Noisy continuous-opinion dynamics

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Abstract. We study the Deffuant *et al* model for continuous-opinion dynamics under the influence of noise. In the original version of this model, individuals meet in random pairwise encounters after which they compromise or not depending on a confidence parameter. Free will is introduced in the form of noisy perturbations: individuals are given the opportunity to change their opinion, with a given probability, to a randomly selected opinion inside the whole opinion space. We derive the master equation of this process. One of the main effects of noise is to induce an order–disorder transition. In the disordered state the opinion distribution tends to be uniform, while for the ordered state a set of well defined opinion clusters are formed, although with some opinion spread inside them. Using a linear stability analysis we can derive approximate conditions for the transition between opinion clusters and the disordered state. The master equation analysis is compared with direct Monte Carlo simulations. We find that the master equation and the Monte Carlo simulations do not always agree due to finite-size-induced fluctuations that we analyze in some detail.

Contents

1. Introduction

The application of techniques and tools from nonlinear and statistical physics to explain the dynamics of opinion changes in a society has become a topic of interest in recent years [1]. A society can be thought of as a complex system composed of a large number of interacting individuals with diverse opinions. These opinions are not necessarily static, and can evolve due to a variety of internal as well as external factors, such as the influence of advertising and acquaintances, amongst others. As a result of this evolution, a consensus opinion could emerge (a vast majority of individuals adopting a similar opinion), or the population could fragment into a number of clusters. To analyze the process of opinion formation, several models inspired from statistical mechanics have been developed. In those, the opinion held by an individual is a dynamical variable which evolves by some rules, usually with an important stochastic ingredient [2]. Models can be divided into two broad categories: *discrete models* where the opinion can only adopt a finite set of values [3]–[5], and *continuous models* where the opinion of an individual is expressed as a real number in a finite interval [6]–[9]. Discrete models are useful when analyzing cases in which individuals are confronted with a limited number of options (a political election, for example) where one is forced to choose amongst a finite set of parties. Continuous models are more suitable for analyzing cases in which a single issue (legalizing abortion, for example) is being considered and opinions can vary continuously from 'completely against' to 'in complete agreement'.

A continuous model introduced by Deffuant *et al* [6] has received much attention recently. This model implements the *bounded confidence* mechanism by which two individuals only influence the opinion of each other if their respective opinions differ less than some given amount. In other words, people holding too distant opinions on an issue will simply ignore each other and will, hence, keep their original opinions. It is only through the interaction of not too distant people that we manage to modify our opinion. This model was in turn inspired by the Axelrod model for the dissemination of culture [10]

and general threshold models [11] and has in turn inspired a large number of extensions and modifications [12]–[20].

In the Deffuant *et al* model individuals meet in random pairwise encounters in a given connectivity network, but the subsequent evolution is completely deterministic. This leads to final states in which either *perfect* consensus has been reached or the population splits into a finite number of clusters such that all individuals in one cluster have *exactly* the same opinion. We believe that such uniform states are not very realistic and some degree of discrepancy must appear within otherwise well defined clusters. In this paper, we introduce an additional element of randomness in the dynamics. The aim is to represent, certainly in a caricaturist manner, the element of *free will* present in all human decisions by which we do not follow blindly the opinion dictated by our relationships. Our aim is to analyze how the interplay between this *free will* and the interactions amongst individuals affects cluster formation in opinion dynamics. In the language of statistical mechanics, what we are doing is to add noise to the deterministic dynamics and analyze which aspects of the model are robust against the introduction of noise. Noise is introduced by allowing an individual opinion to change to another randomly chosen value in the whole opinion space. Under some circumstances this turns out to be equivalent to allowing each agent to return, at some random times, to a specific opinion preferred by that agent.

Our analysis, based upon numerical integrations of the corresponding master equation as well as Monte Carlo simulations, reveals new and interesting phenomenology. There exists a critical value m_c of the noise intensity, which depends on the confidence range, such that for noise larger than this value the system becomes disorganized and cluster formation does not occur. We provide a linear stability analysis that reproduces the order–disorder transition that occurs at m_c. For noise smaller than m_c, the steady state probability distributions in opinion space broaden with respect to the noiseless case, but still have large peaks and cluster formation can be unambiguously defined by looking at the maxima of the distributions. The cluster formation occurs by a series of bifurcations that mimic those that occur in the noiseless case.

An important aspect of our work, that we want to stress here, is that the numerical Monte Carlo simulations do not necessarily agree with the results from the master equation. This is due to the inherent finite-size-induced fluctuations that occur in the simulations. A similar warning is required when one tries to make inferences as regards possible applications of the model to real situations. For example, it is possible to find regions of bistability where dynamical transitions between a single cluster and two clusters occur. These transitions do not occur in the infinite-size thermodynamic limit taken routinely in most studies. This stresses the role that a finite size has in the dynamics of social systems [25].

This paper is organized as follows. The Deffuant *et al* model is briefly reviewed in section 2. The main results are presented in section 3, devoted to studying this model in the presence of noise. In section 4 we use a linear stability analysis to derive approximately the critical value of the noise intensity for the formation of opinion clusters. A summary and conclusions are presented in section 5.

2. Review of the Deffuant *et al* model

Let us consider a population with N individuals. We will denote by x_n^i the number representing the opinion on a given topic that individual i has at time step n. As

mentioned in section 1, the opinion is a real variable in a finite interval and, without loss of generality, we take $x_n^i \in [0,1]$. Initially, it is assumed that the values x_0^i for $i = 1, \ldots, N$ are randomly distributed in the interval $[0,1]$. A dynamics is introduced to reflect that individuals interact, discuss, and modify their opinions. In the original version of the model [6], at time step n two individuals, say i and j, are randomly chosen. If their opinions satisfy $|x_n^i - x_n^j| < \epsilon$, i.e. they are close enough, they are modified as

$$x_{n+1}^i = x_n^i + \mu(x_n^j - x_n^i), \qquad x_{n+1}^j = x_n^j + \mu(x_n^i - x_n^j), \tag{1}$$

but otherwise they remain unchanged. Whether the opinions have been updated or not, time increases: $n \to n + 1$. As a consequence of the iteration of this dynamical rule, the system reaches a static final configuration which, depending on the values of the parameters ϵ and μ, can be a state of full consensus where all individuals share the same opinion, or one of fragmentation with several opinion clusters. It is customary to introduce the time variable $t = n \Delta t$, where $\Delta t = 1/N$, measuring the number of opinion updates per individual, or number of Monte Carlo steps (MCS).

The parameter μ is restricted to the interval $(0, 0.5]$. It determines the time for convergence between individuals as well as the number of final clusters [23, 24]. For small values of μ, the individuals slightly change their opinions during the meeting, while for $\mu = 0.5$ the interacting individuals fully compromise and, after the meeting, they share the same opinion. As in most studies, we will adopt from now on in this paper the value $\mu = 0.5$. The parameter ϵ, which runs from 0 to 1, is the confidence parameter. Starting from uniformly distributed random values for the initial opinions, the typical realization is that for large values, $\epsilon \geq 0.5$, the system evolves to a state of consensus where all individuals share the same opinion and that, decreasing ϵ, the population splits into opinion clusters separated by distances larger than ϵ.

