Noise-Induced Nonequilibrium Phase Transition

C. Van den Broeck
Limburgs Universitair Centrum, B-3590 Diepenbeek, Belgium

J. M. R. Parrondo
Departament de Física Aplicada I, Universidad Complutense Madrid, 28040 Madrid, Spain

R. Toral
Departament de Física, Universitat de les Illes Balears, 07071 Palma de Mallorca, Spain
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We report on a simple model of a spatially distributed system which, subject to multiplicative noise, white in space and time, can undergo a nonequilibrium phase transition to a symmetry-breaking state, while no such transition exists in the absence of the noise term. The transition possesses features similar to those observed at second order equilibrium phase transitions: divergence of the correlation length and of the susceptibility, critical slowing down, and scaling properties. Furthermore, the transition is found to be reentrant: The ordered state appears at a critical value of the noise intensity but disappears again at a higher value of the noise strength.

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Noise is usually thought of as a phenomenon which induces disorder. There are, however, some situations in which it participates in the creation of an ordered state through its interaction with the nonlinearities of the system [1–4]. In this Letter, we present a novel and quite remarkable phenomenon in which multiplicative noise, white in space and time, actually produces an ordered symmetry-breaking state through a genuine nonequilibrium phase transition in a system where no such transition is observed in the absence of noise. The transition possesses characteristic features of a second order phase transition, such as divergence of the correlation length and susceptibility, critical slowing down, and scaling behavior. Furthermore, the transition is found to be reentrant: The ordered state appears above a critical value of the noise intensity, but disappears again through another second order phase transition at a higher value of the noise strength. This phenomenon should not be confused with the so-called noise-induced transitions [4], in which the shape of a probability density qualitatively changes under influence of multiplicative noise. The latter transitions occur in zero-dimensional systems and do not break the ergodicity of the system.

The evidence for the existence of a noise-induced phase transition comes from a mean-field approximation, as well as from a more sophisticated calculation which takes into account the effect of spatial correlations. Furthermore, our theoretical results are supported by extensive simulations in two dimensions. Although the model itself is artificial (it has been chosen mainly for mathematical convenience, while it also seems to be the simplest example displaying a noise-induced phase transition), we believe that the same phenomenon can and will occur in more complicated models of real physical systems. We hope that this Letter will give an impetus for the search of such systems both theoretical and experimental.

We consider the following lattice model. The state of the system is described by a set of scalar variables $\{x_i\}$, $i = 1, \ldots, L^d$, defined on a cubic lattice in d dimensions with lattice points i. The variables x_i evolve in time according to the following set of stochastic differential equations, defined in the sense of Stratonovich calculus [5]:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i - \frac{D}{2d} \sum_{j \in n(i)} (x_i - x_j). \qquad (1)$$

Here $n(i)$ is the set of the $2d$ nearest neighbors of site i, and $\{\xi_i(t)\}$ are Gaussian noises, white in time and space, with zero mean and autocorrelation function given by

$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma^2 \delta_{ij}\delta(t - t'). \qquad (2)$$

Equations of this kind are very general and cover a multitude of different physical phenomena, both equilibrium [when $g(x) = 1$] and nonequilibrium. We would like to track down the existence of nonequilibrium phase transitions, arising from the multiplicative nature of the noise, in models described by these equations. Such a phase transition is characterized by a breaking of the ergodicity and the appearance of multiple steady state probability distributions $P^{\rm st}(\{x_i\})$ (in the limit of an infinite system). As is well known from the theory of equilibrium critical phenomena, it is usually very difficult to establish the existence of phase transitions in a rigorous manner. In the present situation, we have the additional complication that the explicit form of the steady state probability distribution is not known, so that the traditional techniques from equilibrium statistical physics cannot be applied. However, the oldest and simplest ansatz which can reproduce, albeit not always faithfully, the breaking of ergodicity,

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namely, the idea of a Weiss mean field, can readily be adapted to the present problem. This approach has been applied successfully in a number of other stochastic problems [6–16].

For the study of the steady state properties, it is convenient to switch to the following Fokker-Planck equation, which is equivalent to the stochastic differential equation (1):

$$\partial_t P(\{x_i\}, t) = \sum_i \frac{\partial}{\partial x_i}\left[-f(x_i) + \frac{D}{2d} \sum_{j \in n(i)} (x_i - x_j) + \frac{\sigma^2}{2} g(x_i) \frac{\partial}{\partial x_i} g(x_i) \right] P(\{x_i\}, t). \tag{3}$$

