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STOCHASTIC RESONANCE IN BISTABLE AND EXCITABLE SYSTEMS: EFFECT OF NON–GAUSSIAN NOISES

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We analyze stochastic resonance in systems driven by **non–Gaussian** noises. For the bistable double well we exploit a consistent Markovian approximation that enables us to get quasi-analytical results for the signal-to-noise ratio in the case of a colored non–gaussian noise. The results are compared with those coming from numerical simulations of the system. We also study the FitzHugh-Nagumo excitable system in the limit of non–Gaussian white noise. In both systems, we find that, as the noise departs from Gaussian behavior, there is a regime (different for the excitable and the bistable systems) in which there is a notable robustness against noise tuning since the signal-to-noise ratio curve broadens and becomes less sensitive to the actual value of the noise intensity. We also compare our results with some experiments in sensory systems.

Keywords: Stochastic resonance. Non–Gaussian Noises.

Noise–induced phenomena have attracted considerable interest during the last decades. Among them *Stochastic Resonance*, a counterintuitive entanglement between noise an nonlinearity, detaches due, besides other fundamental aspects, to its potential applications for optimizing the response to weak external signals in nonlinear dynamical systems, as well as to its connection with some biological mechanisms. Since its discovery on the early eighties [1, 2], a large number of papers, conference proceedings and reviews have been published [3] showing the many applications in science and technology (e.g. in paleoclimatology, electronic circuits, lasers, chemical systems, etc.) and the connection with some situations of biological interest (noise-induced information flow in sensory neurons in living systems, influence in ion-channel gating or in visual perception [4]). Some recent publications have shown the possibility of achieving an enhancement of the system response by means of the coupling of several units in what conforms an *extended medium* [5], or analyzing the possibility of making the system response less dependent on a fine tuning of the noise intensity, as well as different ways to control the phenomenon [6].

Most studies on stochastic resonance have been made in a paradigmatic bistable one-dimensional double-well system, and with very few exceptions [7], assuming that the noises are Gaussian. However, some experimental results in sensory systems, particularly Refs. [8] and [9], offer strong indications that the noise source in these systems could be non–Gaussian. More specifically, some recent detailed studies on the source of fluctuations in some biological systems [10] clearly indicate that such noise sources in general are non–Gaussian and with a bounded distribution. In this letter, we analyze the role of the non–Gaussian nature of the stochastic terms, comparing our results with those obtained in the classic experiments on a crayfish neural system [8]. We conclude that the consideration of non–Gaussian noises can improve the agreement with the experimental results, hence supporting the hypothesis that the noise in those biological systems is actually non–Gaussian.

Stochastic resonance in a double well potential under the effect of non–Gaussian noises has been studied in [11]. We now sketch the main results of this general study. Let us consider the following Langevin equation:

$$\dot{x} = -\frac{\partial U}{\partial x} + \eta(t) \tag{1}$$

where, as in most stochastic resonance studies, we assume that the potential $U(x,t) = U_0(x)+xS(t)$, consists of a double well function $U_0(x)$ and a linear modulation term, $S(t) = \epsilon\cos(\Omega t)$, of amplitude ϵ and frequency Ω. However, at variance with other studies, we assume that the noise term $\eta(t)$ has a non-Gaussian distribution. Although we believe that our results are quite general, for concreteness and motivated by the work in [12] based on a generalized thermostatistics distribution [13], we consider that the noise term is a Markovian process generated as the solution of the following Langevin equation:

$$\dot{\eta} = -\frac{1}{\tau_0}\frac{d}{d\eta}V_q(\eta) + \frac{1}{\tau_0}\xi(t) \tag{2}$$

being $\xi(t)$ a standard Gaussian white noise of zero mean and correlation $\langle\xi(t)\xi(t')\rangle = 2D_0\delta(t-t')$, and

$$V_q(\eta) = \frac{D_0}{\tau_0(q-1)}\ln\left[1 + \frac{\tau_0}{D_0}(q-1)\frac{\eta^2}{2}\right]. \tag{3}$$

The stationary properties of the noise η, including the time correlation function, have been studied in [14]. The stationary probability distribution is given by:

$$P_q^{st}(\eta) = \frac{1}{Z_q}\left[1 + \frac{\tau_0}{D_0}(q-1)\frac{\eta^2}{2}\right]^{\frac{-1}{q-1}}, \tag{4}$$

where Z_q is the normalization factor. This distribution can be normalized only for $q < 3$. The first moment, $\langle\eta\rangle = 0$, is always equal to zero, and the second moment,

