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Phase transitions induced by microscopic disorder: A study based on the order parameter expansion

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A B S T R A C T

Based on the order parameter expansion, we present an approximate method which allows us to reduce large systems of coupled differential equations with diverse parameters to three equations: one for the global, mean field, variable and two which describe the fluctuations around this mean value. The method is based on a systematic perturbation expansion and can be applied around the vicinity of the homogeneous state. With this tool we analyze phase transitions induced by microscopic disorder in three prototypical models of phase transitions which have been studied previously in the presence of thermal noise. We study how macroscopic order is induced or destroyed by time-independent local disorder and analyze the limits of the approximation by comparing the results with the numerical solutions of the self-consistency equation which arises from the property of self-averaging. Finally, we carry on a finite-size analysis of the numerical results and calculate the corresponding critical exponents.

1. Introduction

The effect of time-dependent noise in extended dynamical systems has been the subject of intensive study in the last years [1]. Besides the expected disordering role, it has been found that some kind of order at the macroscopic level can appear by increasing the intensity of the noise. Examples of this paradoxical result include stochastic resonance [2,3], or enhancement of the effect of an external forcing under the right amount of noise, coherence resonance [4] (also named as stochastic coherence [5]) where a dynamical system displays optimal periodicity at the right noise value, noise sustained patterns, structures and fronts [6,7], phase transitions where a more ordered phase appears when increasing the noise intensity [8,9], etc.

In a very general framework, it has been argued that the resonance with an external forcing can also be achieved when the time-dependent noise is replaced by a more general source of disorder. This includes natural diversity or heterogeneity, competitive interactions, disorder in the network of connectivities, etc. and can appear in driven bistable and excitable systems [10–12], in linear [13] and chaotic [14] oscillators and in a variety of other systems [15–23]. A unifying treatment of the role of noise and diversity for non-forced excitable systems, has been developed in [11].

In this work we examine the effect that structural disorder or diversity, in the form of quenched noise, has on some prototypical models of phase transitions which have been studied thoroughly in the presence of noise. From the practical point of view, the models we will be considering bear some similarities with random field, or impurities, models. As tool of investigation we will refine a previously developed order parameter expansion method of approximating large systems of coupled differential equations [24–28] with diverse parameters. This allows the reduction of the large set of differential equations to just three: one for the global, mean field, value and two which describe the fluctuations around this mean value. Within this approximation (which is valid in the vicinity of the homogeneous state) we will analyze three different models and show its ease to deliver some understanding of the emergent properties of the global behavior. To find the limits of the order parameter expansion method we will compare the results with the solution of the self-consistency equation which arises from the property of self-averaging.

The chosen models are a set of globally coupled ϕ^4-systems both in the presence of additive and multiplicative quenched noise and the canonical model for noise-induced phase transition [8,9]. It will be seen that quenched noise can induce phase transitions (in, out and reentrant) of ordered phases.

The rest of the paper is organized as follows: In the next section we will describe the analytical methods, self-consistency and order parameter expansion; in Section 3 we will apply those methods to the showcase models and compare with the results of numerical

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simulations; the last section closes the paper with the discussion of the presented results.

2. Models and method

The type of models we will be considering in this paper is defined via differential equations for the dynamics of a set of real variables x_i:

$$\dot{x}_i = f(x_i, \eta_i; X), \quad i = 1, \ldots, N. \tag{1}$$

The time derivative $\dot{x}_i(t) = \mathrm{d}x_i(t)/\mathrm{d}t$ depends on the constant parameter η_i, a kind of quenched noise. The values $\{\eta_1, \ldots, \eta_N\}$ are independently drawn from a probability distribution $g(\eta)$ of mean H and variance σ^2. Coupling between the different dynamical equations is provided by the presence of the global variable or mean value $X(t) = \langle x_i(t)\rangle \equiv \frac{1}{N}\sum_{i=1}^N x_i(t)$ in Eq. (1). For a given realization of the η_i's variables the x_i's tend in the limit $t \to \infty$ to some asymptotic, stationary values which, in general, will depend on initial conditions. Some insight can be obtained if we write Eq. (1) as a relaxational dynamics [29] in a potential $V(x_i, \eta_i; X) = -\int^{x_i} \mathrm{d}x_i' f(x_i', \eta_i; X)$:

$$\dot{x}_i = -\frac{\partial V(x_i, \eta_i; X)}{\partial x_i}. \tag{2}$$

If the potential $V(x_i, \eta_i; X)$ is monostable for a particular value of X, then the variable $x_i(t)$ tends during the dynamical evolution towards the single minimum of $V(x_i, \eta_i; X)$. Note that the location of this minimum will change with time as X evolves. If, on the contrary, $V(x_i, \eta_i; X)$ presents several minima, the dynamics will tend towards one of the local minima of the potential.

In the following we will be interested in characterizing the stationary solution by the ensemble average value and fluctuations with respect to realizations of the quenched noise and initial conditions of the global variable X. We first review briefly the self-consistency method and then explain the approximate method based on the order parameter expansion.

