Analysis and Characterization of the Hyperchaos Generated by a Semiconductor Laser Subject to a Delay Feedback Loop

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ABSTRACT

We characterize the chaotic dynamics of semiconductor lasers subject to either optical or electro-optical feedback modeled by Lang-Kobayashi and Ikeda equations, respectively. This characterization is relevant for secure optical communications based on chaos encryption. In particular, for each system we compute as function of tunable parameters the Lyapunov spectrum, Kaplan-Yorke dimension and Kolmogorov-Sinai entropy.

Keywords: Chaotic Lasers, Chaotic Communications, Delay, Feedback

1. INTRODUCTION

In last decade optical chaos encryption[1,2] has arised as a promising technique to improve and complement software or quantum cryptography. In this field, the masking of the message to be encoded is performed at the physical layer by the "mixing" of the signal with a chaotic carrier generated by some nonlinear optical element. The recovery of the message is based on the synchronization phenomenon by which a receiver, quite similar to the transmitter, is able to reproduce the chaotic part of the transmitted signal. After synchronization occurs, the decoding of the message is straightforward by comparing the input and output at the receiver.

A crucial issue in all encryption techniques is their security and how this is related to controlable parameters. The security of data encryption using the before-mentioned chaos methods relies upon two important points: the unpredictability of the carrier signal, and the sensibility exhibited by the dynamics of chaotic systems under parameter mismatch. Due to the second point, only a system very similar to the chaotic transmitter can be used to decode the message in an efficient way. From a practical point of view an exhaustive study of the first point is required to guarantee the security of the tranmission, since it is known that low dimensional chaos would make easy the interception of the message. This work adresses specifically this issue. Here we analyze the statistical properties of the chaotic signal and their dependence on tunable system parameters and type of feedback. In particular, we compute the Lyapunov exponents, the Kaplan-Yorke dimension (d_{KY}) and the Kolmogorov-Sinai entropy (h_{KS}) from appropiate models to describe the dynamics of semiconductor lasers with optical or electro-optical feedback.

For the computation of the Lyapunov exponents we have applied the ideas of Farmer[3] to our cases, integrating the corresponding delay differential equations with an Adams-Bashforth-Moulton fourth order predictor-corrector method. From the Lyapunov spectrum it can be also characterized both the geometrical and dynamical aspects of an strange attractor. The first can be accomplished by the computing the Kaplan-Yorke dimension which is an estimate for the information dimension. This is a measure of the degree of disorder of the points on the attractor or, more precisely, specifies how the amount of information needed to locate the system in the phase space with an accuracy ϵ scales with that resolution. However, the computational effort to compute the information dimension from the very definition or using the correlation integral technique is still nowadays nonattainable for very high dimensional systems. For this reason we use the Kaplan-Yorke conjecture that

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stands for the equality of the information dimension and the following quantity known as the Kaplan-Yorke dimension

$$d_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} \tag{1}$$

where the integer j, that represents the number of degrees of freedom, meets the conditions $\sum_{i=1}^{j} \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$ when the Lyapunov exponents are ordered by their magnitude from positive to negative values.

On the other hand, the degree of chaoticity of a system can be measured from a generalization of the concept of entropy for state space dynamics. The Kolmogorov-Sinai entropy measures the average loss of information rate, or equivalently is inversely proportional to the time interval over which the future evolution can be predicted. Its range of values goes from zero for regular dynamics, it is positive for chaotic systems and infinite for a perfectly stochastic process. The important point here is that the larger the entropy, the larger the unpredictability of the system what is a highly desired property to ensure security in a chaos encryption scheme. The computation of the Kolmogorov-Sinai entropy is again from the Lyapunov exponents through the so-called Pesin identity that states

$$h_{KS} = \sum_{i|\lambda_i > 0} \lambda_i \tag{2}$$

i.e. the Kolmogorov-Sinai entropy is equal to the sum of all the positive Lyapunov exponents. To be precise, the sum of the positive Lyapunov exponents is an upper bound to the Kolmogorov-Sinai entropy but the equality (2) seems to hold in very general situations and it is usually the only way to obtain a good estimation of h_{KS}.

This work is organized as follows. Section 2 and 3 are devoted to the characterization of high dimensional chaos in single-mode laser diodes with electro-optical and all-optical feedback, respectively. Some concluding remarks and future work are given in Section 4.

2. HIGH DIMENSIONAL CHAOS IN SEMICONDUCTOR LASERS WITH ELECTRO-OPTICAL FEEDBACK

The system considered in this section consists of an electrically tunable DBR multielectrode laser diode with a feedback loop formed by a delay line and an optical device whose peculiarity is to exhibit a nonlinearity in wavelength. This system was proposed by Goedgebuer and coworkers as the generator of the chaotic signal for an appropiate chaos encryption scheme.[4] The wavelength of the chaotic carrier is described by the following dynamical equation

$$\tau \frac{d\lambda(t)}{dt} = -\lambda(t) + \beta_\lambda sin^2 \left(\frac{\pi D}{\Lambda_0^2} \lambda(t-T) - \Phi_0 \right) \tag{3}$$