$\langle\eta^2\rangle = \frac{2D_0}{\tau_0(5-3q)} \equiv D$, is finite only for $q < 5/3$. Clearly, when $q \to 1$ we recover the limit of η being a Gaussian colored noise (Ornstein-Uhlenbeck process). For $q > 1$ the distribution extends from $-\infty$ to $+\infty$ and its asymptotic power–law decay is slower than that of a Gaussian distribution of the same variance (for instance, for $q = 2$ it is a Cauchy distribution). Furthermore, for $q < 1$, the distribution has a cut-off and it is only defined for $|\eta| < \sqrt{\frac{2D_0}{\tau_0(1-q)}}$. Finally, the correlation time τ of the process η diverges near $q = 5/3$ and it can approximated over the whole range of values of q as $\tau \approx 2\tau_0/(5-3q)$.

Stochastic resonance shows up as a maximum in the signal to noise ratio, SNR, defined as the ratio of the strength of the output signal and the broadband noise output evaluated at the signal frequency Ω. Analytical predictions can be obtained using the so–called *two–state model* [15] within the *effective Markovian approximation*. We have also performed numerical simulations of the previous Langevin equation with double well potential $U_0(x) = -\frac{x^2}{2} + \frac{x^4}{4}$. The details of this calculation are given elsewhere [11] and here we just sketch the main results as shown in figures 1 and 2.

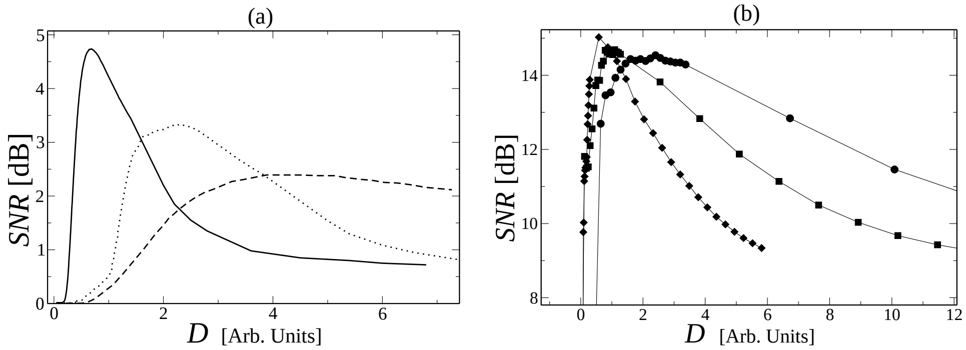

Fig 1. Plots of (a) Theoretical and (b) Simulation results of SNR vs D, for $\tau = 0.16$. Signal parameters are $\epsilon = 0.07$ and $\Omega = 3.6$. In plot (a) it is $q = 0.75$ (solid curve), 1.0 (dotted), and 1.25 (dashed); while in (b) $q = 0.75$ (♦), 1.0 (■), and 1.25 (•).

In figure 1 we depict the SNR, as a function of the noise intensity D, for a fixed value of the time correlation τ and various values of q. The general trend is that the maximum of the SNR curve increases and shifts towards smaller values of D when $q < 1$, this is when the noise, departing from the Gaussian behavior, has a bounded distribution. The opposite tendency occurs when $q > 1$, when the noise distribution has long power–law like tails. The main qualitative features of this approximate analytical result are confirmed by intensive numerical simulations of the system given by Eqs. (1,2), which has been integrated numerically using a stochastic Runge–Kutta method known as the Heun method [16] and averaged over 2000 trajectories. Both the analytical and the numerical results show that the maximum of the SNR curve flattens for increasing values of q, indicating that the system, when departing from Gaussian behavior for $q > 1$, requires a less fine tuning of the noise intensity in order to maximize its response to a weak external signal. This is the main result in this case.

