Physica D 61 (1992) 159–165
North-Holland

PHYSICA D

Noise and pattern selection in the one-dimensional Swift–Hohenberg equation

E. Hernández-Garcia[a], M. San Miguel[a], Raúl Toral[a] and Jorge Viñals[a,b]
[a]*Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain*
[b]*Supercomputer Computations Research Institute, B-186, Florida State University, Tallahassee, FL 32306-4052, USA*

The question of pattern selection in the presence of noise is addressed in the context of the one-dimensional Swift–Hohenberg equation, a model for the onset of convection. We show how noise destroys long-range order in the long-time patterns, so that characterization of the selected pattern in terms of the Fourier mode with the maximum spectral power is not always suitable. The number of zeros of the configurations turns out to be a better quantity. We consider also the decay process after an Eckhaus instability. It is shown that the selected pattern is close to the one of fastest growth during the linear regime, and not to the variationally preferred. This mechanism is robust to small noise.

1. Introduction

The issue of pattern selection has been studied in a wide variety of physical systems and mathematical models [1]. The basic question is to identify the mechanisms by which a system prepared in some initial condition evolves towards a final stationary state which is only a particular one among a set of possible steady states. Typical examples include fingering instabilities in Hele–Shaw-type experiments, dendritic or directional solidification, Rayleigh–Bénard convection in simple or binary fluids, Taylor–Couette flow or a host of convective instabilities in nematic liquid crystals.

It is common to model the occurrence and evolution of this type of instabilities with "amplitude equations". Amplitude equations are simplified mathematical models that describe the slow variations in time and in space of the original variables that characterize the system of interest near the instability. Such equations have been derived and used in a variety of cases to obtain, for example, bifurcation diagrams, small amplitude stationary solutions, etc. In addition, some mathematical models have been introduced such that their associated amplitude equation coincides with the amplitude equation of the process under study. For example, the Swift–Hohenberg (SH) model [2] leads to the same amplitude equation than the hydrodynamic equations (in the Boussinesq approximation) that describe Rayleigh–Bénard convection in a simple fluid. We note that the equivalence between both descriptions has been only established near the onset of the convective instability. Far from threshold, the SH equation is expected to mimic some of the short wavelength aspects of the full hydrodynamic equations, absent in the simpler amplitude equation.

Our study concerns the effect of random fluctuations on the issue of pattern selection. Such fluctuations, usually of thermal origin, have been commonly neglected in the study of fluid systems because of its smallness [3]. The recent work by Rehberg et al. [4] shows that such fluctuations are directly observable in nematic liquid crystals close to the electrohydrodynamic instability. The consideration of noise is essential to identify time scales in transient processes [5]. In addition, the discussion of fundamental questions such as the existence of a real symmetry breaking associated

0167-2789/92/$05.00 © 1992 – Elsevier Science Publishers B.V. All rights reserved

with pattern formation requires the consideration of fluctuations. We restrict in the present work to the one-dimensional situation, which can only model the appearance of straight and parallel rolls in the convective problem. The dynamics in the two-dimensional case is greatly affected by the presence of topological defects, so that our results cannot be directly applied to that case. However, there is experimental evidence of some aspects common both one and two dimensions [6].

It is known that in the absence of fluctuations the nature of the stationary solutions and its selection strongly depends on the choice of boundary conditions. This is obvious in small aspect ratio systems (analogous to small aspect Rayleigh–Bénard convection cells), but even in large aspect ratio systems the election of rigid boundary conditions greatly reduce the number of available steady states [7]. As will be shown below, the presence of noise destroys spatial coherence in the states of the system. Then, the selection process in the bulk is expected to be independent of boundary conditions.

In section 2 we show that the mode with maximum amplitude in the power spectrum of the pattern is not a good characterization of the asymptotic stationary states. The reason is that many neighbouring modes are excited by noise, corresponding to the absence of truly long range order in the system. We estimate a correlation length from a correlation function which gives a quantitative measure of the absence of long-range order. However, the number or zeros of the pattern (corresponding to the number of rolls in the convective problem) is nearly constant at long times, so that it is a good characterization of the finally selected fluctuating pattern.