The process can be described in terms of a master equation for the probability density function $P(x, t)$ for an individual opinion x at time t. The appendix contains a derivation of this master equation in the presence of the additional noise term described in section 3. Equation (A.4) with $m = 0$ is the master equation for the noiseless original Deffuant *et al* model, first obtained in [12]. A detailed analysis based on its numerical integration [12, 20][1] shows that there are four basic modes of cluster appearance, dominance, or splitting, which are called *bifurcations* [12, 20] in this context (see figure 1): nucleation of two minor clusters symmetrically from the center of the opinion interval (type 1, such as the birth of two minor clusters from the boundaries at $\epsilon = 0.5$ in figure 1); nucleation of two major clusters from the central one (type 2, as occurs at $\epsilon \approx 0.266$ in figure 1); nucleation of a minor central cluster (type 3, as occurs at $\epsilon \approx 0.222$ in the figure); and, finally, sudden increase of the mass of that central cluster accompanied by a sudden drift outwards in the location of the two major clusters (type 4, at $\epsilon \approx 0.182$ in the figure). In this sequence 'major' opinion clusters contain a high fraction of the population, while 'minor' clusters contain a much smaller fraction (of the order of 10^{-2} or smaller). The bifurcation pattern repeats itself as ϵ decreases even further.

It is important to emphasize that the situation depicted in figure 1 is the result for steady solutions of the master equation attained at long times starting from a uniform initial distribution. Many other steady solutions of the master equation exist. In

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[1] In order to compare with the results of [12] we note that our parameter ϵ is related to their parameter Δ by $\epsilon = 1/2\Delta$.
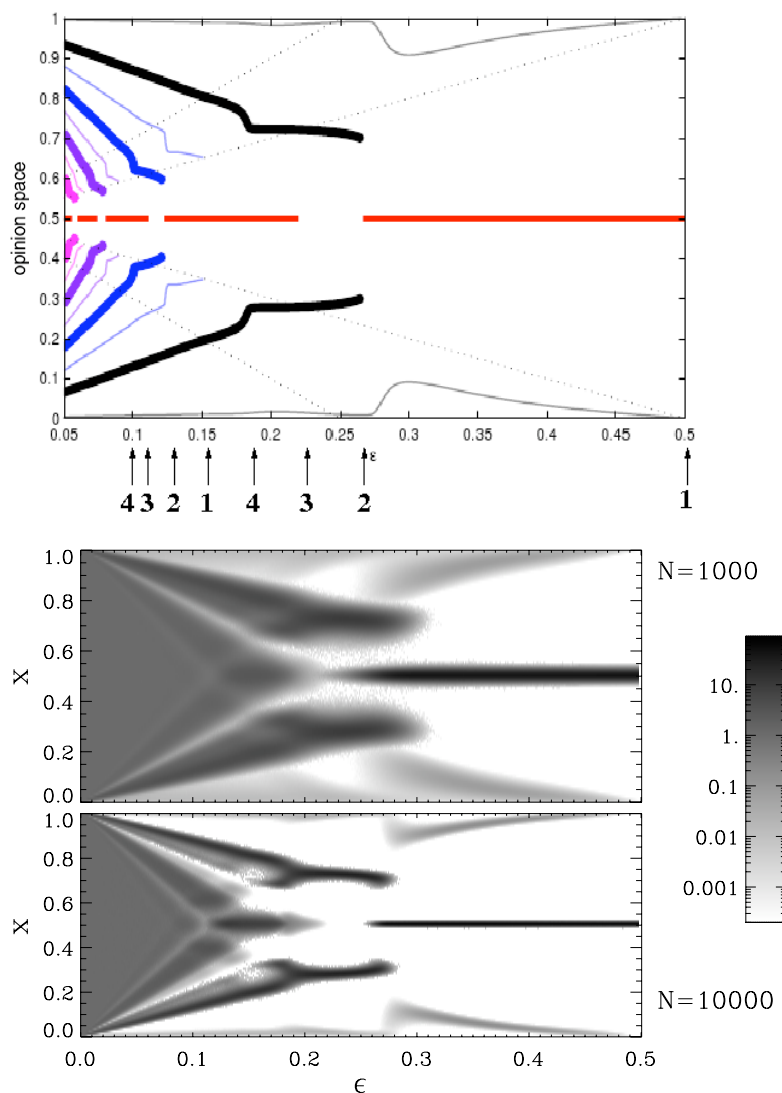
Figure 1. Top panel: bifurcation diagram of the noiseless Deffuant *et al* model reproduced from reference [22], coming from a numerical analysis of the master equation of the model [12, 21]. Lines show the position of the opinion clusters as a function of ϵ. With arrows we indicate the location of the four basic bifurcation types: type 1 at $\epsilon \approx 0.5$, type 2 at $\epsilon \approx 0.266$, type 3 at $\epsilon \approx 0.22$, type 4 at $\epsilon \approx 0.182$, and this pattern repeating as ϵ decreases even further [21]. In the bottom panels we plot in a logarithmic grayscale the asymptotic probability distribution $P_\infty(x)$ (values smaller than 2×10^{-4} are plotted white), as a function of ϵ, resulting from extensive numerical simulations of the microscopic model with $N = 10^3$ and $N = 10^4$ agents, respectively. Note that a single realization using the Monte Carlo microscopic rules of the model leads to a probability distribution which is a sum of delta-functions. The distributions displayed in the panels are the result of an average over 10^5 realizations for $N = 10^3$ and 2×10^4 realizations for $N = 10^4$ and a histogram bin size $\Delta x = 0.01$. In all cases, in the master equation as in the Monte Carlo simulations, the initial condition represents opinions which are random and uniformly distributed in the interval $[0, 1]$.

particular, any combination of delta-functions is a steady solution of the noiseless master equation provided they are separated by more than a distance ϵ. We stress also that this analysis based upon the master equation corresponds to the limit case where the number of individuals N tends to infinity. In Monte Carlo simulations of the microscopic rules, or in practical applications with a necessarily finite number of individuals, some features need to be considered. It is still true that each realization ends up with a small number of clusters (of major type and of minor type), all individuals within a cluster holding exactly the same opinion. However, the exact location of the clusters might vary with respect to the master equation prediction and minor clusters might not appear depending on the particular realization and the total size of the population. Furthermore, there could be realizations in which even the number of observed major clusters differs from the prediction of the master equation. These effects are more pronounced for smaller numbers of individuals. In the same figure 1 we have plotted the distribution of observed clusters, averaged over many realizations for two different numbers of individuals N, where the aforementioned properties can clearly be observed. For instance, for $\epsilon = 0.28$, the master equation predicts that there should be only one major cluster, centered at $x = 0.5$, and two minor clusters. However, in almost half of the realizations with $N = 1000$ individuals the opinions split instead into two major clusters centered around $x = 0.28$ and 0.72 and, eventually, some minor extreme clusters.