where λ is the wavelength deviation from the center wavelength Λ_0, D is the optical path difference of the birefringent plate which constitutes the nonlinearity, Φ_0 is the feedback phase, T is the delay time, τ is time constant in the feedback loop and β_λ is the feedback strength. Since the only nonlinearity in the model comes through the feedback term, the role of the parameter β_λ is twofold: it determines the strength of the feedback as well as the strength of the nonlinearity. Equation (3) is in fact an Ikeda equation and once normalized it takes the form

$$\frac{dx(t)}{dt} = -x(t) + \beta sin^2 \left(x(t-T) - \Phi_0 \right) \tag{4}$$

where the time has been scaled with τ, $x = \pi D \lambda / \Lambda_0^2$ and $\beta = \pi D \beta_\lambda / \Lambda_0^2$. In this dimensionless form, the model has clearly only three independent parameters, β, T and Φ_0 which influence on the dynamics of the system is studied below. It is also worth note that equation (4) follows a period doubling route to chaos when increasing the parameter β.

2.1. Lyapunov exponents

We first analyze the value of the Lyapunov exponents of the model described by equation (4) as function of the feedback strength and the delay time. Figure 1 shows the largest 30 Lyapunov exponents as function of β for delay times T = 5, 10, 20, 50, 100 and 250. For the first three values of the delay time we have explored up to $\beta = 30$ while for the last three only up to $\beta = 6$ because increasing the delay implies more time consuming calculations. In all cases the feedback phase have been fixed at $\Phi_0 = \pi/4$.

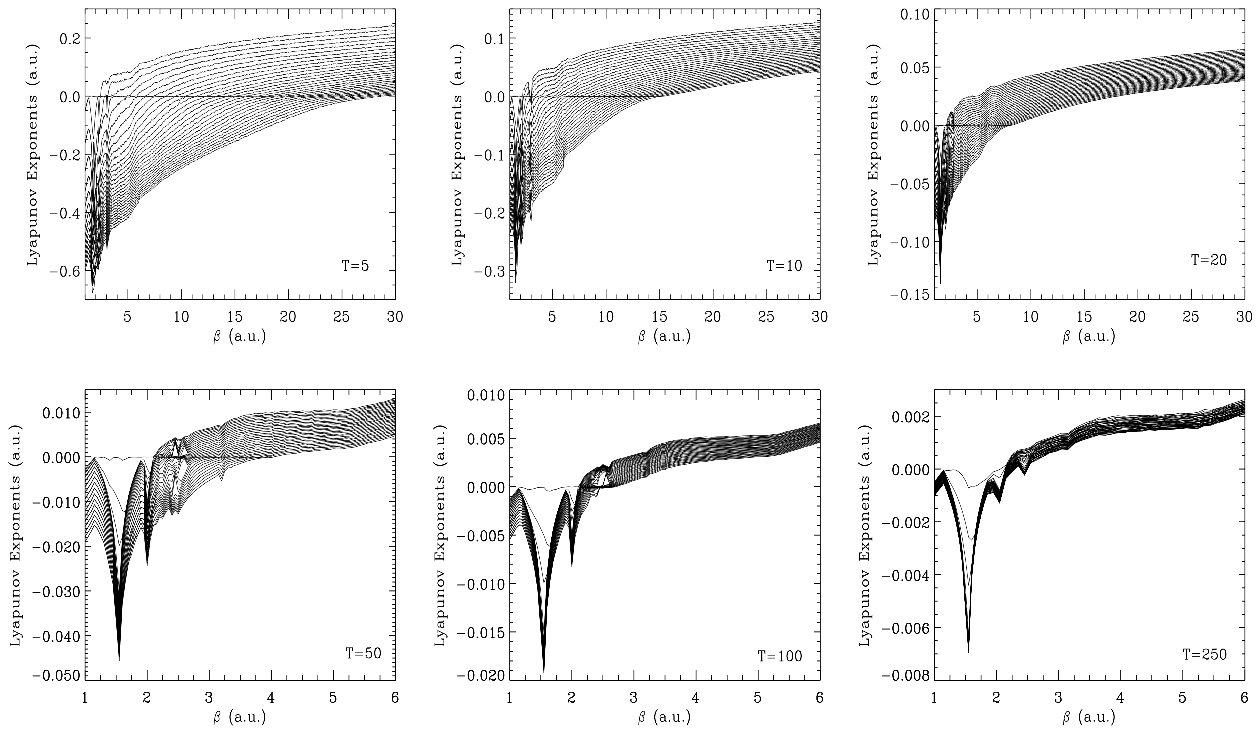

Figure 1. The largest 30 Lyapunov exponents as function of the feedback strength. The delay times are T = 5, 10, 20, 50, 100, 250, from left to right and from top to bottom.

As shown in the figure, the system has at least one positive Lyapunov exponent, and therefore displays chaotic behaviour, for $\beta > 2.1$. This threshold value, that corresponds to the accumulation point in the period doubling cascade, is practically the same for all time delays but it depends on the feedback phase as we will show below. For small values of the feedback strength, $\beta < 3$, the values of the Lyapunov exponents are strongly dependent on β and change with it in an irregular way. The behaviour becomes smoother as β is increased until to grow in a practically linear way. Comparing the figures corresponding to different delay times, it is clear that for a given value of β, the number of positive Lyapunov exponents increases with the delay, in fact it also grows linearly with T. This behaviour is similar to what was described by Farmer for the Mackey-Glass equation.[3] For example, for $\beta = 20$ one finds 20 positive Lyapunov exponents for T=5 while for T=20 one finds 78. However, the value of the positive exponents decreases as the delay is increased. For $\beta = 20$ and T=5 the largest exponent has a value $\lambda_1 = 0.2078$ while for T=20 it takes the value $\lambda_1 = 0.0566$ which is about four times smaller. Therefore, while increasing the delay time increases the number of positive Lyapunov exponents linearly, their value decreases also linearly. This fact, is used in following subsections to explain the behaviour of the Kolmogorov-Sinai entropy as function of the delay.