2.1. Self-consistency

This method uses ideas borrowed from the Weiss molecular field theory [30], which is known to be exact for systems with long-range interaction or, equivalently, in which the interaction occurs through the global variable X, a mean field scenario, as it is our case. Let us denote the stable stationary solution of Eq. (1) by x_i^*. This is nothing but the absolute minimum of the potential $V(x_i, \eta_i; X)$. It will be a function of η_i and the global variable X, i.e. $x_i^* = x^*(\eta_i, X)$. For a given realization of the quenched noise variables η_i's, the value of the global variable must be obtained from the self-consistency relation $X = \frac{1}{N}\sum_{i=1}^N x^*(\eta_i, X)$. It is clear that for large N the sum can be replaced by an integral over the distribution $g(\eta)$ of the independent η_i's variables:

$$X = \int \mathrm{d}\eta g(\eta) x^*(\eta, X). \tag{3}$$

It is then assumed that one can identify the value of X, obtained solving this equation, as the desired ensemble average, i.e. assuming the property of self-averaging [31]. In general, the possible solutions X of the self-consistency Eq. (3) have to be found numerically. A possible scenario is that by changing some parameter (e.g. the root mean square σ or the mean H) of the distribution $g(\eta)$, the solutions bifurcate and the system then undergoes a phase transition between the possible solutions. We will present the results of this procedure in the examples below, but will not give any further details about the (in general, very involved) numerical method used to solve Eq. (3).

2.2. Order parameter expansion

For the development of this approximate method we assume, as in the previous subsection, that the number of degrees of freedom N is very large and then it is possible to substitute the mean value of the distribution $g(\eta)$ by the system average $H = \langle\eta_i\rangle = \frac{1}{N}\sum_{i=1}^N \eta_i$, the variance by $\sigma^2 = \langle(\eta_i - \langle\eta_i\rangle)^2\rangle = \frac{1}{N}\sum_{i=1}^N(\eta_i - \langle\eta_i\rangle)^2$, and similar expressions for other cases.

Our goal is to find an approximate equation describing the dynamics of the mean value variable X. To this end, we will expand the evolution equations in the deviations $\epsilon_i(t) = x_i(t) - X(t)$ of the dynamical variables from the mean value, and in the deviations $\delta_i = \eta_i - H$ of the parameters from their mean value. The Taylor expansion of Eq. (1) around the mean values up to second order gives:

$$\begin{aligned}\dot{x}_i &= f(X, H; X) + \epsilon_i f_x(X, H; X) + \delta_i f_\eta(X, H; X)\\ &\quad + \frac{1}{2}\epsilon_i^2 f_{xx}(X, H; X) + \epsilon_i\delta_i f_{x\eta}(X, H; X)\\ &\quad + \frac{1}{2}\delta_i^2 f_{\eta\eta}(X, H; X) + \cdots.\end{aligned} \tag{4}$$

With the usual notation $f_x(X, H; X) = \left.\frac{\partial f(x,\eta;X)}{\partial x}\right|_{x=X,\eta=H}$, etc. We now take averages and use that $\langle\epsilon_i\rangle = \frac{1}{N}\sum_{i=1}^N \epsilon_i = 0$ and $\langle\delta_i\rangle = \frac{1}{N}\sum_{i=1}^N \delta_i = 0$. Furthermore we have $\langle\delta_i^2\rangle = \sigma^2$ as the parameter distribution's variance. So when we average over Eq. (4) we are left with:

$$\begin{aligned}\dot{X} &= f(X, H; X) + \frac{1}{2}f_{xx}(X, H; X)\langle\epsilon_i^2\rangle + f_{x\eta}(X, H; X)\langle\epsilon_i\delta_i\rangle\\ &\quad + \frac{\sigma^2}{2} f_{\eta\eta}(X, H; X) + O(\langle\epsilon_i^3\rangle, \langle\epsilon_i^2\delta_i\rangle, \ldots).\end{aligned} \tag{5}$$

The evolution of X is then coupled to that of the second moment of the *snapshot probability density* $\Omega = \langle\epsilon_i^2\rangle = \frac{1}{N}\sum_{i=1}^N \epsilon_i^2$ and the so-called *shape parameter* [27] $W = \langle\epsilon_i\delta_i\rangle = \frac{1}{N}\sum_{i=1}^N \epsilon_i\delta_i$. We will now obtain evolution equations for these two variables. We follow closely the method of [25] but keep all terms up to second order in ϵ_i and δ_i. We start by subtracting (5) from (4) to obtain $\dot{\epsilon}_i = \dot{x}_i - \dot{X}$, which can then be replaced in $\dot{\Omega} = \langle 2\epsilon_i\dot{\epsilon}_i\rangle$, $\dot{W} = \langle\delta_i\dot{\epsilon}_i\rangle$. After some algebra, and neglecting terms of order $O(\langle\epsilon_i^3\rangle, \langle\epsilon_i^2\delta_i\rangle, \ldots)$ or higher, we get:

