Linear Instability Mechanisms of Noise-Induced Phase Transitions

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Abstract. We review the role of linear instabilities on phase transition processes induced by external spatiotemporal noise. In particular, we present a detailed linear stability analysis of a standard Ginzburg–Landau model with multiplicative noise. The results show the well-known constructive role of fluctuations in this case. The analysis is performed for both non-conserved and conserved dynamics, corresponding to order-disorder and phase separation transitions, respectively.

1 Introduction

Among all counterintuitive influence that external noise can exert on dynamical systems, spatial ordering has received a special attention in last years. Noise-induced patterns, for instance, have been observed in Swift-Hohenberg models in the presence of external fluctuations [1–3]. The mechanism through which the pattern-forming instability arises is linear, and the role of noise (which as an external noise has to be interpreted in the Stratonovich sense) is to renormalize the coefficients of the corresponding dispersion relation [4, 5]. Subsequent investigations showed the existence of noise-induced phase transitions in field models[1] [6, 7], some of which were attributed to a short-time instability that survives through observable time scales due to entrainment caused by spatial coupling [8]. However, it can also be shown in these cases that the mechanism through which noise destabilizes the disordered phase is again linear [9, 10]. In the following pages, we review in detail the influence of multiplicative noise in the linear destabilization of a homogeneous phase, which leads to the appearance of a noise-induced phase transition. This linear instability mechanism is not unique, since in some cases the destabilization is driven by a nonlinear mechanism [11]. Nevertheless, those situations are still a minority, and will not be considered in the following.

[1] Throughout this paper, we use the term "phase transition" in the statistical-mechanics sense, to denote transitions between *macroscopic* states that display universal properties in the thermodynamic limit.

The stability analysis will be performed for two kinds of dynamics, the first of which is a standard relaxational model that evolves towards one of two phases (ordered or disordered) with no restriction. In the second case, the dynamics is restricted by a conservation law of the spatially averaged field, which leads in the ordered state to a process of phase separation. Whereas the linear stability analysis of the former, non-conserved systems is so far well known [4], that of the conserved case is introduced here in more detail.

2 Linear Stability Analysis

Transitions between two macroscopic phases in a given system occur due to the loss of stability of the initial state for certain values of the control parameters. It is well known nowadays that some types of noise can modify the stability of a state, and thus change the parameter values at which the transition takes place (i.e. the transition point). In many cases, the mechanism through which the destabilization arises is linear, and therefore the location of the corresponding transition point can be found by means of a linear stability analysis. This analysis consists on studying the dynamical behavior of a perturbation applied initially to the state whose stability is being examined. In a linear approximation, valid only at short times, these perturbations either grow or decay exponentially in time. In the first case the initial state is unstable, in the second one it is stable.

In the case of stochastic systems, the linear stability analysis needs to be performed on a statistical moment of the pertubed state. Contrarily to homogeneous (zero-dimensional) systems, the onset of instability for spatially extended systems is the same for all statistical moments (at least when moded–coupling contributions are discarded) [5]. It is especially interesting in this case to perform the analysis on the *structure function*, since this quantity (which is the Fourier transform of the second statistical moment) is proportional to the intensity of scattered light in X-ray and neutron diffraction experiments. Moreover, for conserved systems (whose first moment is constant in time) we are forced to study the second statistical moment (or a higher-order one).

We will now perform the linear stability analysis of the structure function for the particular case of models A and B (using the terminology of critical dynamics [12]) with spatiotemporal multiplicative noise. Model A is the prototype of a non-conserved system, model B of a conserved one.

2.1 Model A

This nonconserved model is defined by

$$\frac{\partial \phi(\boldsymbol{r},t)}{\partial t} = -a\phi - \phi^3 + D\,\nabla^2\phi + \phi\,\xi(\boldsymbol{r},t) + \eta(\boldsymbol{r},t)\,, \tag{1}$$

where both additive and multiplicative noises are Gaussian with zero mean and correlations

$$\langle \eta(\boldsymbol{r},t)\,\eta(\boldsymbol{r}',t')\rangle = 2\,\varepsilon\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(t-t') \tag{2a}$$

$$\langle \xi(\boldsymbol{r},t)\,\xi(\boldsymbol{r}',t')\rangle = 2\,\sigma^2\,c(|\boldsymbol{r}-\boldsymbol{r}'|)\,\delta(t-t')\,, \tag{2b}$$

and ε and σ^2 are the additive and multiplicative noise intensities, respectively. The function $c(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the spatial correlation function of the external noise, which becomes $\delta(\boldsymbol{r}-\boldsymbol{r}')$ in the limit of zero correlation length. This external multiplicative noise can be understood as fluctuations in the control parameter a.

