

# Synchronization Properties of Coupled FitzHugh-Nagumo Systems.

C. J. TESSONE R. TORAL

*Instituto Mediterráneo de Estudios Avanzados (IMEDEA) CSIC-UIB, Ed. Mateu Orfila, Campus UIB, 07122-Palma de Mallorca, Spain*

*Departament de Física, Universitat de les Illes Balears, Campus UIB, 07122-Palma de Mallorca, Spain*

C. R. MIRASSO

*Departament de Física, Universitat de les Illes Balears  
Campus UIB, 07122-Palma de Mallorca, Spain*

J. D. GUNTON

*Department of Physics, Lehigh University  
PA 18015, Bethlehem, USA*

**Summary.** — We study the synchronized/unsynchronized transition as a function of the noise intensity which appears in a system of globally coupled FitzHugh–Nagumo units under the effect of white noise. By the use of the proper definition of an order parameter, we obtain numerically the phase diagram as a function of the noise intensity and coupling constant.

## 1. – Model studied

In the last years, the phenomenon of synchronization in coupled limit cycle oscillators has been extensively investigated[1, 2]. However, not such a thorough study has been carried out for excitable systems, despite the fact that many features of the dynamics of biologically relevant systems, for example, neurons, can be described by simple excitable models. For example, it is found that epileptic crises are characterized by a particularly large amount of neurons firing simultaneously[3].

It is the goal of this paper to delve into some aspects of the synchronization properties of coupled excitable systems under the presence of noise. Although we do not have any specific applications in mind, we believe that our results are quite general, since we use a prototypical model of excitable dynamics. It is known that in those systems noise can induce phenomena such as stochastic resonance [4] (under the presence of an external forcing) or coherence resonance [5, 6, 7]. The latter is a mechanism by which an unforced excitable system shows a maximum degree of regularity in the period between emitted pulses in the presence of the right amount of noise. We focus here on the stationary synchronization properties of the common firings[8], and a more detailed study including the coherence resonance aspects is left for future work. We find that there is a non-equilibrium phase transition between synchronized and desynchronized states. We discuss the proper order parameter to characterize this transition and obtain numerically the phase diagram.

We consider the FitzHugh-Nagumo model which provides the simplest representation of firing dynamics and has been widely used as a model for spiking neurons as well as for cardiac cells[9, 10]. The model is defined in terms of activation  $x$  and inhibition  $y$  variables, as follows:

$$(1) \quad \epsilon \dot{x} = x - \frac{1}{3}x^3 - y$$

$$(2) \quad \dot{y} = x + a + D\xi(t)$$

where, following Ref.[5], a Gaussian white noise  $\xi(t)$  of zero mean and correlations  $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$  has been added to the slow variable  $y$ .  $D$  will be called the noise intensity. The difference in the time scales of  $x$  and  $y$  is measured by  $\epsilon$ , a small number. We work exclusively in the so-called excitable regime, characterized by  $|a| > 1$ .

There is a single stable fixed point  $(x_0, y_0)$  which, in the absence of any external perturbation,  $D = 0$ , is reached independently of the initial condition. When random perturbations are present, the trajectories eventually exit the basin of attraction of the stable fixed point and return to it after making an excursion in phase space, i.e. a pulse.

The next step is to consider an ensemble of  $N$  globally coupled systems:

$$(3) \quad \epsilon \dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{k}{N} \sum_{j=1}^N (x_j - x_i)$$

$$(4) \quad \dot{y}_i = x_i + a + D\xi_i(t), \quad i = 1, \dots, N$$

with independent noises,  $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$ . The systems are globally coupled by a gap-junctional form, as indicated by the last term of Eq.(3), where  $k$  is the coupling strength.