$$\begin{aligned}\dot{X} &= f(X, H; X) + \frac{\Omega}{2} f_{xx}(X.H; X) + f_{x\eta}(X, H; X)\, W\\ &\quad + \frac{\sigma^2}{2} f_{\eta\eta}(X, H; X),\end{aligned} \tag{6a}$$

$$\dot{\Omega} = 2\,\Omega\, f_x(X, H; X) + 2\, W f_\eta(X, H; X), \tag{6b}$$

$$\dot{W} = W f_x(X, H; X) + \sigma^2 f_\eta(X, H; X). \tag{6c}$$

In summary, within this approximation we have obtained a closed set of three differential equations (6a)–(6c). They have the feature of being coupled only in one direction, i.e. $W(t)$ is independent of the others and $\Omega(t)$ depends only on $W(t)$. These equations are valid to study the global behavior in the general case, including non-stationary collective states. Steady state conditions $\dot{W} = \dot{\Omega} = \dot{X} = 0$ lead to $W = -\sigma^2\frac{f_\eta}{f_x}$ and $\Omega = \sigma^2\frac{f_\eta^2}{f_x^2}$, and the equilibrium of variable X is given by the solution of:

$$0 = f + \frac{\sigma^2}{2}\left[f_{\eta\eta} + f_{xx}\frac{f_\eta^2}{f_x^2} - 2\frac{f_{x\eta} f_\eta}{f_x}\right], \tag{7}$$

where we have simplified notation $f = f(X, H; X)$, etc. As before, an analysis of the bifurcations of this equation will allow us to find the possible phase transitions of the model.

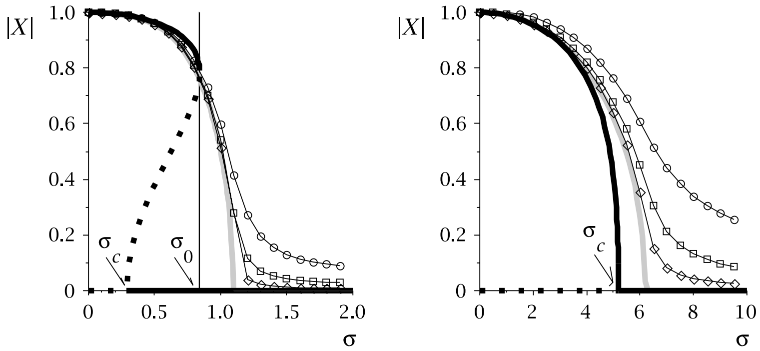

Fig. 1. Bifurcation diagram of the Landau–Ginzburg model with additive quenched noise. Order parameter expansion (thick black lines) predicts a second order transition for $C > a$ (*right*: $C = 10$, $a = 1$) while bistability (first order phase transition) appears for $C < 7a$ (*left*: $C = 1.5$, $a = 1$, the unstable solution is plotted as a dotted line). The self-consistency solution (grey line) does not show bistability in any case. Symbols show the results of numerical simulations of the evolution equations averaged over 10^3 realizations of the quenched noise variables η_i and initial conditions. $N = 10^3, 10^4, 10^5$ (circles, squares, diamonds, respectively).

Our results, Eqs. (6), differ slightly from those in the cited sources. In [24,26,27] the authors require the parameter to be additive, thus setting $f_\eta = 1$. In other works [25,28] any parameter dependence is allowed, but a coherent regime is required, such that terms of order $O(\langle \epsilon_i^2 \rangle)$ and higher are neglected.

3. Examples

After presenting the general development of the order parameter expansion method, we will now apply it to a few models of relevance in the field of phase transitions. Our purpose is to compare the results of our approximation with those coming from the self-consistency equation analysis as well as with numerical simulations of the different models. Solving the self-consistency equation requires in practice a complicated numerical calculation, while our treatment is simple and predicts the existence of phase transitions with reasonable accuracy in some cases.

3.1. Globally coupled Landau–Ginzburg model with additive quenched noise

The Landau–Ginzburg or ϕ^4 scalar field has been studied thoroughly from the analytical and numerical points of view, as a paradigmatic model undergoing a second order phase transition [32]. Here we are interested in this model in the case that the stochastic thermal fluctuations have been replaced by additive quenched noise, as an example of a random field scalar model [33]. The dynamical equations for the set of real variables x_i, $i = 1, \ldots, N$, are:

$$\dot{x}_i = a\,x_i - x_i^3 + C\,(X - x_i) + \eta_i. \tag{8}$$

The study of the model using the self-consistency relation equation (3) can be found in [12]. Here we want to use the order parameter expansion to derive the main properties of this model, in particular the existence of a phase transition as a function of the intensity σ of the fluctuations of the random fields η_i.

