurons [15] and the Yamada model for a self-pulsating semiconductor laser [16]. Coherence resonance has been also observed in dynamical systems close to the onset of a bifurcation [17] as well as in other bistable and oscillatory systems [18, 19]. Experimental evidence for the existence of coherence resonance has been given for a laser system [20] and for excitable electronic circuits [21, 22].

In this paper we will prove that it is possible to display the main features of coherence resonance in chaotic and other bistable systems in which the attractors are not of the fixed-point type. Although we believe that our results are quite general, we will consider specifically a Chua circuit operating in a chaotic regime with two independent, symmetric, attractors. We will argue that the existence of a well-defined characteristic time when moving around

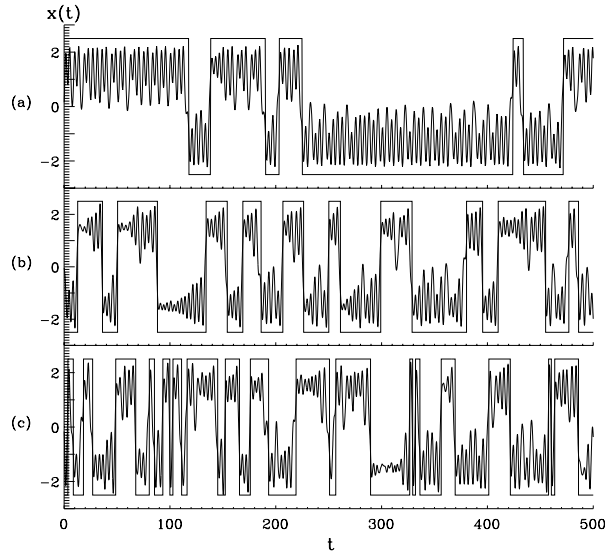


Fig. 1 – Time series for the x variable of the Chua system given by eqs. (1) for three different noise levels: a) $D = 0.02$, b) $D = 0.08$, optimum noise level, and c) $D = 0.16$.

each attractor is a necessary ingredient for the occurrence of coherence resonance. This characteristic time plays the role of the excursion time for excitable systems in the sense that only during a small fraction of this time, when the trajectory comes as close as possible to the other attractor, the fluctuations can induce jumps between the two attractors. We will give numerical and experimental evidence that such a Chua circuit in the presence of noise can undergo oscillations whose regularity is optimal (in a sense to be precisely defined later) for some intermediate value of the noise intensity. Later, we will also show that the basic ingredients for this new kind of coherence resonance are already present in a simpler toy model with two separated limit cycles. That the addition of noise can induce some degree of regularity in a chaotic system has been shown recently in a different context related to the existence of noise-induced synchronization of chaotic systems [23].

Let us consider the Chua system, in its dimensionless form, under the presence of additive noise [24]:

$$\begin{aligned}\dot{x} &= \alpha(y - h(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z + \xi(t),\end{aligned}\tag{1}$$

where $\xi(t)$ is the Gaussian white noise, of zero mean and correlations $\langle \xi(t)\xi(t') \rangle = D^2\delta(t-t')$. The nonlinear function $h(x)$ is given by $h(x) = bx + \frac{a-b}{2}(|x+1| - |x-1|)$. We have taken the values $a = -1/7$, $b = 2/7$, $\alpha = 4.60$, $\beta = 6.02$, $\gamma = 0$, for which the Chua system has two chaotic attractors: a single scroll and its mirror image. Depending on the initial conditions, the system will rotate around one attractor or the other. In other words, in the absence of fluctuations, the attractors are independent and trajectories cannot jump from one to the other. The movement around each attractor has a well-defined mean angular frequency ω_0 , which for these values of the parameters is $\omega_0 \approx 3$.

In fig. 1 we plot three trajectories of the variable $x(t)$ corresponding to increasing levels of