In section 3 we analyze the decay of an Eckhaus unstable periodic pattern in the presence of noise. Since the SH equation derives from a potential, it can be thought that a variational principle will determine the final pattern. It is shown that the selected pattern is close to the mode of fastest growth after the instability, and not to the one that minimizes the potential, even in presence of fluctuations. This result illustrates that, even in this model with variational structure, variational properties do not completely describe the dynamics.

2. Characterization of the selected pattern in the presence of noise

The SH equation describing the temporal evolution of a dynamic variable $y(x, t)$, function of a space variable x and time t, is [2]

$$\frac{\partial y(x,t)}{\partial t} = \left[\gamma^2 - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right] y(x,t) - y(x,t)^3 + \xi(x,t) . \tag{1}$$

γ plays the role of a control parameter and $\xi(x, t)$ is a Gaussian random process that satisfies

$$\langle \xi \rangle = 0,$$
$$\langle \xi(x,t)\, \xi(x',t') \rangle = 2\epsilon \delta(x - x')\, \delta(t - t') . \tag{2}$$

The parameter ϵ describes the intensity of the fluctuating contribution to eq. (1).

In the absence of noise ($\epsilon = 0$), eq. (1) admits stationary solutions of spatial periodicity $2\pi/q$, where $q \in [q_{-L}, q_L]$ and $q_{\pm L} \equiv \sqrt{1 \pm \gamma}$. They have the form

$$y_q(x) = \sum_{i=0}^{\infty} A_i(q) \sin((2i+1)qx) . \tag{3}$$

The coefficients $A_i(q)$ can be approximately found for small γ [8]. For γ arbitrary, they can be found numerically.

Only a subset of those stationary solutions is stable against small perturbations, namely those with $q \in [q_{-E}, q_E]$, $(q_{-L} < q_{-E} < q_E < q_L)$. Solutions with q outside this range are unstable against a modulational instability known as the Eckhaus instability [9–11]. The value of $q_{\pm E}(\gamma)$ is known as the Eckhaus boundary and can be numerically computed. For γ small it can be shown that $q_{\pm E} \approx 1 + (q_{\pm L} - 1)/\sqrt{3}$.

In addition to the periodic solutions in (3), eq. (1) admits as stationary solutions the trivial $y(x) = 0$, which is unstable if $\gamma^2 > 0$, and a family of aperiodic solutions as the one shown in fig. 1. They can be numerically found by solving the stationary version of (1). They turn out to be always dynamically unstable and are expected to be related to the secondary bifurcating solutions presented in [10] and in [12] for different equations.

The question of pattern selection can be formulated as follows: starting with the unstable configuration $y(x) = 0$, what is the state of the system for $t \to \infty$? To answer this question, a possible approach is to take advantage of the fact that the SH equation admits a potential form:

$$\frac{\partial y(x,t)}{\partial t} = -\frac{\delta F[y]}{\delta y(x,t)} + \xi(x,t) . \qquad (4)$$

The Lyapunov functional $F[y]$ is given by

$$F[y] = \int \mathrm{d}x \left[-\tfrac{1}{2}(\gamma^2 - 1)\, y^2 + \tfrac{1}{4} y^4 - \left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \right] \qquad (5)$$

and $\delta/\delta y(x,t)$ stands for the functional derivative with respect to $y(x,t)$. The stationary solutions (3) corresponds to local minima of $F[y]$.

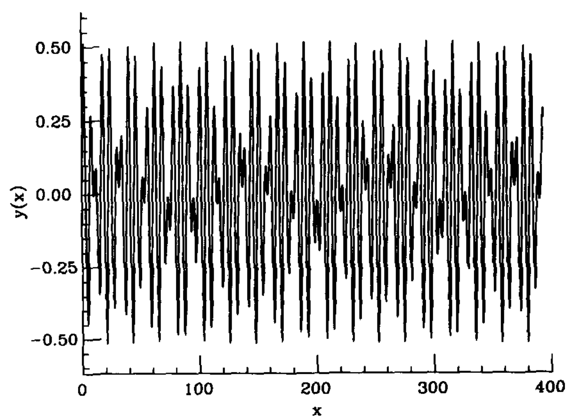

Fig. 1. A non-periodic, stationary but unstable solution of the deterministic SH equation for $\gamma = 0.5$.