The Lyapunov spectra plotted in Figure 1 for different delay times display some degree of self-similarity. In fact, it is possible to rescale the axis corresponding to the Lyapunov exponents multiplying its value by the delay time in such a way that the different pannels in Figure 1 nearly overlap. Figure 2 shows the first and second Lyapunov exponents as function of β for different delay times scaled in this form. The scaling is quite good even for short delay times (T=5) and improves as the delay time is increased. Therefore, we conclude that asymptotically (large T and β) the Lyapunov exponents scale as $\lambda \propto \beta/T$.

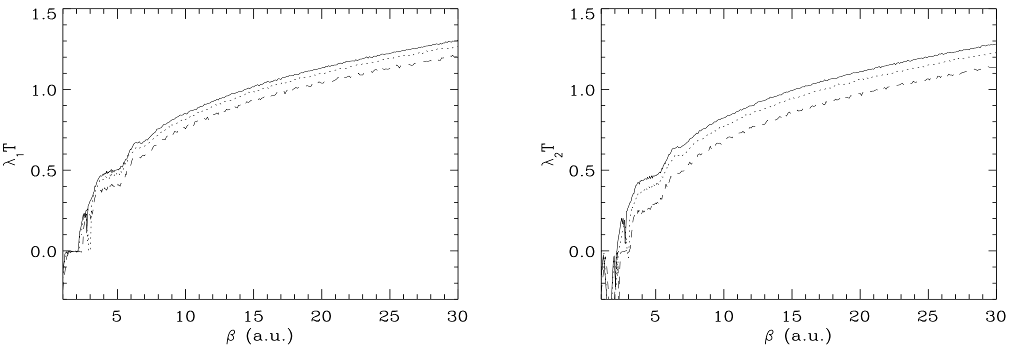

Figure 2. In the left pannel it is shown the largest Lyapunov exponent scaled with the delay time for T=5 (dashed), T=10 (dotted) and T=20 (solid). The same is plotted for the second largest Lyapunov exponent in the right pannel.

We now adress the role of the delay phase Φ_0. We consider a fixed delay time T=20 and plot in Figure 3 the largest 30 Lyapunov exponents as function of the nonlinearity strength β for feedback phases $\Phi_0 = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$. Note that we have only explored this range because of the symmetry of equation (4) to the change $\Phi_0 \mapsto \Phi_0 + \pi$.