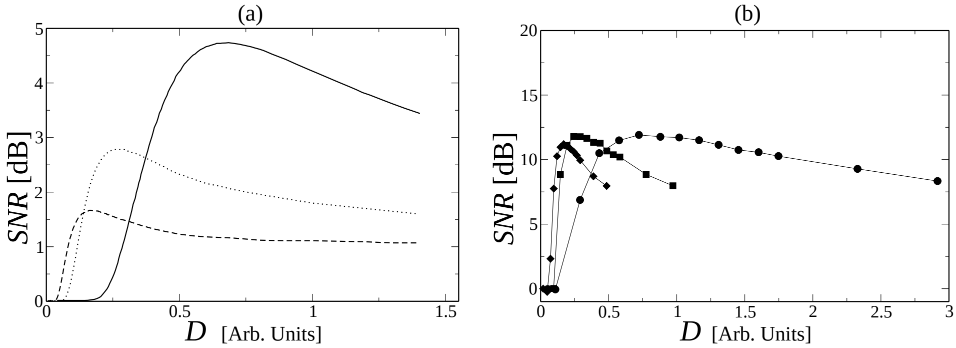

Fig 2. Results of (a) Theoretical and (b) Simulation results of SNR vs D, for a fixed parameter $q = 0.75$. Signal parameters are $\epsilon = 0.07$ and $\Omega = 3.6$. For plot (a) the value of time correlation is $\tau = 0.182$ (solid line), 0.545 (dotted), and 1.091 (dashed); while in (b) $\tau = 0.182$ (●), 0.545 (■), and 1.091 (◆).

Figure 2 shows the SNR as a function of noise intensity D, for different fixed values of time correlation τ and a fixed value of $q = 0.75$. We find here that an increase of the correlation time induces a decrease of the maximum of SNR as well as shifting it towards smaller values of noise intensity. These results can be compared with the known ones in the case of colored Gaussian noises (Ornstein–Uhlenbeck process) [3], where an increase of the correlation time induces a decrease of the maximum of SNR as well, but that maximum is shifted instead towards *larger* values of noise intensity.

After this analytical and numerical study showing the mean features of stochastic resonance under the presence of a colored non–Gaussian noise in a bistable system, we proceed to analyze the situation in which the non–Gaussian noise is applied to an excitable system. Our aim is to make a comparison with the experimental results on a crayfish neural system [8]. With this goal in mind, a suitable model to compare with the crayfish neural system under consideration is the celebrated FitzHugh-Nagumo excitable system. In the same way as was used in Ref. [8], we define the model system through the set of equations

$$\begin{aligned} \tau_v \dot{v} &= v(v-0.5)(1-v) - w, \\ \tau_w \dot{w} &= v - w - b + \epsilon \cos(\Omega t) + \eta(t), \end{aligned} \tag{5}$$

where $v(t)$ is the variable associated to the action potential (in an activator-inhibitor model it corresponds to the activator variable), $w(t)$ is the recovery (inhibitor) variable. The characteristic times for such variables are, respectively, τ_v and τ_w. The recovery variable, $w(t)$, is subjected to a periodic signal whose (small) amplitude we denote by ϵ and its frequency by Ω) and a noise source $\eta(t)$. This random variable is distributed according to the stationary distribution in Eq. 4 but we have set the correlation time τ of the η process to be equal to zero (white–noise limit). In the numerical simulations, this is achieved by using a rejection method [17] to sample directly distribution 4, instead of generating the random variable through the solution of Eq. 2. Notice that the distribution 4 can be characterized by giving just the parameter q and the noise variance D.

Along all the simulations, we fixed $\tau_v = 10^{-3}$s and $\tau_w = 1$ s. While we considered the signal as having frequency $\Omega = 55$ Hz and amplitude $\epsilon = 0.03$ V. All these values agree with those from Ref. [8], however their results are a particular case of ours, i.e. $q = 1$, for the noise source. As before, the numerical simulations of Eqs. 5, have been performed by means of the stochastic Heun method with a time step of $\Delta t = 1$ms. The results shown here have been averaged over 2000 trajectories for each set of parameters.

The time series $v(t)$ was converted into a spike train (mimicking what happens in the nervous system). One spike occurs when $v(t)$ exceeds the threshold potential $V_t = 0.15$ V. Each spike was modelled as a square wave of height $V_s = 1$ V, and duration $t_s = 3$ ms. It is not possible to have two successive spikes with a lag smaller than t_s. After this signal processing, the SNR was evaluated following the usual procedure.

