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# A Nonequilibrium Phase Transition Induced by Multiplicative Noise

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## 1 Introduction

Over the past 15 years, the effect of noise on the behaviour of nonlinear systems has been a major theme of investigation. An important example of a noise-induced phenomenon is the so-called noise-induced transition [1], referring to the situation in which the form of a probability density, describing the steady state properties of a noisy nonlinear system, undergoes a qualitative change. In its simplest version, one considers a system described by a scalar variable obeying the following nonlinear stochastic differential equation:

$$\dot{x} = f(x) + g(x)\xi \ , \tag{1}$$

where $\xi$ is a Gaussian white noise with intensity $\sigma^2$, interpreted in the Stratonovich sense. The steady state probability corresponding to (1) reads

$$P^{\mathrm{st}}(x) \sim \exp\left\{ \int^x dy \frac{f(y) - \frac{\sigma^2}{2} g(y) g'(y)}{\frac{\sigma^2}{2} g^2(y)} \right\} \ . \tag{2}$$

The extrema $\bar{x}$ of this probability density obey the following equation:

$$f(\bar{x}) - \frac{\sigma^2}{2} g(\bar{x}) g'(\bar{x}) = 0 \ . \tag{3}$$

Note the appearance of an additional term, resulting from the noise, which can change the type or degree of nonlinearity of the steady state equation. For example, for

$$\dot{x} = -x + \lambda(1 - x^2) + (1 - x^2)\xi \tag{4}$$

with $|x| < 1$, it is found that the probability density is unimodal, with a maximum at $\bar{x} = 0$ (which corresponds to the deterministic steady state) for $\lambda = 0$ and an intensity of the noise $\sigma^2 < 1$. However the density becomes bimodal for $\sigma^2 > 1$ (with new solutions $|\bar{x}| \neq 0$ appearing, cf. (3)). This phenomenon has been called noise-induced bistability. Nevertheless, one has to keep in mind that this change in the form of the probability density does not correspond to a genuine bifurcation or phase transition with breaking of ergodicity or of symmetry. In the above example, transitions between the $x > 0$ and $x < 0$ "phases" occur constantly, fluctuations in the value of $x$ are very large and one cannot talk about different macroscopic phases. Furthermore the $x \leftrightarrow -x$ symmetry of the model is not destroyed. One can reduce the frequency of the transitions by playing on time scales in more complicated models [2], but this is only a quantitative effect and there is no phase transition in the traditional sense of the word. Our purpose here is to investigate if a genuine phase transition can occur when considering spatially distributed systems. The answer turns out to be "yes", but the conditions for and characteristics of the transition are somewhat unexpected. In particular it is found that model (4) does not undergo a phase transition.

## 2 Mean Field Model

By spatially coupling units $i$ that are described by scalar variables $x_i$ with local dynamics identical to (1), one is led to the following set of stochastic differential equations:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i - \frac{D}{2d} \sum_{j \in n(i)} (x_i - x_j) \ , \tag{5}$$

where $\xi_i$ are uncorrelated Gaussian white noises with strength $\sigma^2$ and $n(i)$ represents the neigbourhood of unit $i$. We will be considering a cubic lattice so that there are exactly $2d$ such neighbours. The multivariate steady state probability associated to (5) is only known for the case of additive noise $g \equiv 1$. To make progress in the multiplicative noise case, we introduce the Weiss mean field approximation and assume that $\sum_{j \in n(i)} x_j = 2d\langle x \rangle$ where $\langle x \rangle$ is the average value which uniform throughout the system [3]. In this way, the equations for all the units decouple, and (5) takes on a form similar to (1), but with $f$ replaced by $f - D(x - \langle x \rangle)$. The solution for the single unit steady state probability thus reads (cf. (2), we dropped the subscripts $i$ for simplicity of notation):

$$P^{\rm st}(x) \sim \exp\left\{ \int^x dy \frac{f(y) - \frac{\sigma^2}{2} g(y)g'(y) - D(y - \langle x \rangle)}{\frac{\sigma^2}{2} g^2(y)} \right\} \ . \tag{6}$$

The value of $\langle x \rangle$ follows from the self-consistent requirement that

$$\langle x \rangle = \int dx \ x \ P^{\rm st}(x) = F(\langle x \rangle) \ . \tag{7}$$

Whenever this nonlinear equation in $\langle x \rangle$ has multiple solutions, the mean field theory predicts symmetry breaking associated to the occurrence of a phase transition. To get an idea of what kind of results to expect, we consider the limit

$D \to \infty$, in which case (7) reduces to the following simple form:

$$f(\langle x \rangle) + \frac{\sigma^2}{2} g(\langle x \rangle) g'(\langle x \rangle) = 0 \ . \tag{8}$$

This equation should be compared with (3). First note the difference in interpretation. The solutions to (8) correspond to the various macroscopic phases of our system, not to extrema of some probability density. Secondly, we note the surprising difference in the sign between (3) and (8). One of the consequences is that models that exhibit noise induced bistability do not present, in their spatially extended version, a phase transition to an ordered phase. We now turn to a model which does exhibit such a phase transition [4].