Numerical simulations of this coupled system of equations<sup>(1)</sup> show that, for some range of parameter values, the different units fire pulses at the same times. Notice that, although some amount of noise is needed in order to induce firings and hence observe synchronized behavior, too a large noise finally degrades the quality of the synchronized state. A general framework to study such synchronization phenomena is given by the work by Kuramoto[1]. He considers coupled phase variables  $\phi_i(t)$  following a stochastic dynamics and discusses the existence of a synchronized regime in terms of the coupling strength and the noise intensity. It turns out that the Kuramoto model displays a genuine phase-transition in which synchronization disappears if the noise surpasses a given critical value. We will show that the same behavior can be observed in our model.

The first step consists in defining phase-like variables  $\phi_i$  for our model. They should satisfy the condition that their variation between 0 and  $2\pi$  represents the pulse movement starting from the fixed point, traveling through all the cycle, and ending again at the fixed point. Several different approaches have been taken in order to evaluate the phases  $\phi_i$ . The most naïve, definition is based upon the fact that the limit cycles in which the variables  $(x_i, y_i)$  evolve are approximately centered around the origin. Then, the easiest choice is

$$(5) \quad \phi_i = \arctan\left(\frac{y_i}{x_i}\right).$$

However, this choice is only valid for particular cases of the parameters of the FitzHugh-Nagumo model. For large noise intensities, for example, the pulses are not so clearly centered around the origin. A definition of more general validity uses the so-called Hilbert transform[2]. Let us consider the variable  $x_i(t)$ . From it we can construct the so-called ‘‘analytic signal’’,  $s_i(t) = x_i(t) + i\hat{x}_i(t)$ , where  $\hat{x}_i(t)$  denotes the Hilbert transform of the function  $x_i(t)$ . For a general function,  $g(t)$ , such a transform is defined as

$$(6) \quad \hat{g}(t) = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{g(\tau)}{t-\tau} d\tau$$

where PV denotes the principal value of the integral. The phase is defined as the argument of  $s_i(t)$ , i.e.

$$(7) \quad \phi_i(t) = \arctan\left(\frac{\hat{x}_i}{x_i}\right).$$

From a computational point of view, it is very costly to perform the convolution involved in the Hilbert transform. We will show now that the same phase can be obtained

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<sup>(1)</sup> The numerical integration of eqs. (3-4) use a stochastic Runge-Kutta method (known as the Heun method [11]) with a time step  $h = 10^{-4}$ .

by a much more efficient procedure. This is based upon the equality

$$(8) \quad g(t) + i\hat{g}(t) = 2\mathcal{F}^{-1} [\mathcal{F} [g(t)] \cdot \Theta(\omega)]$$

involving the Fourier transform operator  $\mathcal{F}$ . Here  $\Theta(\omega)$  is the Heaviside function:  $\Theta(\omega) = 0$  for  $\theta < 0$ ,  $\Theta(\omega) = 1$  for  $\theta \geq 0$  defined in the Fourier space  $\omega$ .

This relation can be proved by replacing

$$(9) \quad g(t) = \int_{-\infty}^{\infty} g(t_o)\delta(t-t_o)dt_o,$$

in the right hand side of (8):

$$\begin{aligned} 2\mathcal{F}^{-1} \left[ \mathcal{F} \left[ \int_{-\infty}^{\infty} g(t_o)\delta(t-t_o)dt_o \right] \cdot \Theta(\omega) \right] &= 2 \int_{-\infty}^{\infty} \mathcal{F}^{-1} [\mathcal{F} [g(t_o)\delta(t-t_o)] \cdot \Theta(\omega)] dt_o \\ &= 2 \int_{-\infty}^{\infty} \mathcal{F}^{-1} \left[ g(t_o)\delta(t-t_o) \frac{e^{it_o\omega}}{\sqrt{2\pi}} \cdot \Theta(\omega) \right] dt_o \\ &= 2 \int_{-\infty}^{\infty} \left( \frac{1}{2}g(t_o)\delta(t-t_o) + \frac{i}{2\pi} \frac{g(t_o)}{t_o-t} \right) dt_o \\ &= g(t) + i\hat{g}(t) \end{aligned}$$