3. Effect of noise

Noise is introduced as a random change of an individual's opinion. Specifically, we modify the dynamics as follows: at time step n the original dynamical rule, equation (1), applies only with probability $1 - m$. Otherwise, a randomly chosen individual i changes opinion to a new value x_{n+1}^i drawn from a uniform distribution in the interval $[0, 1]$ and all other opinions remain unchanged. The probability m is a measure of the noise intensity. Note that, quite generally, this rule is equivalent to allowing each agent to return to a specific, basal, opinion preferred by that agent, provided that the basal opinions are randomly distributed amongst the agents. In this section, we will study in detail the effect of this new ingredient in the dynamical evolution of the model. We analyze both the results coming from a numerical integration of the master equation as well as numerical Monte Carlo-type simulations of the microscopic rules of the model.

3.1. Master equation approach

The appendix contains a derivation of the master equation, equation (A.4), appropriate for this process. We have first obtained the asymptotic distribution $P_\infty(x) = \lim_{t \to \infty} P(x, t)$ of the master equation starting from a suitable initial condition $P(x, t = 0)$. As in other studies, we assume that the initial condition represents a uniform distribution in opinion space, i.e. $P(x, t = 0) = 1$ for $x \in [0, 1]$ and $P(x, t = 0) = 0$ otherwise. For $m = 0$ the steady state distribution $P_\infty(x)$ is a sum of delta-functions located at particular points. In the case $m > 0$ the steady distributions are no longer delta-functions but still are peaked around some particular values if ϵ is not too small or m not too large. We have plotted in figure 2 the master equation steady probability distributions $P_\infty(x)$ as a function of the parameter ϵ for two different values of the noise intensity. In the small noise case
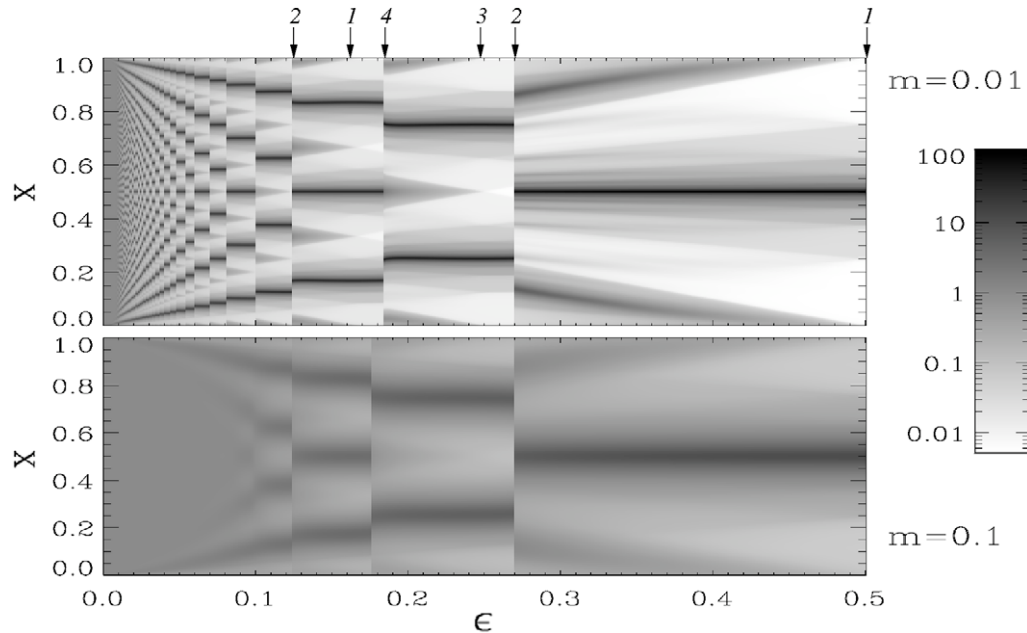
Figure 2. Plot, in a logarithmic grayscale, of the asymptotic probability distributions $P_\infty(x)$ as functions of ϵ, obtained after a numerical integration of the master equation (A.4) for $m = 0.01$, top panel, and $m = 0.1$, bottom panel. We have used a fourth-order Runge–Kutta method with a time step $\Delta t = 0.1$ for the time evolution and Simpson's rule for the integrals in x-space with a discretization $\Delta x = 1/M$, $M = 2000$. We have checked for some test cases that smaller time or discretization steps do not significantly change the results. As in figure 1 we show with arrows the location of the bifurcation points.

$m = 0.01$ it is still possible, for not too small ϵ, to identify the same type of bifurcation as in the noiseless case by looking at the maxima of the probability distributions: a type 1 bifurcation at $\epsilon = 0.5$ where minor clusters begin to form at $x = 0$ and 1; a type 2 bifurcation at $\epsilon \approx 0.2695$ where the distribution switches from having one single maximum at $x = 0.5$ to having two maxima of equal height located at $x \approx 1/4$ and $3/4$; a type 3 bifurcation at $\epsilon \approx 0.250$ where a central maximum begins to grow; a type 4 bifurcation at $\epsilon \approx 0.1835$ where three equally spaced maxima of equal height at $x \approx 1/6$, $x = 1/2$ and $x \approx 5/6$ appear. This pattern of bifurcations repeats as ϵ decreases even further. However, type 1 and type 3 bifurcations are somewhat ambiguous to define since the relative importance of the minor clusters actually increases continuously instead of sharply increasing when new maxima begin to form. We have observed numerically that, at variance with the noiseless case, the location of the major clusters (defined as the absolute maxima of the distribution) does not vary with ϵ until a new bifurcation of type 2 or type 4 is reached. Similarly, we observe that these maxima are regularly located at $x \approx 1/k, 3/k, \ldots, (k-1)/k$ for $k = 2, 4, 6, \ldots$. We have not been able to find a suitable explanation for these interesting properties.

The same general structure can be observed in the case of larger noise $m = 0.1$, although the distributions are much broader now. The location of the main bifurcation

points can be located at $\epsilon \approx 0.2695$ (type 2) and $\epsilon \approx 0.1755$ (type 4). The type 1 and type 3 transitions are very imprecisely defined, especially for smaller values of ϵ.

For both noise values, one observes that clusters become less defined and finally are replaced by a more or less unstructured distribution for ϵ below a critical value ϵ_{c} which increases with m. Alternatively, one realizes the existence of a critical value $m_{\mathrm{c}} = m_{\mathrm{c}}(\epsilon)$, increasing with ϵ, above which the cluster structure disappears from the steady distribution.

A somewhat expected feature that emerges from the data shown in figure 2 is that the width of the steady distributions grows with the noise intensity. An explicit expression for the width of the single maximum present when $\epsilon \geq 1$ (i.e. when all individuals are allowed to interact) could be obtained from the master equation, since in this case the moments form a closed hierarchy. Defining the moments M_1 and M_2 as in the appendix, the variance $\sigma^2 = M_2 - M_1^2$ satisfies

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}t} = -\sigma^2 + \frac{m}{12}. \tag{2}$$

In this limiting case (where the main feature of the model, bounded confidence, has been lost since everybody is able to interact with everybody), the variance reaches a steady state in which the width increases with noise as $\sigma \sim m^{1/2}$.