## 3 Noise-induced Phase Transition

Consider the following model:

$$f(x) = -x(1+x^2)^2 \qquad g(x) = 1 + x^2 \ . \tag{9}$$

$f$ and $g$ have been chosen such that the system displays a perfect $x \leftrightarrow -x$ symmetry. Yet, for an intensity of the noise larger then some critical value $\sigma^2 > \sigma_c^2$, where the value of $\sigma_c^2$ depends on $D$, the mean field theory predicts the appearance of ordered phases with $\langle x \rangle \neq 0$. This is already apparent from (8). By Taylor expansion around $\langle x \rangle = 0$, one finds

$$f(\langle x \rangle) + \frac{\sigma^2}{2} g(\langle x \rangle) g'(\langle x \rangle) = -\langle x \rangle + \sigma^2 \langle x \rangle - 2\langle x \rangle^3 + \sigma^2 \langle x \rangle^3 + \dots$$

so that two new symmetry breaking solutions $\langle x \rangle \sim \pm\sqrt{\sigma^2 - 1}$ appear for $\sigma^2 \geq 1$.

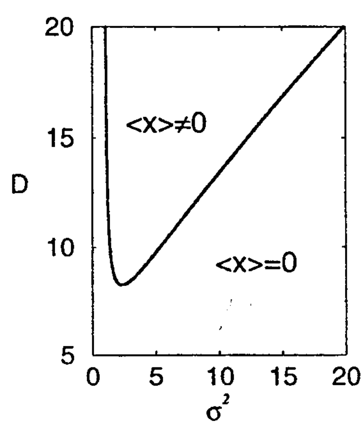


**Fig. 1.** Phase diagram for the noise-induced phase transition as predicted by the mean field theory.

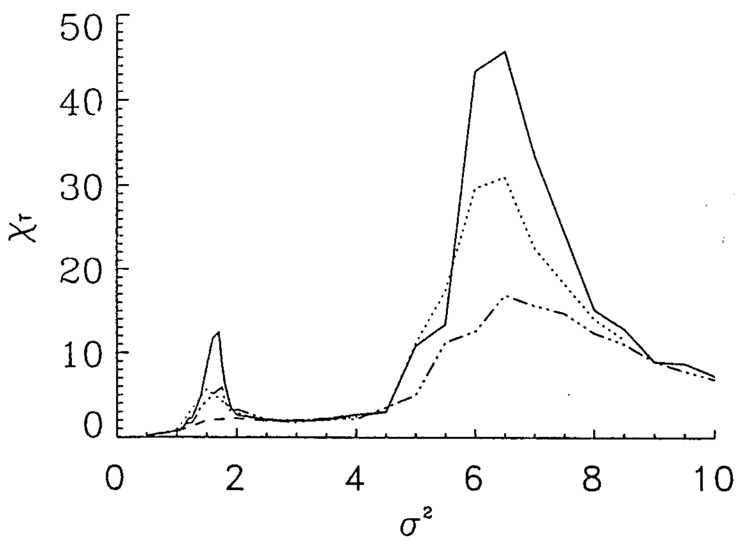


**Fig. 2.** Susceptibility in function of the intensity $\sigma^2$ for the value of $D = 20$ and for system sizes $10 \times 10$ (dashed line), $20 \times 20$ (dotted–dashed), $30 \times 30$ (dotted) and $40 \times 40$ (solid).

A numerical analysis of (7) confirms this result, and allows one to determine the region in parameter space $(D, \sigma^2)$ where the ordered phase appears (see Fig. 1). One concludes from the mean field analysis that the ordered phase appears for a sufficiently strong spatial coupling, and in a window of intermediate noise strengths. In other words, the transition is reentrant. These qualitative features are confirmed by extensive simulations of a 2-dimensional system. Furthermore, these simulations give convincing evidence of the fact that the appearance or disappearance of the ordered state takes place through a genuine second order phase transition with all the properties normally associated to equilibrium phase transitions such as scaling, divergence of the susceptibility and of the temporal and spatial correlations, finite size effects, etc. As an illustration, we have plotted in Fig. 2 the susceptibility in function of the intensity $\sigma^2$ for the value of $D = 20$ and for system sizes up to $40 \times 40$. One clearly sees the development of divergencies at the first and reentrant location of the phase transition. Note that the mean field prediction overestimates the location of the reentrant transition (for more details see [4]). Further and more accurate simulations will be needed to determine the value of the critical exponents or to determine the universality class of this new type of nonequilibrium transition.

## 4 Discussion

The present work ends the speculation about whether multiplicative noise can be an essential ingredient for the formation of structure. The answer is affirmative, but our theoretical results also indicate that this phenomenon cannot be properly discussed within the context of a theory for zero-dimensional systems. For example, the model defined in (9) does not undergo a noise-induced bistability. We therefore believe that the physical ingredients leading to the formation of ordered structures under influence of multiplicative noise are essentially different from those needed or discussed in the context of noise-induced transitions.

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