Following the steps from Section 2, we obtain the set of equations for the order parameter X and the fluctuations W, Ω:

$$\dot{X} = (a - 3\,\Omega)\;X - X^3 + H \tag{9a}$$

$$\dot{\Omega} = 2\,\Omega\left(a - C - 3X^2\right) + 2\,W \tag{9b}$$

$$\dot{W} = W\;\left(a - C - 3\,X^2\right) + \sigma^2. \tag{9c}$$

The steady state for the order parameter, Eq. (7), leads to:

$$0 = \left(a - 3\,\frac{\sigma^2}{\left(3\,X^2 + C - a\right)^2}\right) X - X^3 + H. \tag{10}$$

We now consider the case of zero average field $H = \langle \eta_i \rangle = 0$. In that case, Eq. (10) can have up to five real solutions. The trivial solution $X = 0$, always exists and it is stable (if $C > a$) whenever $\sigma > \sigma_c$, with

$$\sigma_c = \begin{cases} 0 & \text{if } a < 0, \\ \sqrt{\dfrac{a}{3}}(C - a) & \text{if } a > 0. \end{cases} \tag{11}$$

It turns out that for $C > 7a > 0$, the set of Eqs. (18b), (9a) and (9c) contains two additional real stable fixed point solutions $\pm X_0$ for $\sigma \le \sigma_c$. At $\sigma = \sigma_c$ it is $X_0 = 0$ and hence σ_c identifies a second order, continuous, phase transition (see right panel of Fig. 1). If $7a > C > a > 0$ the range of existence and stability of these two additional solutions extends up to $\sigma \le \sigma_0$, where $\sigma_0 \ge \sigma_c$ is given by:

$$\sigma_0 = \sqrt{\frac{4}{243}\,(2\,a + C)^3}. \tag{12}$$

Hence, in the range $\sigma \in [\sigma_c, \sigma_0]$ there is bistability between the $X = 0$ and the $\pm X_0$ solutions. Moreover, two additional symmetric unstable solutions $\pm X_1$ appear in this range. Therefore, the point σ_0 signals the appearance of a first order, discontinuous, phase transition (see Fig. 1, *left*). In that range, the three stable solutions coexist with the two unstable solutions.

From a microscopic point of view, the phase transition from the $|X| > 0$ to the $X = 0$ states can be explained as follows: for $\sigma = 0$, it is $\eta_i = 0$, $\forall i$; all variables end up in the same stationary value $x_i = \sqrt{a}$ or $x_i = -\sqrt{a}$ and the average value satisfies $|X| = \sqrt{a} > 0$. As the noise intensity increases, $\sigma > 0$, the average value $|X|$ tends to zero and the chances that individual values η_i are smaller than $-CX$ grow. This changes the minimum's sign in the (individual) potential. As a consequence the distribution of $\{x_i\}$ becomes bimodal and the mean value approaches zero.

The existence of a phase transition from order to disorder predicted by the order parameter expansion simple approximation scheme is confirmed by the numerical solution of the self-consistency equation (3) [12]. However, the transition appears to be always second order, so indicating the validity of the prediction of the approximate order parameter expansion in the limit of large coupling. In fact, the critical value σ_c predicted by the order–disorder transition, Eq. (11), deviates systematically from the value coming from the numerical integration of the self-consistency equation (3) for large coupling constant C, as shown in Fig. 2, although the relative error between the two values decreases as C increases.

We have also compared these predictions versus the results coming from intensive numerical simulations. In the simulations

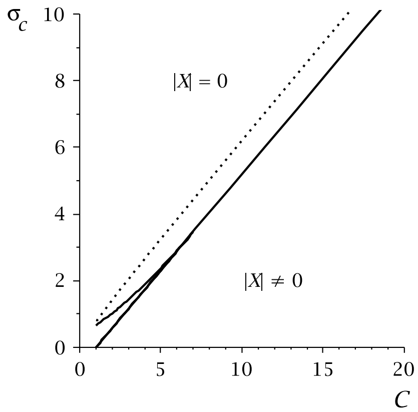

Fig. 2. Critical intensity of the additive quenched noise for the Landau–Ginzburg model versus coupling strength for $a = 1$. Prediction of order parameter expansion (11) as continuous line, exact solution (3) as dotted line. The order parameter expansion predicts a bistability region for $C < 7a$.