Thus one can achieve the calculation of the Hilbert transform by using two Fourier transforms. This leads to a very efficient numerical algorithm since the use of the fast

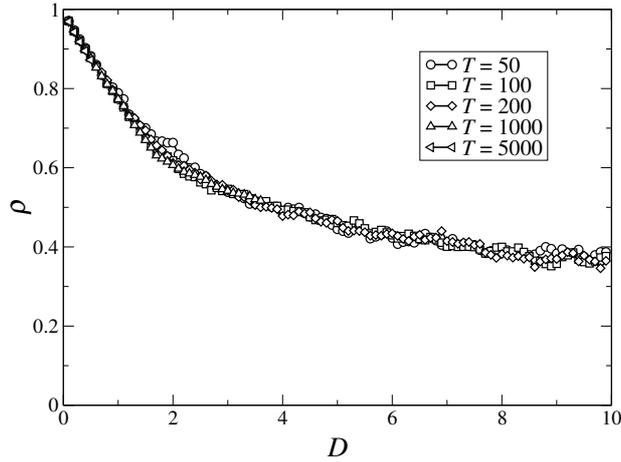


Fig. 1. – Order parameter  $\rho$  as a function of noise intensity for a system of globally coupled FitzHugh-Nagumo systems. Values of the parameters:  $a = 1.$ ,  $\epsilon = 0.01$ ,  $k = 1.$

Fourier transform involves a computer time of order  $\mathcal{O}(T \log T)$  instead of  $\mathcal{O}(T^2)$  which would be the case if one evaluates directly the convolution that defines the Hilbert transform ( $T$  is the length of the time series considered).

## 2. – Synchronization properties

We define an order parameter that allows us to measure the degree of synchronization in the coupled system. In order to follow the Kuramoto scheme, we use the phases  $\phi_i$  introduced before in terms of the Hilbert transform, Eq. 7. Collective amplitude,  $\rho(t)$ , and phase,  $\psi(t)$ , variables are defined as:

$$(10) \quad \rho(t)e^{i\psi(t)} = \frac{1}{N} \sum_{i=1}^N e^{i\phi_i(t)}.$$

Initially, the order parameter  $\rho$  introduced by Kuramoto is defined as the time average:

$$(11) \quad \rho \equiv \langle \rho(t) \rangle_t.$$

In figure 1 we plot  $\rho$  as a function of the noise intensity  $D$  for different number of coupled systems. It turns out that the order parameter continuously decreases with increasing  $D$ , thus showing that the quality of the synchronization worsens for large

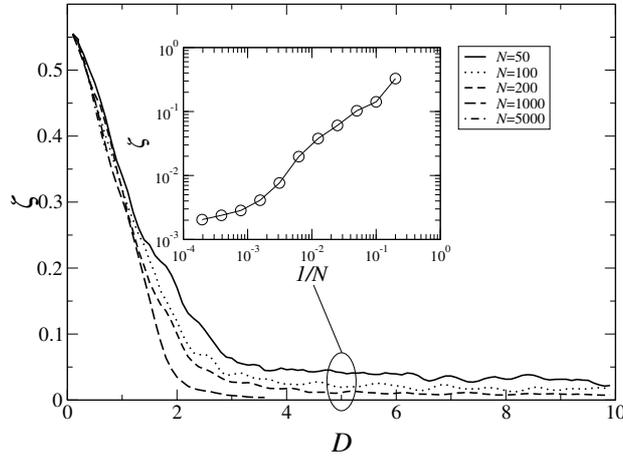


Fig. 2. – In this figure we show the order parameter  $\zeta$  for various system sizes, from  $N = 50$  to  $N = 10000$ . It is also shown the dependence of the order parameter with system size. The parameters of the simulation were  $a = 1$ . and  $k = 1$ . The inset shows that in the unsynchronized state, the finite size effects scale as  $\zeta = \zeta_\infty + \mathcal{O}[N^{-1/2}]$  for sufficiently large  $N$ .