According to (4) for $\epsilon = 0$, the value of $F[y]$ can only decrease during dynamical evolution. Then the system will evolve trying to minimize $F[y]$ until a local minimum is reached. One might expect that in the presence of noise the system will be able to cross the barriers between the basins of attraction of the different minima and, for long times, will settle down into the absolute minimum of $F[y]$. This absolute minimum corresponds to the solution in (3) with a value of $q = q_M$ which is very close but smaller than $q = 1$. For small γ it is [8]

$$q_M \approx 1 - \tfrac{1}{1024}\gamma^4 . \qquad (6)$$

What is missing in the above reasoning is that, in the presence of noise, there are no truly stationary solutions to (1). What we have at long times is a sequence of configurations with stationary statistical properties. Then, the question of pattern selection is ill-posed in its usual formulation. To make this point clear, we have performed simulations for system size L ranging from $L/2\pi = 16$ to 256. This corresponds, for our discretization $\Delta x = 2\pi/32$, to the consideration of 512 to 8192 Fourier modes. More computational details can be found in [13]. Fig. 2 shows a typical time evolution of the power spectrum $P(k,t) = |\hat{y}(k,t)|^2$ at long times ($\hat{y}(k,t)$ denotes the Fourier transform of $y(x,t)$). We see that a set of neighbouring modes are permanently excited by noise. None of them is really *selected*. The width of the range of excited modes turns out to be independent of system size. In [13] we showed that this range of modes is related to a finite correlation length: noise destroys long-range order in this one-dimensional system and the patterns are only coherent over a distance r_0.

The meaning of this correlation length is illustrated in fig. 3. It displays the normalized correlation function

$$G(r) \equiv \lim_{t\to\infty} \frac{\langle y(x+r,t)\, y(x,t)\rangle}{\langle y(x,t)^2\rangle} . \qquad (7)$$

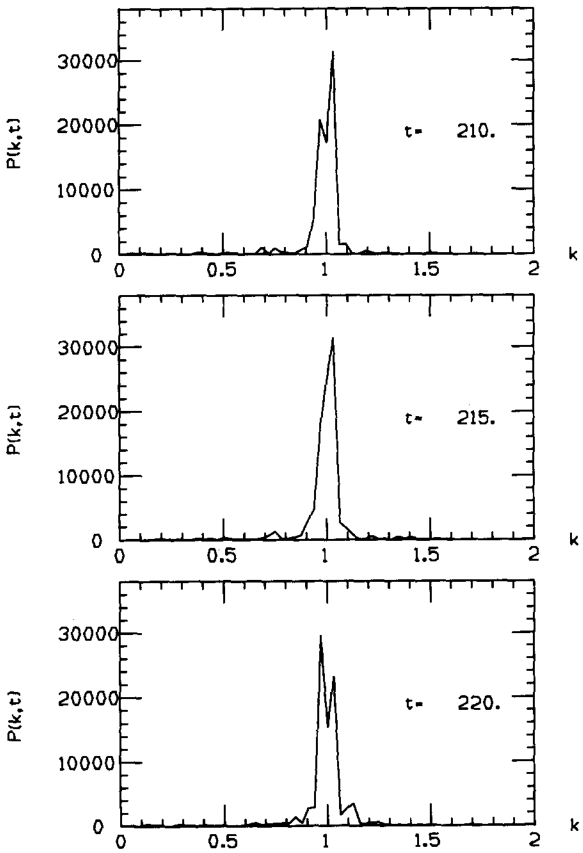

Fig. 2. Time evolution of the power spectrum in a system of size $L = 64\pi$, $\gamma = 0.5$ and noise intensity $32\epsilon/\pi = 0.2$. The distance between discrete modes is $\Delta q = 0.03125$.