we have integrated the full set of Eq. (8) up to the steady state and, then, we have computed the order parameter $m = \langle\langle |X| \rangle\rangle$ and its fluctuations $\chi = \frac{N}{\sigma^2}[\langle\langle X^2\rangle\rangle - \langle\langle |X|\rangle\rangle^2]$. Here $X = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\langle\langle \cdots \rangle\rangle$ denotes an ensemble average with respect to realizations of the random variables η_i and initial conditions. The simulation results for the order parameter are indicated by symbols in Fig. 1. As usual, the transition from order to disorder is smeared out due to finite-size effects but the numerical simulations do approach the results of the self-consistency equation as the number N of variables increases. We have analyzed our data using standard finite-size scaling relations [34,35] and found that the dependence of the order parameter on σ can be well fitted by $m(\sigma, N) = N^{-b/2} f_m(\epsilon N^b)$ with $\epsilon = 1 - \sigma/\sigma_c$, $b \approx 0.33$ and being f_m a scaling function. See evidence in the left panels of Fig. 3 for two different values of the coupling constant. Note that this scaling relation implies that in the thermodynamic limit, the order parameter vanishes as $m(\sigma) \sim (\sigma_c - \sigma)^{1/2}$, the typical mean field result. Similarly, the fluctuations can be fitted by the form $\chi(\sigma, N) = N^c f_\chi(\epsilon N^b)$, with $c \approx 0.67$ and f_χ the appropriate scaling function, as demonstrated in the right panels of Fig. 3 again for two different values of the coupling constant. This implies that in the thermodynamic limit, the fluctuations diverge as $\chi(\sigma) \sim |\sigma_c - \sigma|^{-\gamma}$ with $\gamma = c/b \approx 2$.

3.2. Globally coupled Landau–Ginzburg model with multiplicative quenched noise

We now consider the case in which the quenched noise couples multiplicatively to the variable x_i:

$$\dot{x}_i = (a + \eta_i)\, x_i - x_i^3 + C\,(X - x_i)\,. \tag{13}$$

This model has been studied extensively in the case that the η_i's are independent white noises and it has been found that an increase in the noise intensity leads to a transition from disorder to order [1,36–38]. We want to compare the predictions of the self-consistency equation with the order parameter expansion and numerical simulations to check if a similar result holds in the case of quenched noise. Without coupling ($C = 0$) Eq. (13) is a prototype of supercritical pitchfork bifurcations (see e.g. in [39]) with two possible sets of solutions: $x_i = 0$ is the stable solution whenever $a + \eta_i \leq 0$, or $x_i = \pm\sqrt{a + \eta_i}$ are stable solutions and $x_i = 0$ is unstable for $a + \eta_i > 0$.

To study the consequences of coupling, $C > 0$, we use the above developed order parameter expansion approximation. After

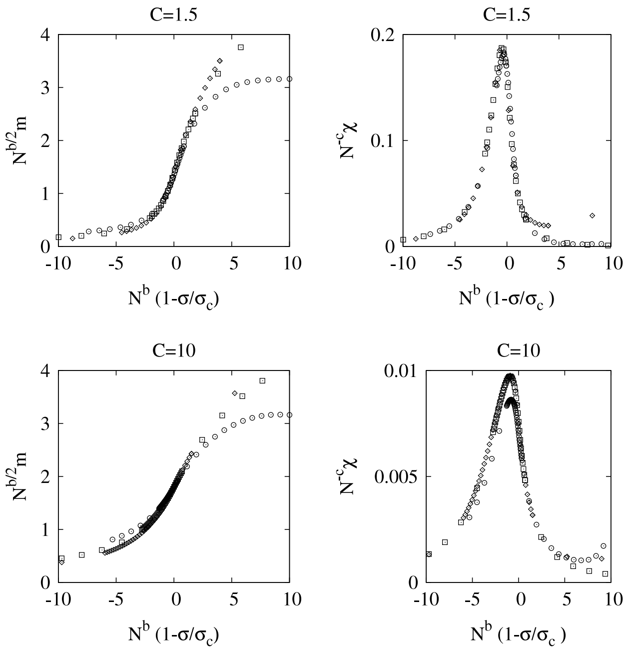

Fig. 3. Finite-size scaling analysis of the Landau–Ginzburg model with additive quenched noise. Rescaled simulation data for low coupling (*top graphs*, $\sigma_c = 1.094$) and high coupling (*bottom graphs*, $\sigma_c = 6.203$). Ensemble average m (*left*) and fluctuations χ (*right*) as defined in the main text. Exponents: $b = 0.33$, $c = 0.67$. Ensemble sizes: $N = 10^3$, 10^4, 10^5 (circles, squares, diamonds). In all cases: $a = 1$. Here, σ_c has been determined to a high degree of accuracy by using the numerical solution of Eq. (3).

setting $H = \langle \eta_i \rangle = 0$, the equations are:

$$\dot{X} = aX - X^3 - 3X\Omega + W, \tag{14a}$$

$$\dot{\Omega} = \left(2a - 6X^2 - 2C\right)\Omega + 2XW, \tag{14b}$$

$$\dot{W} = \left(a - 3X^2 - C\right)W + X\sigma^2. \tag{14c}$$

The equilibrium condition (7) leads to:

$$0 = aX - X^3 - \frac{3X^3\sigma^2}{\left(C - a + 3X^2\right)^2} + \frac{X\sigma^2}{C - a + 3X^2}. \tag{15}$$

Similarly to the uncoupled case this equation has two different regimes of solutions: On one hand, if $a \geq 0$ the stable solutions of Eq. (14) are $X = \pm\sqrt{a}$ for $\sigma = 0$. As σ increases, $|X|$ monotonically increases as well (see Fig. 4, *left*). On the other hand, if $a < 0$ then $X = 0$ is a stable solution for small σ. At some value σ_c it becomes unstable and a fork of solutions grows out of zero (see Fig. 4, *right*). σ_c is determined by Eq. (15) and is related to a and C by:

$$\sigma_c^2 = a\,(a - C)\,. \tag{16}$$

σ_c identifies a second order phase transition from disorder to order (i.e. from $X = 0$ to $X \neq 0$). In this case of $a < 0$, the value σ_c grows monotonously with coupling strength C, a rather counterintuitive observation, since it means that the coupling hinders the ordering and more structural disorder is needed to induce macroscopic order (Fig. 5).