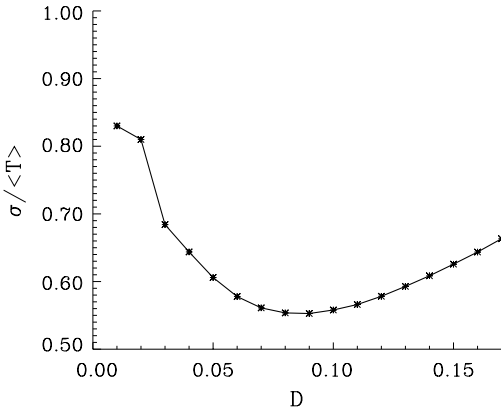


Fig. 2

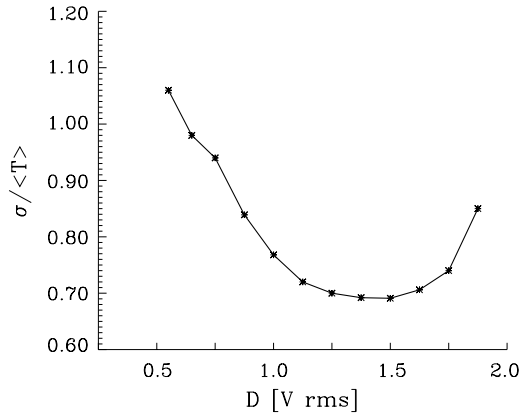


Fig. 3

Fig. 2 – Standard deviation normalized by the mean time $\sigma / \langle T \rangle$ for the Chua system given by eqs. (1) for noise levels ranging from 0.01 to 0.17.

Fig. 3 – Standard deviation normalized by the mean time $\sigma / \langle T \rangle$ in the case of the Chua circuit (see the text for details of the parameters) for noise levels ranging from 0.5 to 1.875 (V rms).

the noise intensity D . Although we present here results for the case where the noise is added only to the z variable (chosen mainly for experimental reasons), the behavior is similar if noise appears either in the x or the y variable. We observe three qualitatively different behaviors when increasing the noise level. When D is very small (fig. 1a) the average residence time, *i.e.* the time between jumps, is large and the system spends most of the time rotating around one of the attractors. For this situation, the dispersion of the residence time is also large. As D increases, approaching an optimum value (fig. 1b), the system jumps between the two attractors more regularly. These jumps occur, as already mentioned, approximately when the trajectory passes closest to the other attractor. Finally, when D is very large, the system jumps more often but these jumps may start from different points on the trajectory and the behavior of the system is more irregular (fig. 1c). The enhanced regularity that occurs for intermediate values of the noise can be clearly observed in fig. 2, where we plot the standard deviation, $\sigma[T]$, of the residence time in each attractor, normalized to its mean value, $\langle T \rangle$, as a function of the noise intensity. This curve exhibits a minimum at a noise level $D \approx 0.08$. The presence of this minimum is the clearest signature of coherence resonance. Another observed indicator of coherence resonance is the existence of a peak in the power spectrum $S(f)$ of the signal at a finite frequency [11,18]. Moreover, our data show a maximum in the ratio between the height and the width of the peak of $S(f)$ at $D \sim 0.07$. Similarly, we observe that the time-correlation function, $C(t)$, has the longest tail and the lowest minimum at values of noise close to the optimal level.

Although there is a good correspondence between a real Chua circuit and the system of differential equations (1), it is obvious that the numerical results deal necessarily with an idealized Chua circuit. In order to analyze the robustness of the observed phenomenon, we have performed experiments in a real Chua circuit constructed according the classical design (see, *e.g.*, ref. [24]) with the following parameters [25]: $C_1 = 20$ nF, $C_2 = 100$ nF, $R = 1100$ Ω , $a = -1/7$, $b = 2/7$. The noise has been generated with a standard Hewlett-Packard function generator and its intensity has been varied from zero to a few volts. As in the numerical study, the original time series for the $x(t)$ variable has been converted into a variable $u(t)$ taking

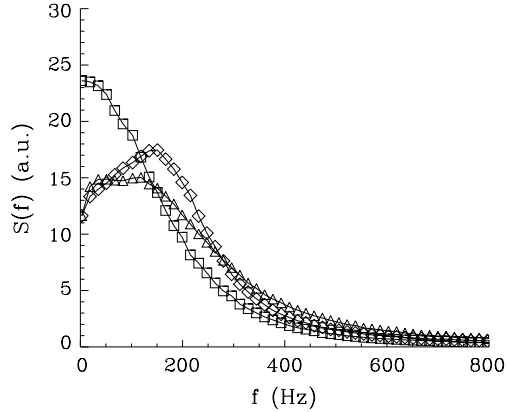


Fig. 4 – Power spectrum $S(f)$ of the digital signal $u(t)$ for the Chua circuit for different noise levels: $D = 0.5$ (V rms) (squares), $D = 1$ (V rms) (rhombuses), $D = 1.5$ (V rms) (triangles). The presence of a peak at a finite frequency is an evidence of coherence resonance.

the values $+1$ and -1 for each of the attractors. Using this variable, we have computed the normalized standard deviation of the residence time, $\sigma[T]/\langle T \rangle$. In fig. 3 we plot this quantity as a function of the noise intensity. Again, a clear minimum appears for an optimal value $D \approx 5500$ mV, so confirming the numerical results. In fig. 4 we plot the power spectrum of the digital variable $u(t)$. We notice the development of a peak at a finite frequency for an intermediate noise level giving further evidence of a regular behavior.