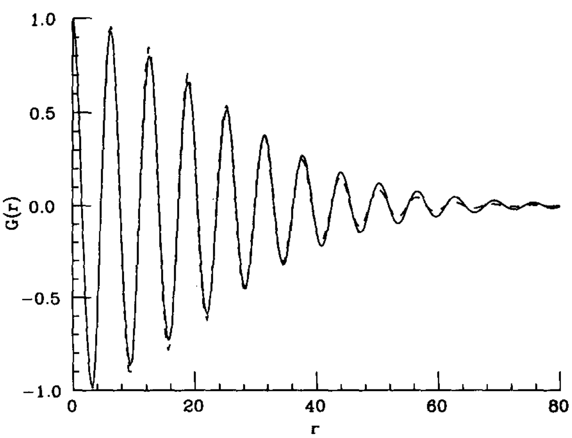

Fig. 3. Solid line: stationary correlation function $G(r)$ for $\gamma = 0.5$, $L = 512\pi$ and $32\epsilon/\pi = 0.1$, averaged over 20 runs. Dashed line: fit to eq. (8), with $q_0 = 1.0$ and $r_0 = 32.0$.

The averages are over 20 realizations of noise histories, and we consider that the limit $t \to \infty$ is achieved when the averages do not show any appreciable time evolution. The function

$$e^{-(r/r_0)^2} \cos(q_0 r) \tag{8}$$

provides a good fit to the data, giving an estimation for r_0. The Fourier transform of (8) is also a good fit of the average power spectrum, except in the tail region. q_0 is a characterization of the wavenumber selected in average but it does not describe single configurations. By fitting $G(r)$ to (8) it is found [13] that $q_0 = 1.00 \pm 0.01$ and that $r_0 \approx \epsilon^{-1/2}$. For the parameters of fig. 3, $L/r_0 \approx 50$.

Despite the absence of long-range order in the long-time configurations for single runs we can still identify a *selected pattern* in the following sense: the number of zeros N of $y(x, t)$, corresponding to the number of rolls in the system, is a remarkably stable quantity. Even when different modes alternate in importance as in fig. 2, N behaves nearly as a constant in time. This is shown in fig. 4 for the same run as in fig. 2. Noise is quite inefficient in creating or destroying rolls in the pattern. In addition, if noise is set equal to zero when the pattern is in such a fluctuating state, the system immediately relaxes to the stationary solution in (3) which has the same number of zeros as the initial fluctuating pattern. Thus we conclude that at long times the system evolves inside the basin of attraction of one of the deterministic solutions y_q in (3), with $q = (2\pi/L)(N/2)$. Such fluctuating evolution prevents coherence of the pattern at long distances, but is unable to significantly affect the value of N.

The observation of the approximate conservation of N is consistent with previous results obtained in the absence of noise [14]. Our data supports that such approximate conservation law is robust to fluctuations after a very early regime in which the amplitude of the pattern is small everywhere. Of course, a value of ϵ too big