The numerical solution of the self-consistency equation (3) is qualitatively similar to the results of the order parameter expansion approximation, however $|X|$ does not increase monotonically with increasing σ. It rather reaches a maximum and decreases after that approaching zero asymptotically. Note that his is not a (reentrant) phase transition since $|X| = 0$ is only reached for $\sigma \to \infty$.

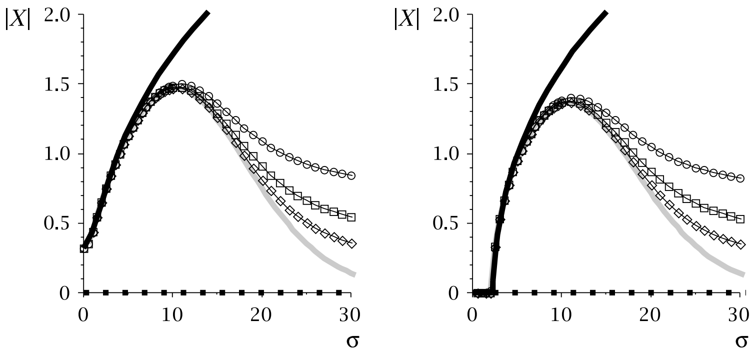

Fig. 4. Bifurcation diagram of the Landau–Ginzburg model with multiplicative quenched noise. Positive values of a show order without noise (*left:* $a = 0.1$), whereas negative values show order only with a finite value of the noise intensity (*right:* $a = -0.5$). In both panels it is $C = 10$. The order parameter expansion approximation scheme gives a monotonous solution while the exact solution of Eq. (3) reaches a maximum and decreases for large σ (grey line). Symbols are the result of direct numerical simulations of Eq. (13) averaged over 10^3 realizations of the quenched noise variables η_i and initial conditions. $N = 10^3, 10^4, 10^5$ (circles, squares, diamonds).

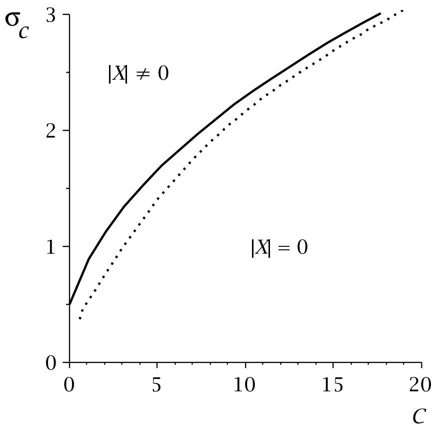

Fig. 5. Critical noise for bifurcation versus coupling strength for $a = -0.5$ for the Landau–Ginzburg model with multiplicative quenched noise. The prediction by the order parameter expansion is shown as continuous line, the exact solution (3) is shown as dotted line.

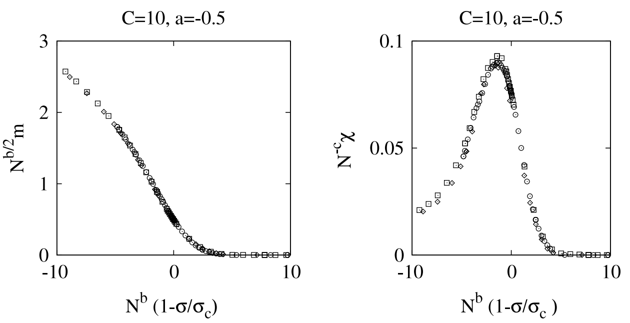

Fig. 6. Finite-size scaling analysis of the Landau–Ginzburg model with multiplicative noise for $a = -0.5$, $C = 10$. Rescaled ensemble average m (*left*) and fluctuations χ (*right*) of 10^3 numerical simulations with $N = 10^3, 10^4, 10^5$ (circles, squares, diamonds). Critical point, as from Eq. (3), is $\sigma_c = 2.169$; exponents: $b = 0.5$, $c = 0.5$.