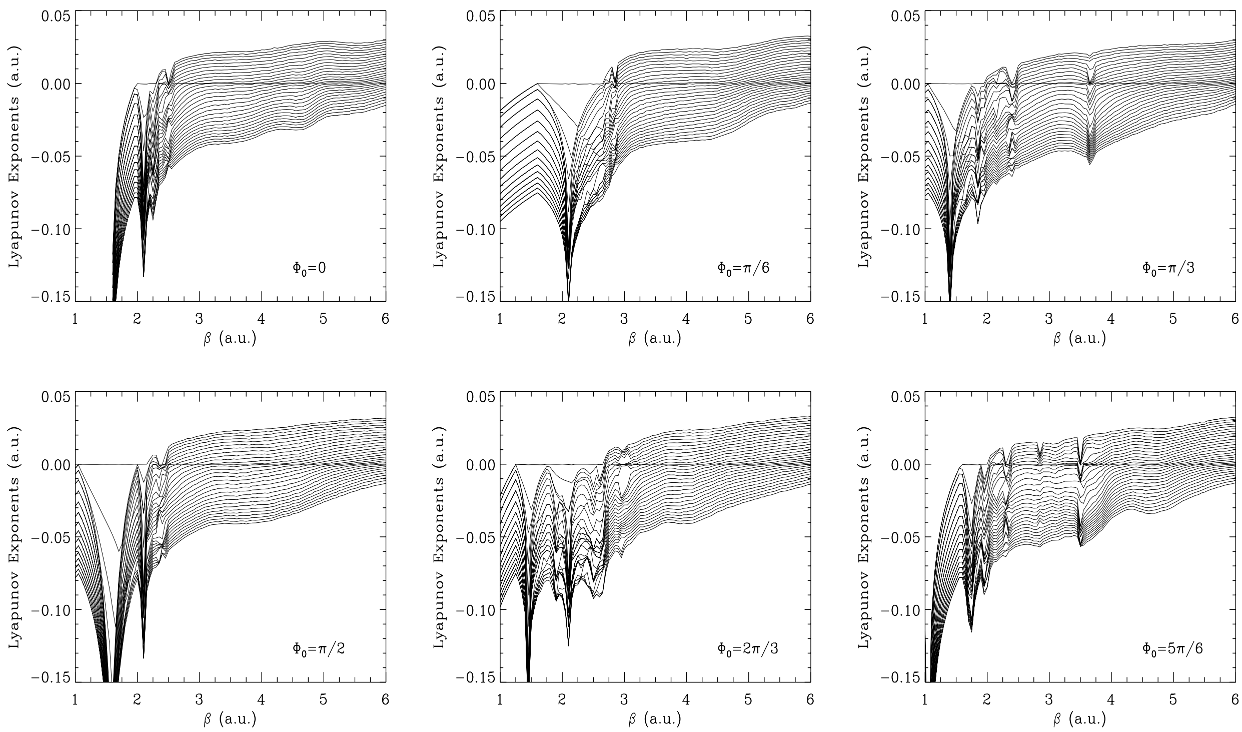

Figure 3: Lyapunov spectra as function of feedback strength for different feedback phases.

For small values of β, all the Lyapunov exponents are negative indicating that the system evolves toward a stable fixed point and remains there. Depending on the value of the feedback phase the fixed point becomes unstable at different values of β. In the case of T=20, for $\Phi_0 = 0$ the fixed point is stable up to $\beta = 2$ while for $\Phi_0 = \pi/3$ or $\pi/2$ it becomes unstable to a limit cycle for β just above 1. The positivity of one Lyapunov exponent, what signals the transition to chaos, is also clearly phase dependent. Depending on the value of the phase some periodic windows may appear within the chaotic regions which are signaled by the largest Lyapunov exponent becoming zero again. The periodic windows can be quite narrow and are located at specific values of β. The number of periodic windows as well as their location depends on the value of the phase. As β is increased the influence of the feedback phase decreases and for $\beta > 5$ the value of the Lyapunov exponents becomes practically independent of Φ_0.

2.2. Information dimension

In this section we focus on the dimension of the chaotic attractor computed through the Kaplan-Yorke conjecture stated in the introduction. Figure 4 shows the information dimension (conjectured to be equal to d_{KY}) as function of the feedback strength for delay times T = 5, 10, 20, 50, 100 and 250 scaled with T.

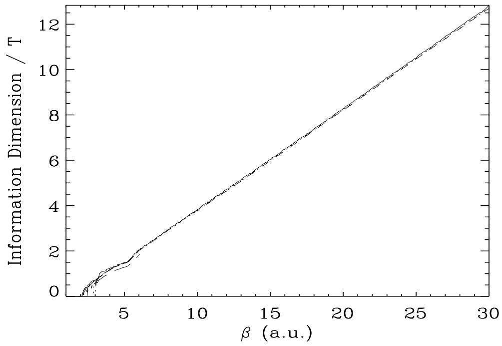

Figure 4: Information dimension for T = 5, 10, 20, 50, 100 and 250 scaled with the delay time.

As shown, for large values of the β parameter, the dimension grows linearly with the feedback intensity. It also grows linearly with the feedback time, in accordance to what was observed in the Mackey-Glass model. Therefore, for values of β large enough, the Kaplan-Yorke dimension should follow an equation of the form

$$d_{KY} = CT\beta \tag{5}$$

where C is a constant. Dimensions as large as 250 can easily achieved for T = 20 and $\beta = 30$ or for weaker feedback strength with longer delays such as $\beta = 4$ with T = 250.

2.3. Kolmogorov-Sinai entropy

In the following we study the Kolmogorov-Sinai entropy h_{KS} that, as it was before-mentioned, measures the unpredictibility of a system. Figure 5 shows the Kolmogorov-Sinai entropy as function of the feedback strength for six different delay times. It is clear from the figure that all the plots overlap, which indicates that the entropy saturates with the delay, and this effect is already achieved with delay T = 5. In this case, the growth of the number of positive Lyapunov exponents when the delay is increased is compensated by the fact that their magnitude decrease in an inversely way, what results in a basically constant value for h_{KS}. Figure 6 also indicates that the Kolmogorov-Sinai entropy grows with the feedback strength and that for large values of the β parameter the entropy obeys

$$h_{KS} = C'\beta \tag{6}$$

where C' is a constant independent of the delay time.

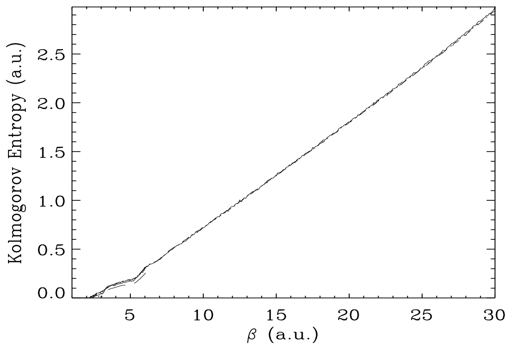

Figure 5: Kolmogorov-Sinai entropy for T = 5, 10, 20, 50, 100 and 250.