3.2. Comparison with Monte Carlo simulations

Once the master equation predictions have been established, and before comparing with the results coming from the Monte Carlo numerical simulations using the microscopic rules, a word of warning is, as in the noiseless case, required. In simulations with a finite number N of individuals the dynamics of the probability distribution as well as its asymptotic, steady state, values might not coincide with the analysis of the master equation. We have found this deviation to be more pronounced in the case of being close to a bifurcation point. For example, in figure 3 we plot the time evolution of the probability coming from Monte Carlo simulations of the model for different system sizes and the results from the master equation in the case $\epsilon = 0.28$, close to a type 2 bifurcation point. It can be seen that, although the Monte Carlo simulation and the master equation agree initially very well, they start to deviate after a time that depends on the number of individuals N: the larger N, the longer the time for which the Monte Carlo simulations are faithfully described by the master equation. In this particular case, $\epsilon = 0.28$, it can be seen that the master equation agrees with the Monte Carlo simulations up to a time $t \sim 10$ for $N = 10^3$ and a time $t \sim 100$ for $N = 10^5$. In view of this difference, it is surprising that the Monte Carlo steady state distributions show only small (although observable) finite-size effects. Quite similar functions describe the steady state data for both $N = 10^3$ and 10^5; see figure 4. As can be seen in figure 3, however, while the numerical solution of the master equation tends to the steady state distribution $P_\infty(x)$, the Monte Carlo simulations tend to another distribution, $P_{\mathrm{st}}(x)$. These two distributions are very different: $P_\infty(x)$ has a large maximum (large cluster) at $x = 0.5$ and two much smaller maxima at $x \approx 0.127$ and 0.873, whereas $P_{\mathrm{st}}(x)$ has two equal maxima at $x \approx 0.25$ and 0.75. Although it appears surprising at first, it turns out that the steady state distribution $P_{\mathrm{st}}(x)$ coming from the Monte Carlo simulations of the model is also very close to a steady
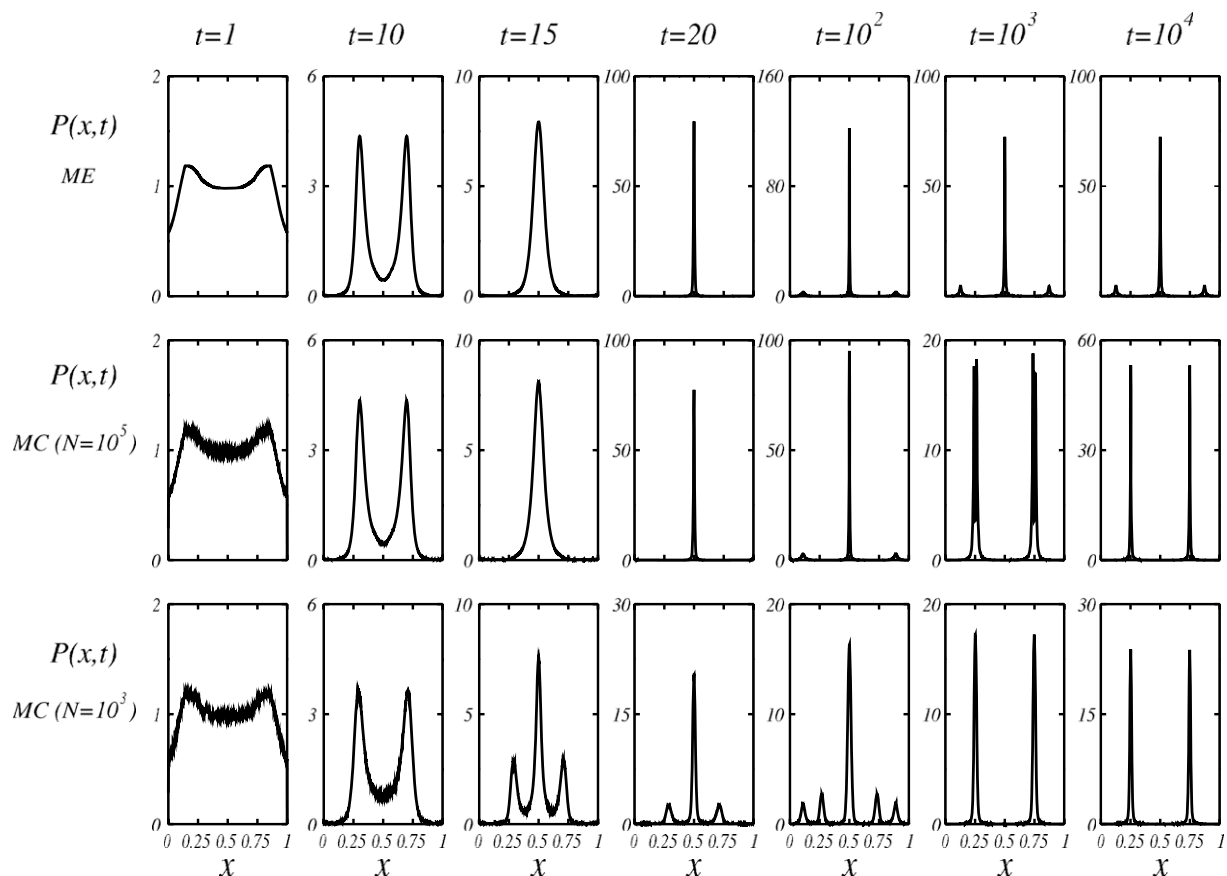
Figure 3. Probability distribution function $P(x,t)$, for intermediate time steps, from Monte Carlo (MC) simulations (histograms binned with bin size $\Delta x = 5 \times 10^{-4}$) for two different system sizes $N = 10^3$ and 10^5, and the master equation (ME) integrations of equation (A.4) starting with a flat distribution with $\epsilon = 0.28$ and $m = 0.01$. The distributions are averages over 10^5 realizations for $N = 10^3$ and 10^4 realizations for $N = 10^5$.

state solution of the master equation (A.4) having two major clusters. However, this last steady state solution cannot be obtained as an asymptotic solution of the master equation $\lim_{t \to \infty} P(x,t)$ starting from a uniform initial condition $P(x, t = 0) = 1$ for $x \in [0,1]$. It turns out that it is reached when starting instead from an initial condition asymmetric with respect to the center of the interval.