The simulation results for the order parameter are shown as symbols in Fig. 4. At this scale no finite-size effects can be seen at the phase transition. In a thorough data analysis with finite-size scaling relations at σ_c, in the way we did in the first example, we found exponents of $b \approx c \approx 0.5$ to fit the order parameter and fluctuations (see Fig. 6). These scaling relations imply, again in the thermodynamic limit, that the order parameter vanishes and the fluctuations diverge as $m(\sigma) \sim (\sigma_c - \sigma)^{1/2}$ and $\chi(\sigma) \sim |\sigma_c - \sigma|^{-\gamma}$, with $\gamma = c/b \approx 1$ respectively.

3.3. *Canonical model for noise-induced phase transitions*

As the last example we will study a model for which a genuine phase transition induced by multiplicative noise has been shown [8,9] with the feature that the ordered phase is reentrant, it only exists for intermediate noise intensities. The equation for an individual element is:

$$\dot{x}_i = -x_i \left(1 + x_i^2\right)^2 + \left(1 + x_i^2\right) \eta_i + C\,(X - x_i) \tag{17}$$

and the reduced system according to Section 2.2 (again setting $\langle \eta_i \rangle = 0$) reads:

$$\dot{X} = -X\left(1+X^2\right)^2 + \frac{1}{2}\left[-12\,\left(1+X^2\right)X - 8X^3\right]\Omega + 2XW \tag{18a}$$

$$\dot{\Omega} = 2\,\Omega\left[-\left(1+X^2\right)^2 - 4X^2\left(1+X^2\right) + C\right] + 2\,W\left(1+X^2\right) \tag{18b}$$

$$\dot{W} = W\left[-\left(1+X^2\right)^2 - 4X^2\left(1+X^2\right) + C\right] + \sigma^2\left(1+X^2\right). \tag{18c}$$

The equilibrium condition (7) becomes

$$0 = -X\left(1+X^2\right)^2 + \frac{\left(-\left(6+6X^2\right)X - 4X^3\right)\sigma^2\left(1+X^2\right)^2}{\left(1+6X^2+5X^4+C\right)^2} + 2\,\frac{X\sigma^2\left(1+X^2\right)}{1+6X^2+5X^4+C}. \tag{19}$$

Eqs. (18) have the stable solution $X = 0$ for $\sigma < \sigma_c$ or a pair of symmetric solutions $X \neq 0$ for $\sigma > \sigma_c$, when $X = 0$ becomes unstable (see left panel of Fig. 7). The value of σ_c indicates the location of a second order phase transition. It follows from analyzing the Jacobian of (18a)–(18c) and calculates to:

$$\sigma_c = \frac{1+C}{\sqrt{2C-4}}. \tag{20}$$

Accordingly, a minimal coupling $C > 2$ is necessary to induce the phase transition. An analysis of this relation shows that σ_c has a minimum with respect to C. Therefore, see Fig. 8, the transition is predicted to be reentrant with respect to C: the ordered phase only exists in a range of values for C, with the surprising prediction that too a large coupling destroys the ordered state. The predictions of the order parameter expansion are in qualitative agreement with those obtained after solving the self-consistency equation. However, whereas the order parameter expansion predicts incorrectly that the order parameter monotonously

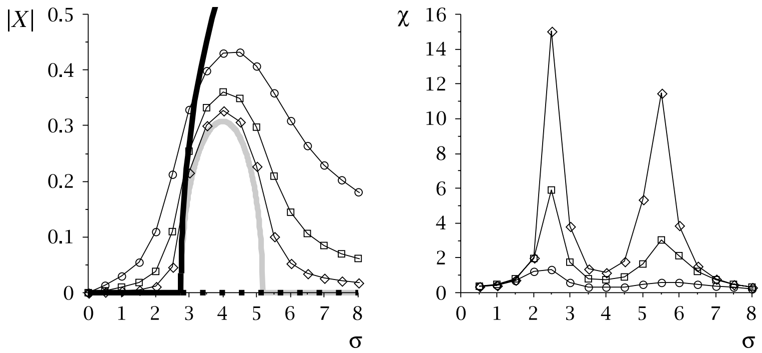

Fig. 7. Bifurcation diagram of model (17) (*left*). Order parameter expansion (thick black line) and exact solution (grey line) together with the ensemble average of 10^3 numerical simulations for $N = 10^3, 10^4, 10^5$ (circles, squares, diamonds). Coupling is $C = 10$. On the *right* the unscaled fluctuations are shown.

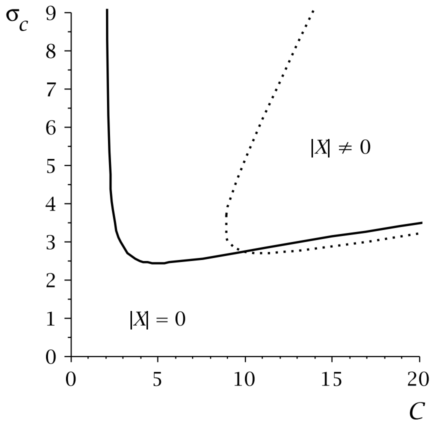

Fig. 8. Critical noise intensity versus coupling strength for model (17). The prediction (20) of the order parameter expansion (continuous line) and the exact solution (3) as dotted line. Only the latter shows reentrance with respect to σ.

increases with σ, as shown in Fig. 7, the self-consistency equation instead predicts that the transition is reentrant also with respect to the quenched noise intensity σ, see Fig. 8. Both reentrant behaviors were observed in the case of time-dependent noise [8,9].