By integration of Eq. (3) over all variables, with the exception of x_i, and using the fact that the steady state properties are isotropic and translationally invariant, one obtains the following exact steady state equation for the one-site probability:

$$0 = \frac{\partial}{\partial x_i}\left[-f(x_i) + D[x_i - E(x_i)] + \frac{\sigma^2}{2} g(x_i) \frac{\partial}{\partial x_i} g(x_i) \right] P^{\mathrm{st}}(x_i), \tag{4}$$

where

$$E(x_i) = \langle x_j | x_i \rangle = \int dx_j\, x_j P^{\mathrm{st}}(x_j | x_i), \qquad \text{with } j \in n(i), \tag{5}$$

represents the steady state conditional average of x_j at a neighboring site $j \in n(i)$, given the value x_i at site i. The solution to Eq. (4) is readily found to be (we drop the subscript i for simplicity of notation)

$$P^{\mathrm{st}}(x) = \frac{1}{Z} \exp\left[\frac{2}{\sigma^2} \int_0^x dy \frac{f(y) - \frac{\sigma^2}{2} g(y) g'(y) - D[y - E(y)]}{g^2(y)} \right], \tag{6}$$

where Z is a normalization constant. This result is exact, but we still have to determine the unknown function $E(y)$. At this stage, we introduce the Weiss mean-field approximation: We neglect the fluctuations in the neighboring sites so that $E(y) = \langle x \rangle$, independent of y. The value of $\langle x \rangle$ then follows from the self-consistency condition

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx\, x P^{\mathrm{st}}(x) = F(\langle x \rangle). \tag{7}$$

When this nonlinear equation has multiple solutions, one concludes that there are several corresponding steady state probabilities $P^{\mathrm{st}}(x)$, and the mean-field theory predicts a phase transition with breaking of ergodicity. The latter is usually accompanied by a breaking of the symmetry. For example, if f is an odd function of x and g is even, it follows from Eq. (1) that any realization $\{x_i(t)\}$ is equally probable as $\{-x_i(t)\}$. Therefore one would expect that $\langle x \rangle = 0$. However, with the appearance of multiple solutions, this symmetry need not be fulfilled by the separate solutions, and one typically finds "ordered" phases with an order parameter $m = |\langle x \rangle| \neq 0$.

In order to study the solutions of Eq. (7), the specific form of the functions f and g has to be specified. To illustrate the phenomenon of a noise-induced phase transition we focus on the simplest possible example that we were able to find [17]:

$$f(x) = -x(1 + x^2)^2, \qquad g(x) = 1 + x^2. \tag{8}$$

In the case of additive noise, $g(x) = 1$, this model does not undergo a phase transition. We will now show that multiplicative noise with the choice of $g(x)$ given above does produce a phase transition. One can easily verify that $F(\langle x \rangle)$ is a smooth, odd function of $\langle x \rangle$, so that $F(0) = 0$ and $\langle x \rangle = 0$ is always a solution of Eq. (7). Two new symmetry breaking solutions will appear when the first derivative $F'(0)$ becomes equal to 1. This condition leads to the following relation between D and σ^2:

$$\frac{1}{N} \frac{D}{\sigma^2} \int_{-\infty}^{+\infty} dx \frac{\exp[-x^2/\sigma^2 - (D/\sigma^2) x^2/(1 + x^2)]}{1 + x^2} \left[\frac{x^2}{1 + x^2} + x \arctan x \right] = 1 \tag{9}$$

with

$$N = \int_{-\infty}^{+\infty} dx \frac{\exp[-x^2/\sigma^2 - (D/\sigma^2) x^2/(1 + x^2)]}{1 + x^2}. \tag{10}$$

The resulting D versus σ^2 curve is represented in Fig. 1 (dashed line) and corresponds to a line of second order phase transitions separating the single phase region $\langle x \rangle = 0$ from the bistable region with stable solutions $\langle x \rangle \neq 0$. The positive nonzero solution $m = |\langle x \rangle|$ of Eq. (7) is represented in Fig. 2 (dashed line) in a function of σ^2 for $D = 20$. The ordered phase appears in the window $1.14 \leq \sigma^2 \leq 19.9$. One concludes that the Weiss mean-field approximation predicts the existence of a symmetry-breaking phase transition with the following general features. Firstly, the ordered phase only appears for a sufficiently strong spatial coupling. Secondly, the transition is second order; the order parameter m increases continuously from a zero value at the critical point as one enters the ordered phase. Thirdly, the phase transition is reentrant: The ordered phase only appears for a window of intermediate noise strength and is destroyed through

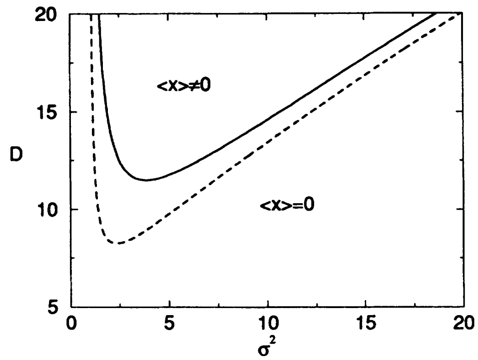

FIG. 1. Phase diagram for the noise-induced phase transition as predicted by the mean-field theory (dashed line), cf. Eq. (9), and the more sophisticated approach (full line), based on Eqs. (13) and (14) for dimension $d = 2$.

another second order phase transition at a higher value of the noise intensity.