Summing up, for $\epsilon = 0.28$ there are two steady state solutions of the master equation, $P_{\text{st}}(x)$ and $P_\infty(x)$. Starting from a uniform initial condition, $P_\infty(x)$ is the one reached asymptotically as a solution of the master equation. However, $P_{\text{st}}(x)$ is, up to finite-size effects, the one reached instead in the Monte Carlo simulations. We interpret this in terms of the relative stability of the two solutions: introducing a fluctuation $\delta P(x)$ on the solution $P_\infty(x)$ it is then possible to reach the solution $P_{\text{st}}(x)$, but not the reverse. This fluctuation $\delta P(x)$ needs to be asymmetric, $\delta P(x) \neq \delta P(1-x)$, and it appears naturally in Monte Carlo simulations because of the finite number of individuals N and it is more probable for smaller values of N. This explains why the system with smaller N
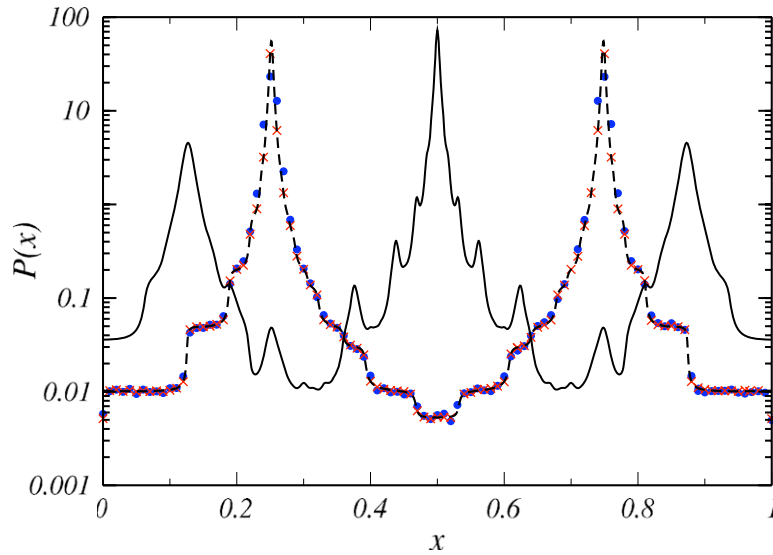
Figure 4. The symbols denote the steady state probability distribution function $P_{\mathrm{st}}(x)$, obtained from Monte Carlo simulations for system sizes $N = 10^3$ ($\cdot$) and $N = 10^5$ ($\times$) in the case $\epsilon = 0.28$, $m = 0.01$. The dashed line is obtained as an asymptotic solution of the master equation starting with a non-uniform initial condition. These three distributions are very similar in this logarithmic scale, although there are differences in the height of the maxima (see figure 3). It turns out that close to each of the two maxima, the distributions can be well fitted with a Lorentzian function [19]. The solid line is the distribution $P_\infty(x)$ coming from a numerical solution of the master equation (A.4) starting with a uniform distribution.

deviates earlier from the solution of the master equation. If one induces artificially[2] such a non-symmetric perturbation in the solution $P_\infty(x)$ or, alternatively, one starts with a non-uniform, asymmetric initial condition, then the master equation tends to $P_{\mathrm{st}}(x)$.

The existence of more that one steady solution of the master equation seems to be a general feature. In fact, if the distributions in figure 2 are recalculated by slowly increasing and decreasing ϵ without resetting the initial condition to $P(x, t = 0) = 1$ after each change in ϵ, we observe the hysteresis behavior typical of bistability occurrence close to first-order transitions. Which one of the possible steady solutions is observed in the Monte Carlo simulations depends on the parameter ϵ. It could even happen that the inherent fluctuations of a finite system take it from one solution to another and back. This sort of bistability is observed, for instance, in Monte Carlo simulations at $\epsilon = 0.31$. As shown in figure 5, the evolution displays multiple jumps between two solutions: one with two maxima of equal height and another one with a large central maximum and two smaller maxima near the edges of the opinion interval. The existence of jumps induced by finite-size fluctuations has been observed in a microscopic model of catalytic surface reaction [26, 27] whose rules are based on an Ising-type model including explicit noise terms.

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[2] If one is not careful enough, the perturbation might also appear as a numerical instability of the integration method.
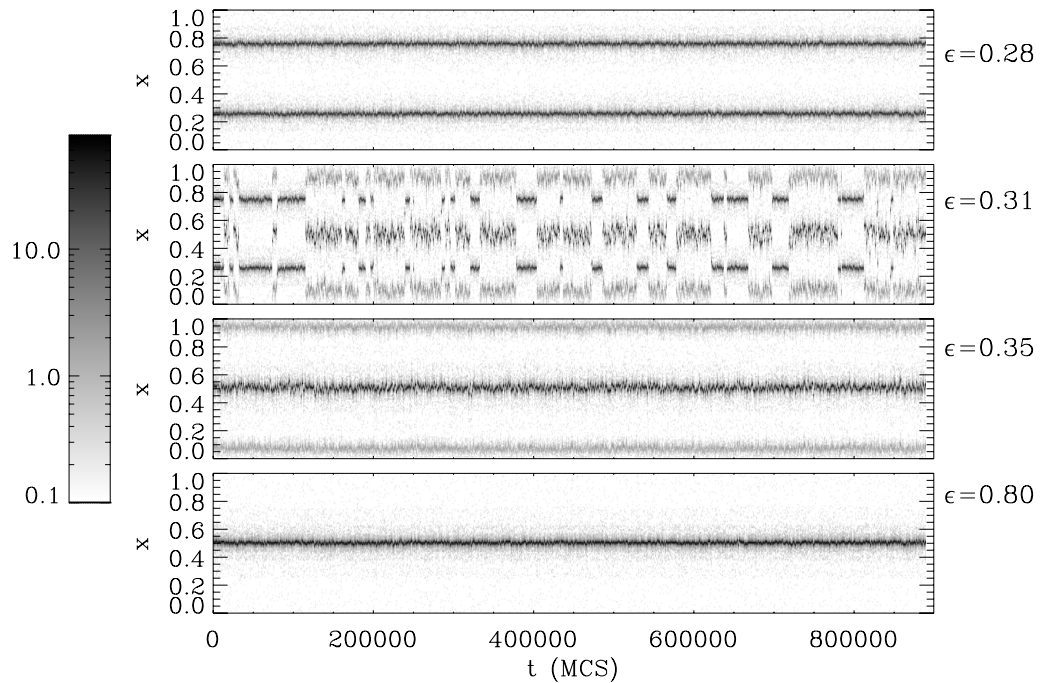
Figure 5. Time series of the opinion distributions $P(x, t)$ at four values of ϵ and $m = 0.01$ and $N = 1000$. The grayscale is logarithmic, with values smaller than 0.1 plotted in white. Simulations are initialized with random homogeneous distributions 1000 MCS before the first time shown in the plots. The first three panels are near a type 2 bifurcation. At $\epsilon = 0.28$ the system appears polarized into two opinion clusters. At $\epsilon = 0.35$ a single major opinion has been established, with two lateral minor clusters. At $\epsilon = 0.31$ the system fluctuates between these two states. A single cluster is observed at large ϵ (bottom panel).