Again we have compared the predictions with the numerical integration of the set of Eq. (17). The simulation results are shown as symbols in Fig. 7. Due to finite-size effects the theoretical results are approached with increasing number N of particles, reentrance and the dependence of σ_c from C are observed. Analyzing the data as we have done with the other examples, we find exponents for the scaling relations of $b \approx 0.33$ and $c \approx 0.67$. As in the first case this implies the relations $m(\sigma) \sim (\sigma_c - \sigma)^{1/2}$ and $\chi(\sigma) \sim |\sigma_c - \sigma|^{-\gamma}$, $\gamma = c/b \approx 2$ in the thermodynamic limit. Fig. 9 summarizes the fitted simulation data.

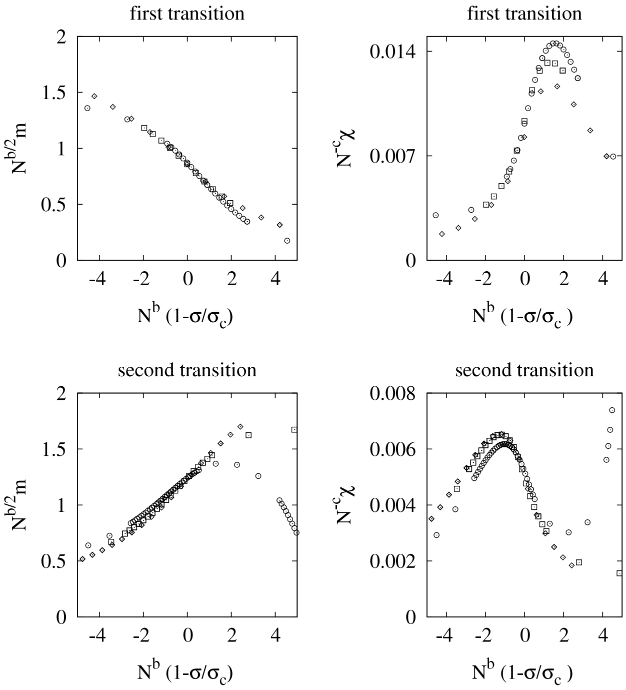

Fig. 9. Rescaled simulation data for the first (*top*) and the second (*bottom*) phase transition ($C = 10$). Mean value (*left*) and fluctuations (*right*) for ensembles of $N = 10^3, 10^4, 10^5$ (circles, squares, diamonds). Critical points, as from Eq. (3), are $\sigma_c = 2.749$ for the first and $\sigma_c = 5.169$ for the second transition. Exponents are: $b = 0.33$, $c = 0.67$.

4. Conclusions

In this paper we have constructed an approximate analytical scheme based on the order parameter expansion [24–28] to study the macroscopic behavior of extended systems which are globally coupled. We have used the method to study in detail the phase diagram of three widely used models of phase transitions in scalar systems: the Landau–Ginzburg scalar model with both additive and multiplicative quenched noise and a genuine model for noise-induced phase transitions where time-dependent noise has been replaced by quenched, time-independent noise coupled multiplicatively to the dynamical variable [8].

We have compared the results of our simple approach with those coming from a numerically involved, but in principle exact, treatment based on the self-consistency relation and with extensive numerical simulations of the corresponding dynamical equations for each model. In the case of additive noise, the main result is that macroscopic order is destroyed when increasing the intensity of the quenched noise. In the other two cases, when noise appears multiplicatively, we find that macroscopic order appears for an intermediate value of the intensity of the quenched noise. Since the quenched noise can represent, for instance, diversity or heterogeneity, it appears paradoxically that some amount of structural disorder is needed in order to observe macroscopic order.

Furthermore it has been shown numerically, that all investigated models follow a finite-size scaling law and the exponents have been determined. It suggests a common universality class for the Landau–Ginzburg model with additive quenched noise and the canonical model for noise-induced phase transitions, whereas the Landau–Ginzburg model with multiplicative quenched noise yields different exponents. A more detailed analysis of the finite-size relations and their possible dependence with the system parameters will be presented elsewhere [40].

The method of order parameter expansion, which we lead consistently up to terms of second order, is a tool which reduces large systems to only a couple of reduced variables. The advantage is its very easy management. In this paper we have proven that reliable conclusions can be drawn with that method in some cases. Since it is an expansion around mean values, designed to be applied near the homogeneous state, the method yields good results for low values of the intensity of the quenched noise or for high synchronization of the subunits. Otherwise, the method might not be reliable. As a consequence the reentrant phase transitions were not predicted in the studied cases for multiplicative noise. It is an open issue how to modify the method in order to predict the reentrant transitions.

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