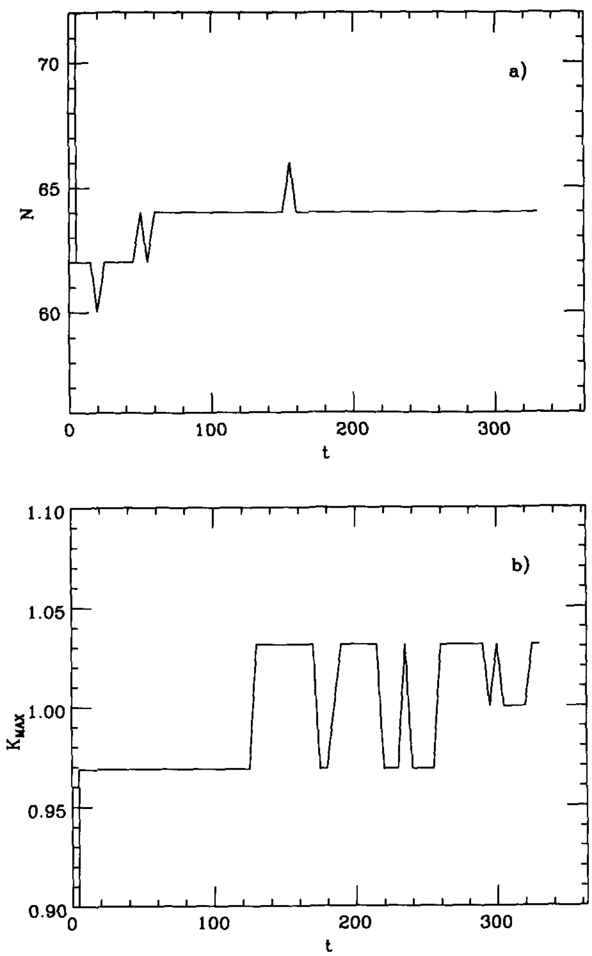

Fig. 4. Time evolution of (a) the number of zeros N, and (b) the mode with maximum power in the spectrum K_{MAX}, for the same simulation as in fig. 2.

induces changes in N even at long times, but then the correlation length r_0 is no more than a few rolls, so that we can not talk of an *ordered pattern*.

Simulations [13] starting from $y(x) = 0$ give a value to the selected N consistent with the corresponding to the most stable pattern (6). However, as seen in fig. 4 for a particular run, this value of N is already selected at relatively early times, when linear processes are still important. By linearizing eq. (1) around the unstable homogeneous state $y(x) = 0$, it can be seen that the mode of fastest growth after the instability is $q = 1$. Our system sizes are not enough to differentiate q_M in (6) from $q = 1$. For example, if $\gamma = 0.5$, $q_M \approx 0.99994$. With our discretization procedure, a simulation considering 10^6 modes, far from our capabilities, would be needed to discriminate $q = 1$ from q_M. A possible interpretation of our results is that the selection process is not due to the variational structure of the SH equation, but to the fast creation in the linear regime of a number of zeros corresponding to $q = 1$. In this regime, when the amplitude of the pattern is small, noise along the evolution path is efficient in creating and destroying rolls, so that the initial condition is rapidly forgotten. This contrasts with the results in [14], where the absence of noise along the path allowed dependence of the selected number of zeros on the initial condition. After the linear regime, when the pattern is well developed everywhere, noise is unable to change N. Results in the next section support this view of the selection process.

3. Pattern selection from an Eckhaus unstable configuration

The arguments in the previous section make it interesting to study a situation in which variational effects could be distinguished from those of fast transients. One such situation is the SH dynamics starting from one of the solutions in (3) with q in the Eckhaus unstable band. By linearizing around this unstable initial state, the evolution of perturbations can be analyzed. For small γ, results from the lower order amplitude equation [11, 15] imply that the fastest growing perturbation is a modulation of the unstable pattern which appears in the power spectrum as two sidebands of wavenumber $q \pm K_0$ around the initial fundamental wavenumber q. K_0 is given for small γ by

$$K_0^2 \approx (q-1)^2 - \frac{[3(q_E-1)^2-(q-1)^2]^2}{4(q-1)^2} . \qquad (9)$$

For γ arbitrary, a detailed numerical analysis can be performed to identify K_0. It will be presented elsewhere. The result is that, in contrast with the decay from $y(x) = 0$, here $q \pm K_0$ can be clearly distinguished from the variationally preferred wavenumber, q_M.

Fig. 5 shows the time evolution of the spectrum averaged over 20 runs. γ is 0.75 and the initial pattern is the solution in (3), numerically computed, with fundamental wavenumber $q = 1.23$, in the Eckhaus unstable band. It has 630 rolls. Eq. (1) is discretized in 8192 lattice points, a distance $\Delta x = 2\pi/32$ apart. Noise intensity was set to $2\epsilon/\Delta x = 0.1$. It is seen how a broad sideband grows and replaces the initial configuration. As in the previous section, the final spectrum is not characterized by a single wavenumber, but it is clear that the region of excited modes (note that the scale is logarithmic) does not include $q_M \approx 1$ and that it is centered around the fastest growing mode in the initial regime. Typical final configurations have around 536 zeros, which are to be compared with the 512 zeros associated with the variationally preferred wavenumber $q_M \approx 1$. Quantitative comparison of the finally selected pattern, defined from the number of zeros of $y(x, t)$, and $q - K_0$ numerically calculated is good when the initial condition is far from the Eckhaus boundary. Close to q_E fluctuations become more important and deterministic theory neglects important effects. A detailed study for several values of the initial condition will be presented elsewhere.