3. HIGH DIMENSIONAL CHAOS IN SEMICONDUCTOR LASERS WITH OPTICAL FEEDBACK

A prototypical model to describe single-mode semiconductor lasers subject to optical coherent feedback are the Lang-Kobayashi equations[5] for complex slowly varying amplitude of the electric field $E(t)$ and the carrier number inside the cavity $N(t)$

$$\dot{E}(t) = \frac{(1+i\alpha)}{2}\left[G - \frac{1}{\tau_{ph}}\right] E + \kappa E(t-\tau) e^{-i\Omega\tau} \quad (7)$$

$$\dot{N}(t) = \frac{I}{e} - \frac{N}{\tau_n} - G|E|^2 \quad (8)$$

where $G \equiv g(N - N_o)/\left(1 + s|E|^2\right)$ is the optical gain, Ω is the frequency of the free-running laser, κ is the feedback coefficient and τ is the external cavity round-trip. We consider the following values for the internal parameters: α=5 is the linewidth enhancement factor, $g = 1.5 \times 10^{-8}$ is the differential gain parameter, $s = 5 \times 10^{-7}$ is the gain saturation coefficient, $\tau_{ph} = 2$ ps is the photon lifetime, $\tau_n = 2$ ns is the carrier lifetime and $N_0 = 1.5 \times 10^8$ is the carrier number at transparency.

The Lang-Kobayashi model includes the feedback after one roundtrip in the external cavity and therefore it may not be valid in regimes of strong optical feedback where multiple external cavity roundtrips should be accounted for. In this section we consider feedback coefficients up to 30 ns^{-1} (corresponding to a reflectivity for the external mirror less than 3%) what consequently locates our study under the assumptions of this model. We should also note that at difference with the model (3), in equations (7-8) the feedback term is linear while the nonlinearities of the system are local in time.

In the following subsections we will analyze the dependence of the chaos characteristics on the parameters that are easily accesible from a practical point of view, namely, the pump current I, the feedback strenght κ, the delay time τ and the feedback phase $\Phi \equiv \Omega\tau \ mod(2\pi)$. The feedback phase can be changed from 0 to 2π by changing the roundtrip cavity length within one optical wavelength, which practically implies a negligible change in τ. Therefore, in practice the feedback phase and the cavity length can be adjusted independently.

3.1. Lyapunov exponents

We first analyze the value of the Lyapunov exponents as function of the feedback strength κ and delay time τ. In Figure 6 are represented the Lyapunov exponents as function of κ for τ = 100, 200, 300 and 1000 ps in the case where the pump is set at 1.5 times the threshold. Note that at this pump, the frequency of the relaxation oscillations is about 4.1 GHz and therefore with the former range of values considered for the delay time we are both exploring the situations of short cavity regime (external cavity frequency larger than relaxation oscillations frequency) and long cavity regime (external cavity frequency smaller than relaxation oscillations frequency).

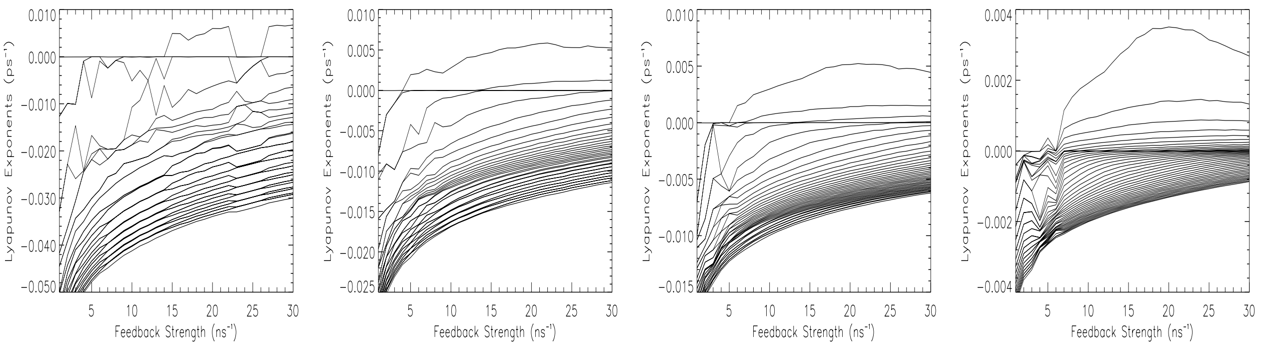

Figure 6. Largest 30 Lyapunov exponents as function of the feedback strength. From left to right, τ = 100, 200, 300 and 1000 ps. The pump current is I = $1.5\mathrm{I}_{th}$ the feedback phase has been fixed at $\Phi_0 = \pi/2$.

For very short external cavities (τ = 100 ps) the Lyapunov exponents as function of the feedback strength are quite irregular and at most only one positive exponent is obtained. In this regime there is also a strong dependence on the phase of the feedback as we will show later in this subsection. For longer cavities the behaviour becomes more regular and more Lyapunov exponents arise. However, there is a significant difference with respect to the electro-optical feedback case described in section 2. As the feedback strength is increased the value of the largest Lyapunov exponent goes through a maximum (at about $\kappa = 20\ \mathrm{ns}^{-1}$) and the decreases, therefore in the case considered here increasing the feedback strength does not imply larger values for the Lyapunov exponents. This will have significant consequences in the Kolmogorov-Sinai entropy, as it will be discussed later.

In Figure 7 we plot the dependence of the value of the Lyapunov exponents as function of the delay time when the pump has been fixed at 1.5 I_{th} and the feedback is $\kappa = 10\ \mathrm{ns}^{-1}$. The number of positive Lyapunov exponents increases with the delay, although the magnitude of the positive exponents decreases, similarly to what was found in the case of electro-optical feedback. Also as in the previous case for large delays (above > 300 ps) the number of positive Lyapunov exponents and their value depend almost linearly with the delay time.