The linear stability analysis is performed in a discrete version of the model. In a d–dimensional discrete lattice of mesh size Δx, (1) takes the form:

$$\frac{d\phi_i}{dt} = -a\,\phi_i - \phi_i^3 + D\sum_j \widetilde{D}_{ij}\,\phi_j + \eta_i(t) + \phi_i\,\xi_i(t)\ , \tag{3}$$

where $\phi_i \equiv \phi(\boldsymbol{r}_i)$, $\boldsymbol{r}_i = \Delta x\,\boldsymbol{i}$, $\boldsymbol{i} \in [0, L-1]^d$ and L is the number of cells on each side of the regular lattice. The sum runs over the whole lattice, and only one index is used to label all cells, independently of the dimension of the system. $\widetilde{D}_{ij}$ accounts for the discretized Laplacian operator

$$\nabla^2 \;\rightarrow\; \sum_j \widetilde{D}_{ij} = \frac{1}{\Delta x^2}\sum_j \left(\delta_{nn(i),j} - 2d\,\delta_{ij}\right)\ , \tag{4}$$

where $nn(i)$ represents the set of all sites which are nearest neighbors of cell i. The discrete noises $\eta_i(t)$ and $\xi_i(t)$ are still Gaussian with zero mean and correlations

$$\langle \eta_i(t)\,\eta_j(t')\rangle = 2\,\varepsilon\,\frac{\delta_{ij}}{\Delta x^d}\,\delta(t-t') \tag{5a}$$

$$\langle \xi_i(t)\,\xi_j(t')\rangle = 2\,\sigma^2\,c_{|i-j|}\,\delta(t-t')\ , \tag{5b}$$

where $c_{|i-j|}$ is a convenient discretization of the function $c(|\boldsymbol{r}-\boldsymbol{r}'|)$, which in the limit of zero correlation length becomes $\delta_{ij}/\Delta x^d$.

In order to study the stability of the homogeneous state ($\phi_i(t) = 0\ \ \forall i$), we linearize (3) and look for the dynamical equation of the two point correlation function $\langle\phi_i\phi_j\rangle$:

$$\frac{d}{dt}\langle\phi_i\phi_j\rangle = -2a\,\langle\phi_i\phi_j\rangle + D\sum_s\Big(\widetilde{D}_{is}\langle\phi_s\phi_j\rangle + \widetilde{D}_{js}\langle\phi_s\phi_i\rangle\Big) + \\ +\langle\xi_i\phi_i\phi_j\rangle + \langle\xi_j\phi_j\phi_i\rangle + \langle\eta_i\phi_j\rangle + \langle\eta_j\phi_i\rangle. \tag{6}$$

The last four terms of this equation can be calculated with the help of Novikov's theorem [13], which in our case takes the forms:

$$\langle\xi_i(t)\phi_i(t)\phi_j(t)\rangle = \sum_s\int dt'\,\langle\xi_i(t)\,\xi_s(t')\rangle\left\langle\frac{\delta(\phi_i(t)\phi_j(t))}{\delta\xi_s(t')}\right\rangle \tag{7a}$$

$$\langle \eta_i(t)\phi_j(t)\rangle = \sum_s \int dt' \langle \eta_i(t)\eta_s(t')\rangle \left\langle \frac{\delta\phi_j(t)}{\delta\eta_s(t')} \right\rangle . \tag{7b}$$

Now, by formally integrating the linear terms of (3),

$$\phi_i(t) = \phi_i(0) + \int_0^t dt' \left(-a\,\phi_i(t') + D \sum_j \widetilde{D}_{ij}\,\phi_j(t') + \eta_i(t') + \phi_i(t')\,\xi_i(t') \right) \tag{8}$$

we can calculate the response functions at equal times

$$\left.\frac{\delta\phi_i(t)}{\delta\xi_s(t')}\right|_{t'=t} = \phi_i(t)\,\delta_{is} , \qquad \left.\frac{\delta\phi_i(t)}{\delta\eta_s(t')}\right|_{t'=t} = \delta_{is} . \tag{9}$$

Making use of these expressions, (7a) and (7b) become

$$\begin{aligned}\langle \xi_i(t)\phi_i(t)\phi_j(t)\rangle &= \sum_s \sigma^2\, c_{|i-s|}\, (\delta_{is}\langle\phi_s\phi_j\rangle + \delta_{js}\langle\phi_s\phi_i\rangle) \\ &= \sigma^2\,\langle\phi_i\phi_j\rangle\left(c_0 + c_{|i-j|}\right) \end{aligned} \tag{10a}$$

$$\langle \eta_i(t)\phi_j(t)\rangle = \sum_s \frac{\varepsilon}{\Delta x^d}\delta_{is}\delta_{js} = \frac{\varepsilon}{\Delta x^d}\delta_{ij} . \tag{10b}$$

For the second statistical moment we thus have

$$\begin{aligned}\frac{d}{dt}\langle\phi_i\phi_j\rangle = -2a\,\langle\phi_i\phi_j\rangle + D\sum_s \left(\widetilde{D}_{is}\langle\phi_s\phi_j\rangle + \widetilde{D}_{js}\langle\phi_s\phi_i\rangle\right) + \\ +2\sigma^2\,\langle\phi_i\phi_j\rangle\left(c_{|i-j|} + c_0\right) + 2\frac{\varepsilon}{\Delta x^d}\,\delta_{ij} .\end{aligned} \tag{11}$$