To gain more insight into the dynamics of this chaotic system we consider a simplified two-variable system (x_1, x_2) with two stable limit cycles. The first one, C_1 , is around the unstable fixed point $(1, 0)$ and the second one, C_2 , around the unstable fixed point $(-1, 0)$. There are no other stable fixed points or limit cycles in the system. We assume that the limit cycles are circumferences of radius R close to but smaller than 1, and that they have a constant angular speed ω_0 . Under these circumstances, which limit cycle is chosen as a dynamical attractor depends exclusively on the initial condition [26]. Let us add now some noise to the dynamics. If the noise intensity is small, the modification to the trajectories will be small. Moreover, the probability that noise induces a jump between the attractors is only significant near the closest points in the limit cycles, *i.e.*, when the trajectory passes closest to the origin of the coordinate system. If the system does not jump at this point then it has to wait for a complete rotation for another chance to jump. Hence, the rotation period $2\pi/\omega_0$ plays the role of the excursion time t_e in the excitable system in the sense that, for moderate levels of noise, the system cannot jump to the other attractor during this time.

Let us define the variable $u = \text{sign}(x_1)$. This is equal to $+1$ when the system is in attractor C_1 and -1 when in attractor C_2 . The evolution of $u(t)$ can be described by a series of time intervals alternating the values of $+1$ and -1 . According to the previous argument, for small noise, the duration T of each time interval is a random variable taking values which are an integer multiple of $2\pi/\omega_0$. The probability that the system jumps between the attractors exactly after n cycles follows the geometric distribution $P(T = 2\pi n/\omega_0) = p(1-p)^{n-1}$, where p is the probability that the system jumps between the two attractors during the time they are closest at each cycle. This probability p will be a function of the noise intensity D and also of the angular speed ω_0 determining the time the system is ready for jumping. Using this distribution, the relative fluctuation of the time between jumps is $\sigma[T]/\langle T \rangle = \sqrt{1-p}$.

Therefore, for small D , p will be small and R will initially decrease with D . Since, according to the general argument of [11] developed at the beginning, R will eventually grow for large D , we expect a minimum in a plot of $\sigma[T]/\langle T \rangle$ vs. D . When re-examining figs. 2 and 3, it can also be observed that both limits, low and high noise intensity, behave roughly as estimated for the simpler model described above, indicating that our simplified model captures the main ingredients of coherence resonance in the more complicated chaotic system. However, there is an important difference between the simple dynamical model and the Chua system. In the former, the trajectory is always on the limit cycle and the system is ready to jump to the other attractor at any cycle. On the contrary, in the Chua system, after the trajectory jumps from one attractor to the other, the motion usually starts close to the center of the attractor, in an inner orbit, and it is not ready to jump to the other attractor until the outer orbits are reached. This fact can be guessed from fig. 1 when looking at the times the system jumps from one attractor to the other. This difference might be responsible for the fact that in the Chua model R does not clearly approach to 1 as D is decreased. On the other hand, if noise is arbitrarily increased, the Chua system saturates to a unique limit cycle thus losing its characteristic behavior.

In conclusion, we have shown, both numerically and experimentally, that coherence resonance can be observed in a chaotic system. We have also shown that a simple model, composed of two separate limit cycles, is able to exhibit coherence resonance. Within this model we were able to predict, for instance, the limits of the normalized standard deviation of the residence time by a simple analytical approximation. The behavior of the chaotic Chua system follows qualitatively the results derived in the simple model, with coherence resonance illustrated by the dependence of several different quantities on the noise intensity. Finally, we consider particularly interesting the fact that the combination of noise and chaos can lead to some degree of regularity in the system.

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$$\begin{aligned}\dot{r} &= -2r(r - R), \\ \dot{\theta} &= \omega_0,\end{aligned}$$

where $r = \sqrt{(x_1 - s)^2 + x_2^2}$, $\theta = \arctan\left(\frac{x_2}{x_1 - s}\right)$, and $s = 1$ for $x_1 \geq 0$ and $s = -1$ for $x_1 < 0$. A similar behavior occurs in a model for a Degenerate Optical Parametric Oscillator, where the phenomenon of coherence resonance can also be observed (RABIOSI I., TORAL R. and SAN MIGUEL M., work in progress).