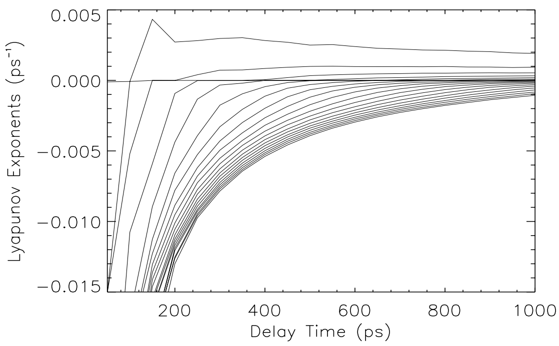

Figure 7: Largest 20 Lyapunov exponents as function of the delay time for I = $1.5\mathrm{I}_{th}$, κ = 10 ns $^{-1}$ and $\Phi_0 = \pi/2$.

We now address the role of the delay phase. We plot in Figure 8 the first 20 Lyapunov exponents as function of the feedback phase for delay times τ = 100, 200, 300 and 1000 ps. For small values of the delay (τ = 100 ps and 200 ps which correspond to the short cavity regime), there is a strong dependence on the feedback phase. In that regimes, the system can exhibit periodic or chaotic behaviour depending on the specific value of Φ_0. Similarly to what is obtained for the Ikeda equation (4), for larger delay times there is practically no dependence on the feedback phase.

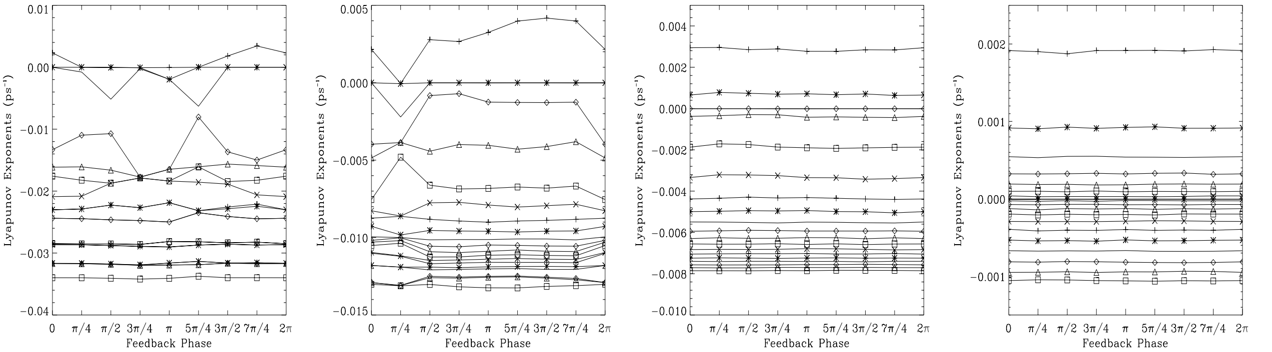

Figure 8. The first 20 Lyapunov exponents as function of the feedback phase for pump I = $1.5\mathrm{I}_{th}$. From left to right, τ = 100, 200, 300 and 1000 ps.

The dependence on the pump current is shown in detail in Figure 9 for a feedback coefficient $\kappa = 20$ ns^{-1} with delay time $\tau = 300$ ps (left pannel) and $\tau = 1000$ ps (right pannel). The largest Lyapunov exponent goes in both cases through a maximum for I = $1.5\mathrm{I}_{th}$ and decreases until reaches a zero value (what indicates a return to a regular dynamics) for I $> 2.8\mathrm{I}_{th}$ when τ is 300 ps and for I $> 3.1\mathrm{I}_{th}$ when τ is 1000 ps.

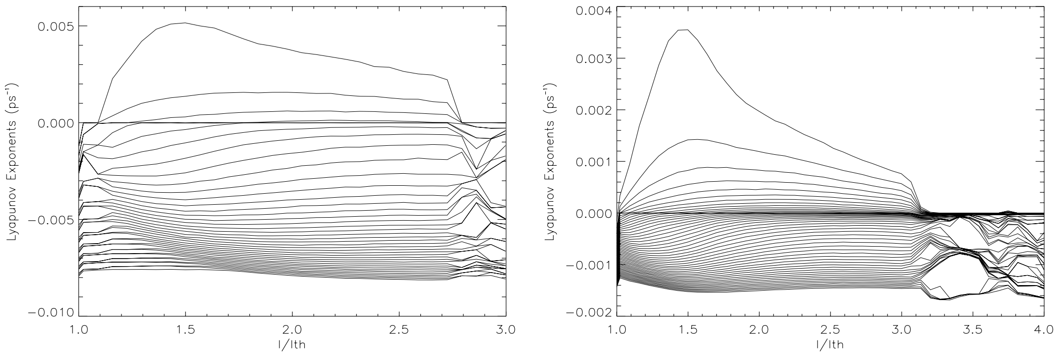

Figure 9. The first 40 Lyapunov exponents as function of the current pump for τ = 300 ps (left) and τ = 1000 ps (right). The feedback strength has been taken as $\kappa = 20$ ns^{-1}.