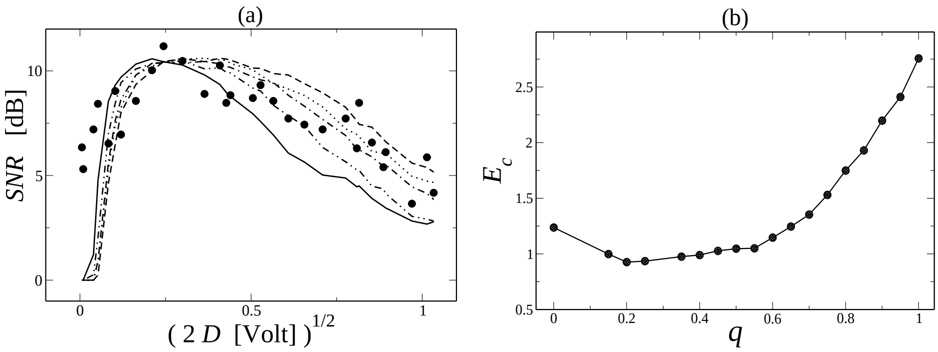

Fig 3. (a)The SNR for the FitzHugh-Nagumo model is plotted as a function of $\sqrt{2D}$ (the noise standard deviation). The case $q = 1$ is shown with a solid line, while the different values of q are $q = 0$ (dashed line), $q = 0.2$ (dotted), $q = 0.4$ (dash-dotted), $q = 0.6$ (double dot-dashed). Simulation results are compared with the experimental data of Ref. [8] ($\bullet$). (b)The mean square error, E_c, of the fittings as a function of q.

We have compared the experimental results for output SNR, with those of simulations for different values of the parameter q, as a function of noise intensity D. The results are shown in figure 3(a). The case $q = 1$ exactly corresponds to the results of Ref. [8], and from such a comparison it becomes apparent that the fit results acceptable near the SNR maximum of experimental data. But, for large noise intensities, this case ($q = 1$) underestimates the values of SNR.

The same figure shows the results for different values of q. It appears that for a large range of values of q the results fit in good agreement the experimental data in the SNR peak. Again, for low noise intensities, our simulations do not show any improvement with respect to those obtained in the Gaussian case studied in [8]. However, when decreasing the value of q from $q = 1$ the underestimation of SNR for large noise intensities disappears until for $q \to 0$ there is now an overestimation of the experimental results. In order to quantify the goodness of the fit and to determine the optimal value of q, we compute the mean square error E_c, calculated as the following integral on the noise intensity variable D:

$$E_c(q) = \int_{D_a}^{D_b} \left[SNR(D,q) - SNR_e(D)\right]^2 \, dD, \tag{6}$$

where D_a and D_b are the minimum and maximum values of D for the experimental data, respectively. $SNR_e(D)$ is the SNR of the experimental data and $SNR(D,q)$ represents the output SNR obtained with the simulations for the parameter q. The results are depicted in figure 3(b). In this plot, it is apparent that the case $q = 1$ gives the worst fittings among all, and for a value of $q \approx 0.2$, the error reaches a minimum.

Summarizing, motivated by some experimental results in sensory systems [8, 9], we have analyzed the problem of stochastic resonance when the noise source is non–Gaussian in two paradigmatic models: the bistable doble well and the excitable FitzHugh-Nagumo systems. Although we believe that similar results will be obtained for most distributions showing a similar non–Gaussian behavior, we have chosen, for concreteness, a noise source (white or colored) with a probability distribution based on the generalized thermostatistics [13]. In the colored case and making use of a path integral approach, we have applied an effective Markovian approximation that allows us to get some analytical results for the bistable systems. In addition, we have performed exhaustive Monte Carlo simulations. Even though the agreement between theory and numerical simulations is only partial and qualitative, the effective Markovian approximation turns out to be extremely useful to (qualitatively) predict general trends in the behavior of the system under study.

Our numerical and theoretical results for the bistable system indicate that: (i) for a fixed value of τ, the maximum value of the SNR increases with decreasing q; (ii) for a given value of q, the optimal noise intensity (that one maximizing SNR) decreases with increasing τ and the corresponding value of the SNR is approximately independent of τ. Moreover, in the case of an asymptotic power–law decay for the non-Gaussian noise variable ($q > 1$), we find that the SNR becomes less dependent on the precise value of the noise intensity.

The same lack of dependance on the noise intensity is found in the excitable system. However, at variance with the previous result in the bistable double well system, we find that this effect occurs for bounded distributions, $q < 1$. The reduction in the need of *tuning* a precise value of the noise intensity is of particular relevance both in technology and in order to understand how a biological system can exploit the stochastic resonance phenomenon. As was indicated in Ref. [9], non–Gaussian noises could be an intrinsic characteristic in biological systems, particularly in sensory systems [4, 8, 9]. The present results indicate that the noise model used here offers an adequate framework to analyze such a problem.

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