Previous theoretical analysis of the selection process neglected fluctuations during the transient. The simulations presented in [11] for the amplitude equation found that the final wavenumber was systematically in between the initial state and the one predicted by linear theory. These authors interpret their results in terms of a stabilization of the pattern after the firsts changes in N (phase slips in their model) have occurred, so that the driving force for further changes in N disappears. The evolution stops in a minimum of the Lyapunov functional

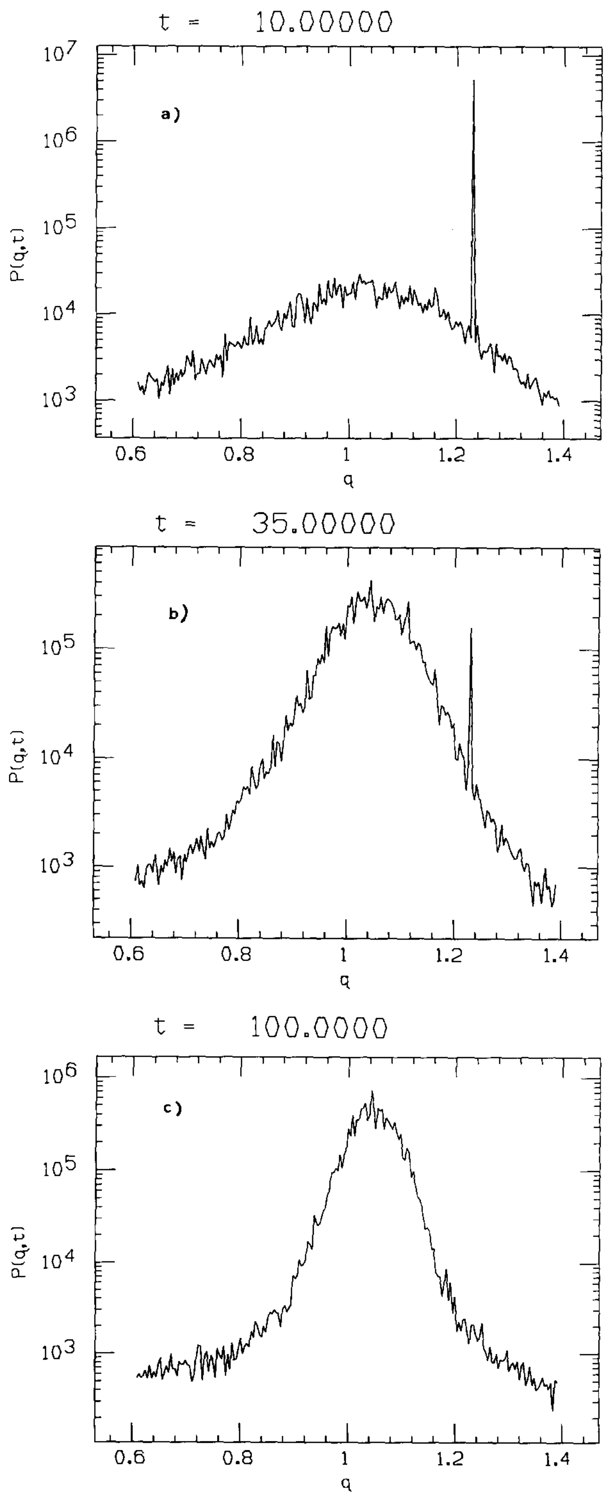

Fig. 5. Decay of an Eckhaus unstable state. (a) $t = 10$, (b) $t = 30$, and (c) $t = 100$.

different from the associated to $q - K_0$. We see from our simulations that noise along the trajectory is still inefficient in driving the system towards the more stable states, as the one characterized by q_M.