3.2. Information dimension

Figure 10 indicates how the information dimension of the system grows almost in a linear way with the feedback (at least up to $\kappa = 30$ ns^{-1}) for large enough delay times. In the case of very short external cavities ($\tau = 100$ ps) the dimension changes irregularly, as expected from the value of the Lyapunov exponents shown in the first pannel of Figure 6.

When the delay is varied, the information dimension increases linearly with it in accordance to what is obtained in the case of electro-optical feedback and in the Mackey-Glass model.[3]

The dependence of the dimension on the pump current is shown in Figure 11 for a feedback strength $\kappa = 20$ ns^{-1}. For moderately short ($\tau = 300$ ps) and also for long cavities ($\tau = 1000$ ps) the dimension goes through a maximum value when the pump is increased that is located at I = $2.2\mathrm{I}_{th}$. For larger pump values in both cases the dimension fails to zero in correspondence with the periodic behaviour indicated by the Lyapunov exponents.

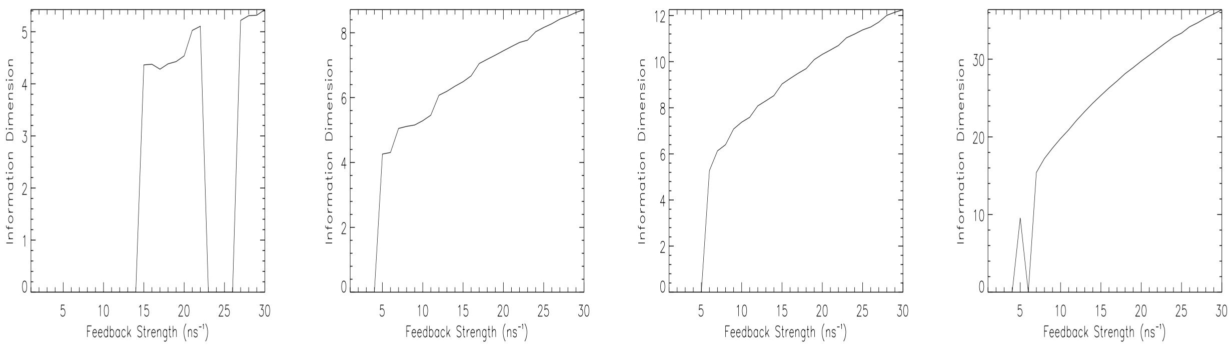

Figure 10. Information dimension as function of the feedback for pump I = $1.5I_{th}$. From left to right, τ = 100, 200, 300 and 1000 ps.

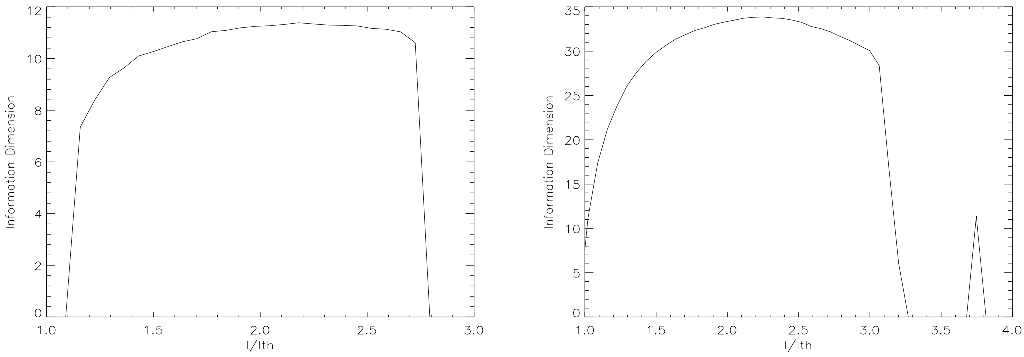

Figure 11. Information dimension as function of the pump for τ = 300 ps (left) and τ = 1000 ps (right). The feedback is 20 ns^{-1}.

3.3. Kolmogorov-Sinai entropy

Figure 12 (left) shows the Kolmogorov-Sinai entropy as function of the feedback strength for pump I=$1.5I_{th}$. The different symbols correspond to different delay times from 200 ps to 1000 ps. The three curves basically coincide, what indicates the saturation of the entropy with the delay time as it is clearly shown in Figure 12 (right). As it happens in the case of electro-optical feedback, increasing the delay increases the information dimension because we have more positive Lyapunov exponents. However, as their value becomes smaller, the Kolmogorov-Sinai entropy remains basically constant.

There is however, an important difference with respect to the electro-optical feedback case, namely, that now the entropy does not increases linearly with the strength of the feedback, it rather reaches a maximum and the it decreases. When the entropy is studied varying the pump current, there is clearly a maximum for the entropy that is reached at I = $1.5I_{th}$ as it is observed in Figure 13.

Therefore, the conclusion is that is not easy to increase the value of the entropy in the case considered in this section. For a given pump value, increasing the feedback level beyond an optimal value leads to a decreasing of the entropy. For a given feedback strength, increasing the pump beyond an optimal value, also leads to a decreasing value for the entropy. A possibility is to simultaneous increase the pump and the feedback level. However, from a practical point of view the pump level can not be increased beyond certain limit without damaging the semiconductor laser.