Looking at figure 2, one can see that for small ϵ or, alternatively, for noise intensity m larger than a critical value m_c which increases with ϵ, the bifurcations become blurred and the maxima of the distributions are not evident, implying the inhibition of cluster formation. This happens at $\epsilon \lesssim 0.1$ for $m = 0.1$ and at $\epsilon \lesssim 0.01$ for $m = 0.01$. A similar effect can be observed in the Monte Carlo simulations and can be described in terms of an order–disorder transition: order identified with the state with well defined opinion clusters and disorder identified with the state without clusters. In figure 6 we plot the steady state probability distribution in a disorder case: $m = 0.01$ and $\epsilon = 0.008$. Note the absence of peaks indicating the presence of clusters, other than the small peaks at the borders of the finite opinion interval coming from edge effects.

To define this transition in a more quantitative way, we have used the so-called cluster coefficient G_M, which aims to characterize the existence of clusters [28]. The definition of the cluster coefficient G_M starts with dividing the opinion space $[0, 1]$ into M equal boxes and counting the number of individuals l_i which, at time step n, have their opinion in the box $[(i - 1)/M, i/M]$. The value of M must not be so large that particles are artificially considered to be part of a single cluster, nor so small that statistical errors are large within one box. Otherwise, the exact value of M is not important, For our problem, the values $M = 100$ and 200 give almost undistinguishable cluster coefficients, as shown in figure 7.
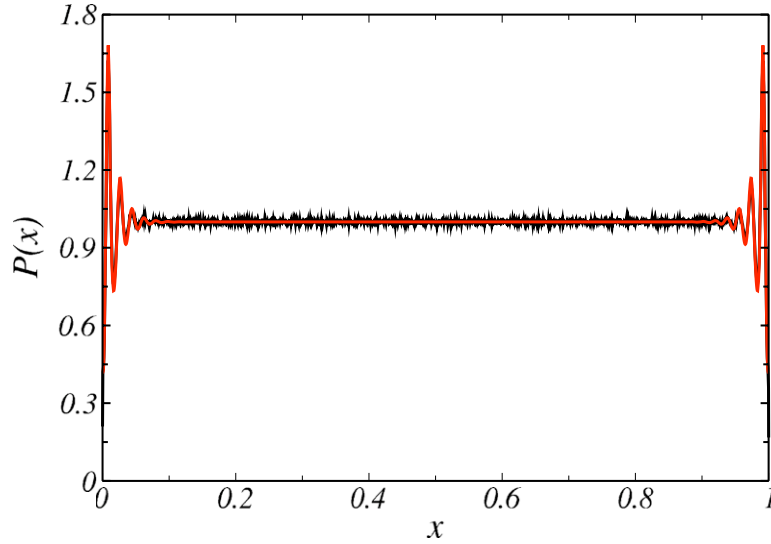
Figure 6. Plot of the steady state probability distribution the $P_{st}(x)$ in the case $m = 0.01$ and $\epsilon = 0.008$. One can note the absence of clusters (other than those coming from edge effects at the borders of the finite opinion interval) both in the master equation (black line, same integration details as in figure 2) and in the Monte Carlo simulations (red line, system size $N = 10^4$ agents averaged over 10^4 realizations). According to the linear stability analysis (see section 4), the critical value for cluster formation is $\epsilon_c = 0.0088$.

One next introduces an entropy-like measure $S_M = -\sum_{i=1}^{M}(l_i/N)\ln(l_i/N)$. Note that $0 \leq S_M \leq \ln M$, and that the minimum value S_M is obtained when all the individuals are in just one box, while the maximum value, $S_M = \ln M$, is reached when $l_i = N/M$, i.e. when the opinions are uniformly distributed in the interval $[0,1]$. Finally, the opinion cluster coefficient is defined as [28]

$$G_M = M^{-1}\langle e^{\overline{S}_M}\rangle, \tag{3}$$

where the overbar denotes a temporal average in steady conditions and $\langle \cdot \rangle$ indicates an average over different realizations of the dynamics. Note that $1/M \leq G_M \leq 1$. Large values $G_M \approx 1$ indicate that the opinions are evenly distributed over the full opinion space (a situation identified with disorder), while small values of G_M indicate that opinions peak around a finite set of major opinion clusters (a situation identified with order).

The data coming from Monte Carlo simulations (see figure 7) show that G_M is an increasing function of the noise intensity m and saturates to its maximum value $G_M \approx 1$ for large enough values of m. The transition from cluster formation to disorder will be defined, somewhat arbitrarily but precisely, as the value m_c of the noise intensity for which the cluster coefficient reaches the value $G_M = 0.9$. For small values of the confidence parameter ϵ the transition to the homogeneous state is abrupt and occurs for small values of m. If one increases ϵ, the transition becomes less abrupt and a higher noise intensity is needed to obtain the homogeneous, cluster-free, state. This last feature can be explained using the linear stability analysis that we shall develop in section 4.
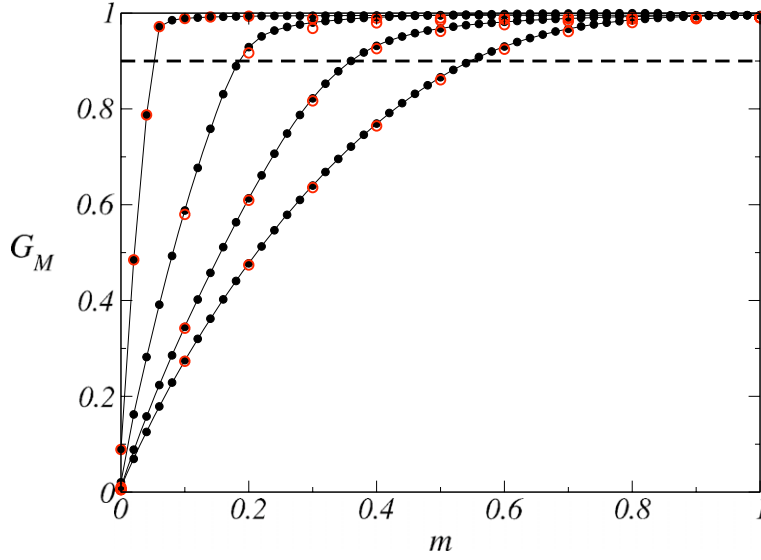
Figure 7. Opinion cluster coefficient G_M versus m for $\epsilon = 0.05$, 0.2, 0.4, and 0.6, from left to right, respectively as obtained from Monte Carlo simulations with $N = 10^5$ individuals (dots; the solid line is a guide to the eye). Solid dots correspond to $M = 100$ and open dots to $M = 200$. The location of the order–disorder transition is defined as the value m_c for which $G_M = 0.9$. Note that, except at some particular points (see figure 8(b)), m_c is an increasing function of ϵ and that the transition is very abrupt for small ϵ.