We now briefly present the main steps and final results of a more sophisticated approach, which involves the approximate calculation of the spatial correlation function. The details of these calculations will be presented elsewhere. The approach is based on the following ansatz for the conditional average (cf. [18,19]):

$$\langle x_j | x_i = y \rangle = \langle x \rangle + c_{ij}(y - \langle x \rangle), \tag{11}$$

where $c_{ij} = \langle \delta x_i \delta x_j \rangle / \langle \delta x^2 \rangle$ is the spatial correlation coefficient between sites i and j and $\delta x_i = x_i - \langle x \rangle$. In particular, the ansatz Eq. (11) implies that

$$E(y) = \langle x \rangle + c(y - \langle x \rangle) \tag{12}$$

where c is the nearest neighbor correlation coefficient $c = c_{ij}$ with $j \in n(i)$. Note that the mean-field approximation corresponds to the choice $c = 0$. The variable c appears

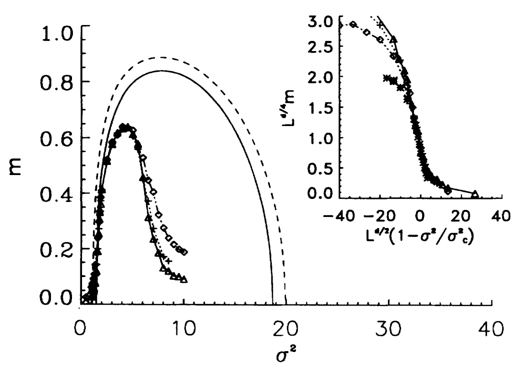

FIG. 2. Order parameter m versus intensity of the noise for $D = 20$, according to the mean-field theory (dashed line), the more sophisticated approach for $d = 2$ (full line), and 2D simulations for system sizes 10×10 (stars), 20×20 (diamonds), 30×30 (crosses), and 40×40 (triangles). The inset shows that the data are consistent with a finite-size scaling theory description (for $d = 2$) using the mean-field values for the exponents.

as a second unknown in the explicit form of the steady state probability, cf. Eqs. (6) and (12). As was the case for the determination of $\langle x \rangle$, the value of c also follows from a self-consistent relation which is found as follows. Combining the ansatz (11) with the Fokker-Planck equation (3), one obtains a closed equation for the correlation c_{ij}, which can be solved without any further approximation. In particular, one obtains the following result for c (in dimension $d = 2$):

$$c = \frac{\beta}{\pi D} K\left[\left(\frac{D}{\gamma + D}\right)^2\right] - \frac{\beta}{2D} + \frac{\alpha\langle x \rangle}{\gamma}, \tag{13}$$

where K is the complete elliptic integral of the first kind [20], and α, β, and γ are given by

$$\alpha = -\frac{\langle f(x) + (\sigma^2/2)g(x)g'(x)\rangle}{\langle \delta x^2 \rangle}, \tag{14a}$$

$$\beta = \frac{\sigma^2 \langle g^2(x) \rangle}{\langle \delta x^2 \rangle}, \tag{14b}$$

$$\gamma = -\frac{\langle \delta x [f(x) + (\sigma^2/2)g(x)g'(x)]\rangle}{\langle \delta x^2 \rangle}. \tag{14c}$$

The averages appearing in the right-hand side of these equations have to be calculated with the steady state probability given by Eq. (6), which itself depends on $\langle x \rangle$ and c. As a result, Eqs. (7), (13), and (14) form a set of nonlinear self-consistent relations determining the value of $\langle x \rangle$ and c. The final results, obtained by a numerical solution of these equations, have been included in Figs. 1 and 2 (full lines). The results are in qualitative agreement with those of the Weiss mean-field approximation (dashed lines), but the parameter region in which the ordered phase is predicted is found to be somewhat smaller. For example, for $D = 20$, the phase transition points are located at $\sigma^2 \approx 1.50$ and $\sigma^2 \approx 18.7$.

The above presented theoretical results have been compared with extensive simulations of a two-dimensional system, obtained through a numerical integration of the set of stochastic differential equations (1) using the Euler method [21–23]. After the system had reached the stationary state, average values were computed using a sample of 16 384 configurations. The results obtained for the particular case $D = 20$ agree qualitatively with the theoretical results, cf. Fig. 2. The existence of an ordered phase, appearing for intermediate strength of the noise, is confirmed. The discrepancy between mean-field theory and numerical results and the large finite-size effects at the reentrant transition can be understood from the fact that the intensity of the noise is quite large at this point. As far as the critical properties in the vicinity of the phase transition points are concerned, it is very difficult to decide on the basis of our simulations whether the model introduced here belongs to any of the existing universality classes. This is so because, on top of the usual problems associated with the study of the critical behavior for equilibrium systems, namely, the growth of statistical errors due to critical fluctuations, we do not know the exact location of the critical points, while the statistics have to be generated by the numerical integration of a set of stochastic differential equations. Nevertheless, the standard equilibrium finite-size scaling theory seems to apply and predicts the location of the critical points (through the method of the cumulants) at $\sigma^2 \approx 1.50$ and $\sigma^2 \approx 7.0$. Furthermore, the order parameter can be scaled reasonably well by using mean-field exponents, cf. inset of Fig. 2.

As a conclusion, we have found strong evidence for the existence of a reentrant noise-induced nonequilibrium phase transition, in which the appearance of an ordered phase results from a nontrivial cooperative effect between multiplicative noise, nonlinearity, and diffusion.

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