4. CONCLUSIONS

As the systems we are dealing with are delayed systems, the number of positive Lyapunov exponents grows linearly with the delay time. This is a general characteristic of delayed systems. The Kaplan-Yorke dimension

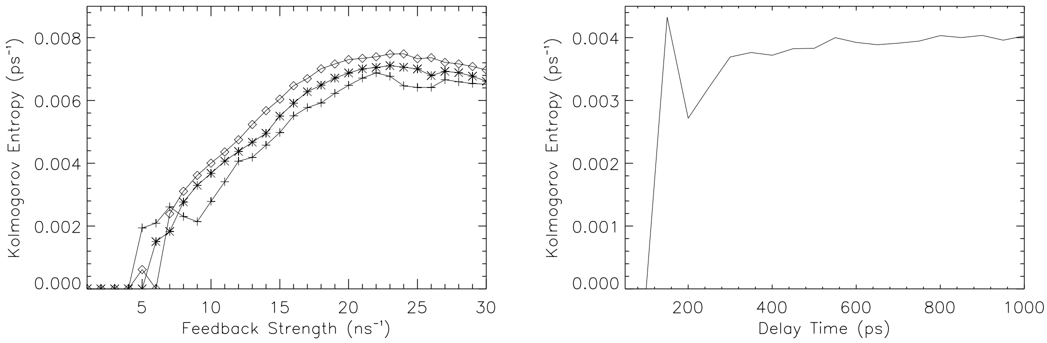

Figure 12. (Left) Kolmogorov-Sinai entropy as function of the feedback strength for delay times $\tau = 200$ ps (crosses), 300 ps (asterisks) and 1000 ps (diamonds). (Right) Kolmogorov-Sinai entropy as function of the delay time for $\kappa = 10$ ns^{-1}.

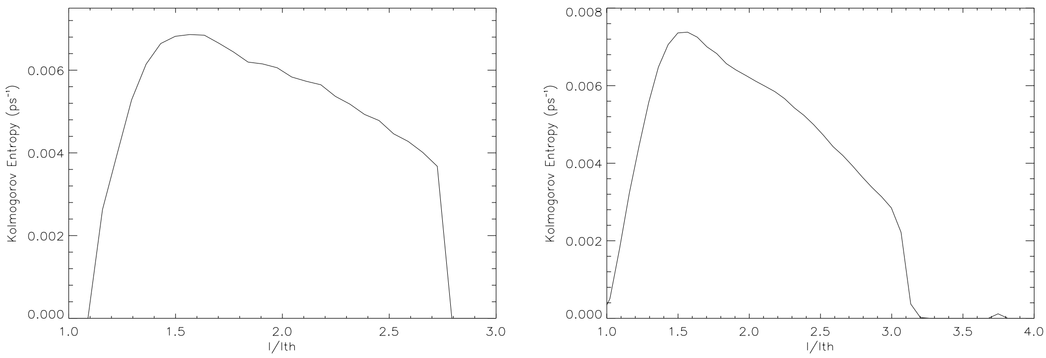

Figure 13. Kolmogorov-Sinai entropy as function of the pump for $\tau = 300$ ps (left) and $\tau = 1000$ ps (right). The feedback is 20 ns^{-1}.

increases also linearly with the delay time. Therefore, very large dimensionalities can be achieved. However, the Lyapunov exponents that become positive as the delay time is increased have a very small absolute value. This, together with the fact that the largest positive Lyapunov exponent decreases as the delay time increases, yields saturation in the Kolmogorov-Sinai entropy. This is also what happens in delayed maps, where the number of peirodic orbits and the topological entropy is bounded when the delay time is increased.[6] Therefore, although the system has a larger dimensionality when increasing the delay, its behaviour does not become more unpredictable. Consequently, for the purpose of using this chaotic output as a carrier for encoding a message, these results suggest that increasing the delay time beyond the value at which the entropy saturates will neither yield a better masking nor improve the security.

In the electro-optical case, the feedback is nonlinear while the laser operates in the linear regime. The number of positive Lyapunov exponents as well as their value increases with the feedback strength in a linear way. Therefore, the Kaplan-Yorke dimension and the Kolmogorov-Sinai entropy grow also linearly with the feedback strength. A clear way to achieve a better masking and more secure encoding is to increase the nonlinear feedback strength.

In the all optical case, the feedback is linear and nonlinearities come from the laser itself. Keeping a constant pump value and increasing the feedback level, the number of positive Lyapunov exponents and their value increases up to a certain value of the feedback strength. Beyond this value, the largest Lyapunov exponent starts to decreases. For a slightly larger value, the second largest Lyapunov exponent also start to decrease, and so on. As a consequence, the Kaplan-Yorke dimension does not grow linearly with the feedback strength any more and the Kolmogorov-Sinai entropy reaches a maximum and then decreases for larger feedback values. So, for a given pump value, there is an optimal feedback strength for masking. Keeping the feedback strength fixed and increasing the pump current, the Kolmogorov-Sinai entropy also goes through a maximum at an optimal pump value. It has been also observed in the model that the entropy fails to zero beyond a given pump value, that depends on the specific values considered for the feedback and delay times. This fact means the return to regular dynamics after a large range of pump currents where the system was operating in the coherence collapse regime.

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