4. Linear stability analysis

We have shown that opinion clusters still form in the presence of small amounts of noise, but an unstructured state without clusters dominates the opinion space for noise larger than a critical noise intensity m_c. Although the transition to cluster formation is a nonlinear process, one can still derive approximate analytical conditions for the existence of cluster formation in the parameter space (ϵ, m) by performing a linear stability analysis of the unstructured solution of equation (A.4). This is greatly simplified if one neglects the influence of the boundaries and assumes that the interval $[0, 1]$ is wrapped on a circle, i.e. there are periodic boundary conditions at the ends of the interval. This would be a reasonable approximation for describing the distribution far from the boundaries if ϵ, which fixes the interaction range, is sufficiently small. In this case the homogeneous configuration $P_h(x) = 1$ is an approximation to the unstructured steady solution of the master equation. Analysis of its stability begins by introducing $P(x, t) = P_h(x) + A_q e^{iqx + \lambda_q t}$, where q is the wavenumber of the perturbation, λ_q its growth rate and A_q the amplitude. After introducing this *ansatz* in equation (A.4) we find the dispersion relation giving the growth rate of mode q:

$$\lambda_q = 4\epsilon(1 - m)\left[\frac{4\sin(q\epsilon/2)}{q\epsilon} - \frac{\sin(q\epsilon)}{q\epsilon} - 1\right] - m. \tag{4}$$

This is plotted in figure 8(a), for several values of m. The maximum value of λ_q occurs at $q_{max} = 2.7907/\epsilon$. It turns out that the maximum value $\lambda_{q_{max}}$ is negative for $m > m_c$ and
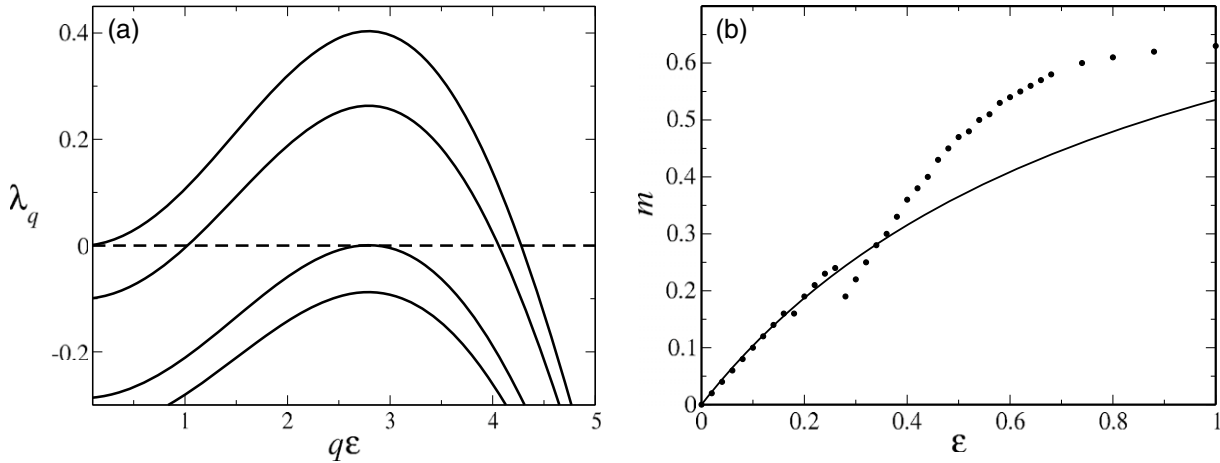
Figure 8. (a) Growth rate, equation (4), of a perturbation to the homogeneous solution as a function of $q\epsilon$ for $\epsilon = 0.35$ and $m = 0.0, 0.10, 0.287$, and 0.35 from top to bottom. It shows that λ_q becomes negative for high values of m. (b) Phase diagram on the plane (ϵ, m) as obtained from Monte Carlo simulations with $N = 10^5$ individuals (dots), using the criterion based on the cluster coefficient $G_{M=100}$ as described in the text. The solid line is the prediction from the linear stability analysis. Clusters appear below the lines, whereas the unstructured state is stable above.

positive for $m < m_c = \epsilon/(a + \epsilon)$ with $a \approx 0.8676$. Alternatively, for fixed m the maximum growth rate is negative for $\epsilon < \epsilon_c = am/(1 - m)$, and positive for $\epsilon > \epsilon_c$. Therefore, the homogeneous state is unstable and cluster formation is possible only for $m < m_c$ or $\epsilon > \epsilon_c$. The numerical values are $\epsilon_c = 0.096$ for $m = 0.1$ and $\epsilon_c = 0.0088$ for $m = 0.01$, in reasonable agreement with the behavior observed for the master equation dynamics in figure 2. Comparison with Monte Carlo simulations is performed in figure 8(b) where we plot the critical value m_c obtained from the cluster coefficient G_M as described earlier. We see in the figure that the agreement is very good for small ϵ but deviates for larger values. This is consistent with the fact that neglecting boundary effects is expected to be appropriate only for small ϵ. Finally, it is possible to estimate roughly the number of clusters n using a simple argument: n is related to the wavelength of the maximum growth as $n = q_{\max}/2\pi$ or $n = 0.444/\epsilon$ for $\epsilon > \epsilon_c$. This result is in qualitative agreement with the $1/2\epsilon$ rule, which says that the number of major clusters after cluster formation is roughly determined as the integer part of $1/2\epsilon$ (see [20, 21] for details).

5. Summary and conclusions

In this paper we have studied the Deffuant *et al* model for continuous-opinion dynamics in the presence of noise. Besides the usual rules of the model, we give each individual the opportunity to change, with a certain probability m, their opinion to a randomly selected opinion inside the whole opinion space. The final behavior depends of the confidence or interaction parameter ϵ and the noise intensity m.

We have first reviewed the original noiseless version of the model. We have shown that for small number of individuals N and depending of the particular realizations, the exact location of the opinion clusters might vary with respect to the predictions of the master equation. In particular, minor clusters might not appear and there could be realizations in which even the number of observed major clusters differs from the prediction of the master equation.

We have derived (appendix) a master equation for the probability density function $P(x,t)$ which determines the individual density or distribution in the opinion space. Numerical integration of this equation from uniform initial conditions reveals that for $m > 0$ the steady distributions are no longer delta-functions as for $m = 0$, but are still peaked around some well defined maximum value, with some non-vanishing width. By looking at those maxima we are able to identify the same kinds of bifurcations as in the noiseless case [12, 20]. At variance with the noiseless case, the location of the maxima (the central opinion of the clusters) does not depend on ϵ until a new bifurcation point is reached.

We have also found that in the noisy case the asymptotic steady state probability distributions reached by Monte Carlo simulations might not coincide with the ones obtained from the master equation dynamics starting from the same symmetric initial condition. This deviation is more pronounced in the case of being close to a bifurcation point. In particular, we have presented a situation where, starting from a uniform initial condition, a particular stationary distribution, $P_\infty(x)$, is actually reached by the master equation but another distribution, $P_{st}(x)$, is the one reached instead in Monte Carlo simulations. The time at which the Monte Carlo simulations begin to deviate from the master equation depends on the system size: the smaller the size, the earlier the deviation occurs although the final Monte Carlo distribution $P_{st}(x)$ shows only small size effects. Remarkably, $P_{st}(x)$ turns out to be close to another steady solution of the master equation. Thus, the discrepancy observed during the dynamics does not seem to be simply a trivial finite-N effect. We interpret it in terms of the relative stability of the two solutions by adding an asymmetric perturbation to $P_\infty(x)$. It is then possible to reach the solution $P_{st}(x)$ obtained from Monte Carlo simulations, but not the other way around. Asymmetric fluctuations appear naturally in Monte Carlo dynamics because of the finite number of individuals, and are larger for smaller N. We have also shown that the fluctuations present in Monte Carlo simulations are even able to induce jumps from one solution to another and back.