noise intensity. The dependence of the order parameter  $\rho$  for relatively large system size ( $N > 100$ ) disappears, showing that finite size effects are very small for these systems sizes. Figure 1 shows that the Kuramoto order parameter for this coupled FitzHugh-Nagumo model does not decay to zero with increasing noise intensity. This is due to the fact that for an excitable system, most of the time all the units oscillate near the fixed point. Hence, the order parameter  $\rho$  is different from zero, even in the case in which all the units are uncoupled and fire unsynchronizedly. Since we are interested in measuring the deviations from this unsynchronized state, we use a different order parameter,  $\zeta$ , first introduced in reference [12]

$$(12) \quad \zeta = \left\langle \left| \rho(t) e^{i\psi(t)} - \left\langle \rho(t) e^{i\psi(t)} \right\rangle_t \right| \right\rangle_t.$$

In the case of complete desynchronization,  $\rho(t) e^{i\psi(t)}$  is almost constant except for finite system-size fluctuations (see the inset in figure 2):

$$(13) \quad \rho(t) e^{i\psi(t)} = \rho e^{i\langle\psi(t)\rangle_t} + \mathcal{O}[N^{-1/2}].$$

Replacing into Eq. 12, we get that  $\zeta \cong \mathcal{O}[N^{-1/2}]$ , and  $\zeta = 0$  in the thermodynamic limit. Therefore, any value of  $\zeta > 0$  indicates that the units have synchronized, and the larger the value of  $\zeta$  the higher the degree of synchronization.

In figure 2, we plot the new order parameter  $\zeta$ , for the same values of the parameters as in the previous figure. In this case, we notice the vanishing of the order parameter, indicating clearly the existence of a phase transition at a critical value  $D_c \approx 2.1$  separating the regime of synchronization/desynchronization. Note that the location of this transition could not be easily derived from the data in figure 1. The complete phase diagram for a wide range of values of noise intensity  $D$  and coupling constant  $k$  is plotted in figure 3.

### 3. – Conclusions

In summary, we have shown that an ensemble of globally coupled FitzHugh–Nagumo excitable systems subjected to independent noises experience a loss of synchronization for increasing noise intensity. Paradoxically, it is noise what initially induces the firings and sets the possibility of observing synchronized pulses.

The synchronization/desynchronization transition requires a proper definition of the order parameter for its characterization, since the usual measures used in coupled oscillators do not properly identify the transition point. We have found that a modified definition of the usual Kuramoto order parameter clearly displays such a transition. This order parameter is obtained from phase-like variables defined through the use of the Hilbert transform and we have given details of a numerically efficient method to compute the phase variables. Further work will aim to characterize this non-equilibrium transition and its universality class. Preliminary results show that the transition is present in locally coupled systems in  $d = 2$  dimensions, but not in  $d = 1$ .

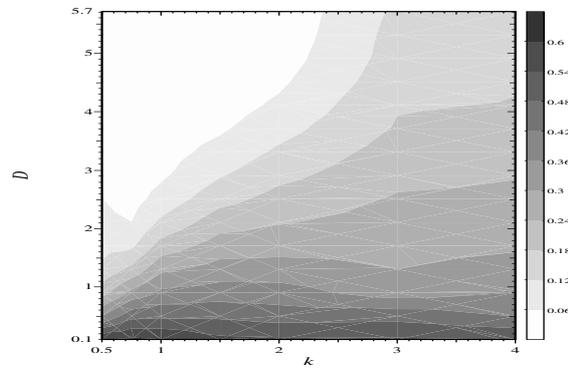


Fig. 3. – The order parameter  $\zeta$  is plotted against the control parameters  $k$ , and  $D$ , it is apparent a phase transition to desynchronization. In the simulations  $a = 1$ .

As stated before, we do not have any specific applications in mind, but since the FitzHugh–Nagumo equations have been widely used to model some biological systems, we believe that our results can be relevant when analyzing the collective response of such systems in a noisy environment.

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