In the experiments by Lowe and Gollub [15] in an electrohydrodynamic system, it was found that, although a mode near $q - K_0$ dominated at intermediate times, a mode near q_M was finally selected. These observations might be a consequence of the two-dimensional nature of the experimental system, where dislocation motion plays an important role and fluctuations are more efficient. It is interesting to note that simulations of the SH model in two dimensions and in the absence of noise [16] end up in the state $q - K_0$, in contrast with the one-dimensional case [11].

The state we find to be selected, the one whose N corresponds to the mode of fastest growth, is probably only a very long-lived metastable state, and strong enough fluctuations will induce the system to abandon it in accessible time scales. But it should be stressed that such big noise will greatly reduce the correlation length r_0 and induce frequent changes in N, so that the concept of "pattern selection" will loose its meaning. In such a situation, probabilistic arguments based on the entropy of the different configurations will be probably more relevant than variational ones.

Acknowledgements

This work has been supported by NATO, within the program "Chaos, order and patterns; aspects on nonlinearity", project number 890482, by the Supercomputer Computations Research Institute, which is partially funded by the US Department of Energy contract No. DE-FC05-85ER25000, by the Dirección General de Investigación Científica y Técnica, contract number PB 89-0424, and by the Universitat de les Illes Balears. Most of the calculations reported here have been performed in the 64k-node Connection Machine at the Supercomputer Computations Research Institute. We are indebted to Dr. Ken Elder and Dr. Martin Grant for many useful discussions.

References

[1] A.C. Newell, G. Ahlers, in: Lectures in the Science of Complexity, ed. by D.L. Stein (Addison-Wesley, Redwood, 1989);
J.E. Wesfreid, H.R. Brand, P. Manneville, G. Albinet and N. Boccara, eds. Propagation in Systems Far from Equilibrium (Springer, Berlin, 1988).
[2] J. Swift and P.C. Hohenberg, Phys. Rev. A 15 (1977) 319.
[3] G. Ahlers, M.C. Cross, P.C. Hohenberg and S. Safran, J. Fluid Mech. 110 (1981) 297.
[4] I. Rehberg, F. Hörner, L. Chiran, H. Richter and B.L. Winkler, preprint;
I. Rehberg, F. Hörner, S. Rasenat and L. Chiran, Physica D, these proceedings.
[5] H. Xi, J. Viñals, and J.D. Gunton, Physica A 177 (1991) 356.
[6] S. Rasenat, E. Braun and V. Steinberg, Phys. Rev. A 43 (1991) 5728.
[7] Y. Pomeau, and S. Zaleski, J. Phys. (Paris) 42 (1981) 515;
M.C. Cross, P.G. Daniels, P.C. Hohenberg and E.D. Siggia, J. Fluid Mech. 127 (1983) 155.
[8] Y. Pomeau, and P. Manneville, J. Phys. (Paris) 40 (1979) L609.
[9] W. Eckhaus, Studies in Non-linear Stability Theory, Springer Tracts in Natural Philosophy, Vol. 6 (Springer, Berlin, 1965).
[10] L. Kramer and W. Zimmermann, Physica D 16 (1985 221.
[11] L. Kramer, H.R. Schober and W. Zimmermann, Physica D 31 (1988) 212.
[12] K. Tsiveriotis, R.A. Brown, Phys. Rev. Lett. 63 (1989) 2048.
[13] J. Viñals, E. Hernández-García, M. San Miguel and R. Toral, Phys. Rev. A 44 (1991) 1123.
[14] H.R. Schober, E. Allroth, K. Schroeder and H. Müller-Krumbhaar, Phys. Rev. A 33 (1986) 567.
[15] M. Lowe and J.P. Gollub, Phys. Rev. Lett. 55 (1985) 2575.
[16] E. Bodenschatz, M. Kaiser, L. Kramer, W. Pesch, A. Weber and W. Zimmermann, in: New Trends in Nonlinear Dynamics and Pattern Forming Phenomena, eds. by P. Coullet and P. Huerre (Plenum, New York, 1989).