An order–disorder transition to cluster formation induced by noise has been characterized using the so-called cluster coefficient G_M for the simulations performed with Monte Carlo dynamics. We have found that G_M is an increasing function of the noise intensity m and saturates to its maximum value $G_M \approx 1$ for large enough values of m. For small values of the confidence parameter ϵ the transition to the disordered state is abrupt and occurs for small values of m. If one increases ϵ, the transition becomes less abrupt and a higher noise intensity is needed to obtain this state. Using a linear stability analysis of the unstructured (no clusters) solution of the master equation we have derived approximate conditions for opinion cluster formation as a function of the relevant parameters of the system. We have found qualitative agreement between the linear stability analysis and numerical simulations. The agreement is better for small values of ϵ where boundary effects, neglected to make feasible the linear analysis, are less

important. However, we should emphasize that the pattern selection of this model is, with noise and without it, intrinsically a nonlinear phenomenon and obtaining the exact critical conditions for opinion cluster formation remains a challenge.

Our work stresses the importance that fluctuations and finite-size effects have in the dynamics of social systems for which the thermodynamic limit is not justified [25]. Further work will address the effect that these ingredients have in the dynamics of continuous-opinion models in the presence of an external influence, or forcing, representing the role of advertising.

Acknowledgments

Appendix

Here we derive the master equation, i.e. the evolution equation for $P_n(x)$, the probability density function (pdf) of the opinions at step n for the model introduced in this paper. Note that $P_n(x)$ is constructed from the histogram of all individual opinions x_n^i at step n. Let us first find the evolution of the pdf for those two particular individuals i, j that have been selected for updating at step n according to the basic rule, equation (1). We will denote by $P_{n+1}^i(x)$ the pdf of the opinion of individual i at the step $n+1$, i.e. the probability that x_{n+1}^i adopts the value x. According to that rule, it is straightforward to derive the evolution equation

$$
P_{n+1}^i(x) = \int_{|x_n^i - x_n^j| < \epsilon} dx_n^i \, dx_n^j P_n(x_n^i) P_n(x_n^j) \delta \left(x - \frac{x_n^i + x_n^j}{2} \right)
$$
$$
+ \int_{|x_n^i - x_n^j| > \epsilon} dx_n^i \, dx_n^j P_n(x_n^i) P_n(x_n^j) \delta(x - x_n^i), \tag{A.1}
$$

and a similar expression for $P_{n+1}^j(x)$. The integrals over x_n^i and x_n^j both run over the interval $[0, 1]$. In this equation an independence approximation for the variables x_n^i, x_n^j has been implicitly assumed, i.e. their joint pdf $P_n(x_n^i, x_n^j)$ is supposed to factorize as $P_n(x_n^i, x_n^j) = P_n(x_n^i) P_n(x_n^j)$. This is an uncontrolled approximation whose validity can only be established by an independent comparison with the Monte Carlo simulation of the microscopic rules. For the individuals $k \neq i, j$ whose opinion does not change at time step $n+1$ we have simply $P_{n+1}^k(x) = P_n^k(x)$.

The pdf $P_{n+1}(x)$ has several contributions: (i) with probability $1 - m$ two individuals, say i, j, are chosen for updating according to the basic evolution rule, equation (1), and $N - 2$ variables remain unchanged. (ii) With probability m one individual is chosen for updating according to the noise rule and $N - 1$ variables remain unchanged; the new opinion of the selected individual is sampled from an, in principle, arbitrary distribution $P_a(x)$, although in this paper we have taken throughout that $P_a(x)$ is the uniform distribution $P_h(x) = 1$ in the interval $[0, 1]$. After consideration of these contributions

we are led to the evolution equation

$$P_{n+1}(x) = (1-m)\left[\frac{N-2}{N}P_n(x) + \frac{1}{N}P_{n+1}^i(x) + \frac{1}{N}P_{n+1}^j(x)\right]$$
$$+ m\left[\frac{N-1}{N}P_n(x) + \frac{1}{N}P_a(x)\right]. \tag{A.2}$$

Replacing $P_{n+1}^i(x)$ and $P_{n+1}^j(x)$ from equation (A.1) one obtains after some algebra

$$P_{n+1}(x) = P_n(x) + \frac{(1-m)}{N}\left[4\int_{|x-x'|<\epsilon/2}\mathrm{d}x'P_n(2x-x')P_n(x')\right.$$
$$\left. - 2P_n(x)\int_{|x-x'|<\epsilon}\mathrm{d}x'P_n(x')\right] + \frac{m}{N}\left[P_a(x) - P_n(x)\right]. \tag{A.3}$$

The integrals over x' run over the interval $[0,1]$, and it has to be imposed that $P_n(x) = 0$ if $x \notin [0,1]$. We now take the continuum limit $P_n(x) \to P(x,t)$ with a time $t = n\,\Delta t$ and taking the limit $\Delta t = 1/N \to 0$ as $N \to \infty$, to obtain

$$\frac{\partial P(x,t)}{\partial t} = (1-m)\left[4\int_{|x-x'|<\epsilon/2}\mathrm{d}x'P(2x-x',t)P(x',t)\right.$$
$$\left. - 2P(x,t)\int_{|x-x'|<\epsilon}\mathrm{d}x'P(x',t)\right] + m\left[P_a(x) - P(x,t)\right], \tag{A.4}$$

which is the master equation of the Deffuant *et al* model in the presence of noise and the basis of our analysis. The noiseless case, $m = 0$, was first obtained in [12]. We note here the symmetry property of the master equation: if the initial condition is symmetric around the central point $x = 1/2$, namely we have that $P(x, t = 0) = P(1-x, t = 0)$, then this property holds for any later time, $P(x,t) = P(1-x,t), \forall t > 0$.

The time evolution of the first moments of $P(x,t)$ can be computed from the master equation. Defining the moments as $M_k(t) = \int \mathrm{d}x\, x^k P(x,t)$ one finds easily that $\mathrm{d}M_0/\mathrm{d}t = 0$ (normalization condition) and that the first moment evolves as $\mathrm{d}M_1/\mathrm{d}t = m(M_1^a - M_1)$, where M_1^a is the first moment of the distribution $P_a(x)$. Therefore, if $m > 0$, the average opinion tends to $M_1 = M_1^a$ independently of the initial condition, and it is always conserved in the noiseless case $m = 0$. Expressions for higher-order moments can only be obtained in the special case $\epsilon \geq 1$, as discussed in the main text.

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