The structure function can be defined as

$$S_\mu(t) = \frac{1}{(L\Delta x)^d}\left\langle \widehat{\phi}_\mu(t)\widehat{\phi}_{-\mu}(t)\right\rangle , \tag{12}$$

where $\widehat{\phi}_\mu(t) \equiv \widehat{\phi}(\boldsymbol{k}_\mu, t)$ is the Fourier transform of $\phi_i(t)$

$$\widehat{\phi}_\mu(t) = \Delta x^d \sum_i e^{-i\boldsymbol{r}_i\cdot\boldsymbol{k}_\mu}\phi_i , \qquad \phi_i(t) = \frac{1}{(L\Delta x)^d}\sum_\mu e^{i\boldsymbol{r}_i\cdot\boldsymbol{k}_\mu}\widehat{\phi}_\mu , \tag{13}$$

with $\boldsymbol{k}_\mu = \frac{2\pi}{\Delta x L}\boldsymbol{\mu}$, and $\boldsymbol{\mu} \in [0, L-1]^d$. From definitions (13) the following relations can be easily verified

$$\sum_\mu e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)} = L^d\delta_{ij} \qquad \sum_i e^{-i(\boldsymbol{k}_\mu - \boldsymbol{k}_\nu)\cdot\boldsymbol{r}_i} = L^d\delta_{\mu\nu} . \tag{14}$$

By using definition (13) we can write the dynamical equation for the structure function as

$$\frac{dS_\mu(t)}{dt} = \left(\frac{\Delta x}{L}\right)^d \sum_{i,j} e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_j - \boldsymbol{r}_i)}\frac{d}{dt}\langle\phi_i\phi_j\rangle , \tag{15}$$

and substituing (11)

$$\frac{dS_\mu}{dt} = -2a\,S_\mu(t) + D\left(\frac{\Delta x}{L}\right)^d \sum_{i,j,s} e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)}\left(\widetilde{D}_{is}\langle\phi_s\phi_j\rangle + \widetilde{D}_{js}\langle\phi_s\phi_i\rangle\right) +$$
$$+2\sigma^2\,c_0\,S_\mu(t) + 2\sigma^2\left(\frac{\Delta x}{L}\right)^d \sum_{i,j} e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)} c_{|i-j|}\langle\phi_i\phi_j\rangle + 2\varepsilon\,. \quad (16)$$

Now we have to rewrite the Laplacian terms. A Fourier-transformed Laplacian operator does not couple variables with different moment. In fact, it can be seen that the Laplacian term leads to

$$\left(\frac{\Delta x}{L}\right)^d \sum_{i,j,s} e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)}\widetilde{D}_{is}\langle\phi_s\phi_j\rangle = \widehat{\widetilde{D}}_\mu\,S_\mu(t)\;, \quad (17)$$

where the following relation has been taken into account

$$\sum_s \widetilde{D}_{is}\langle\phi_s\phi_j\rangle = \frac{1}{(L\Delta x)^{2d}}\sum_{\nu,\rho} e^{i\boldsymbol{k}_\nu\cdot\boldsymbol{r}_j} e^{i\boldsymbol{k}_\rho\cdot\boldsymbol{r}_i}\widehat{\widetilde{D}}_\rho\langle\widehat{\phi}_\rho\widehat{\phi}_\nu\rangle\;, \quad (18)$$

and $\widehat{\widetilde{D}}_\mu = \frac{1}{\Delta x^2}\left(\sum_{|\boldsymbol{i}|=1}\cos(\boldsymbol{k}_\mu\cdot\boldsymbol{r}_i) - 1\right)$ can be understood as the Fourier transform of the discrete Laplacian. On the other hand, using definition (13) and relations (14), the last contribution of the multiplicative noise in (16) can be written as

$$2\sigma^2\left(\frac{\Delta x}{L}\right)^d \sum_{i,j} e^{i\boldsymbol{k}_\mu\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)} c_{|i-j|}\langle\phi_i\phi_j\rangle = \sigma^2\frac{1}{(L\Delta x)^d}\sum_\nu \widehat{c}_\nu\,S_{\mu-\nu}(t)\;, \quad (19)$$

Therefore, taking into account (17) and (19), the equation for the structure function becomes finally

$$\frac{dS_\mu(t)}{dt} = -2\omega_\mu\,S_\mu(t) + 2\,\varepsilon + 2\sigma^2\,\frac{1}{(L\Delta x)^d}\sum_\nu \widehat{c}_\nu S_{\mu-\nu}(t)\;, \quad (20)$$

with the additional dispersion relation $\omega_\mu = a - \sigma^2\,c_0 - D\widehat{\widetilde{D}}_\mu$. In the continuous and thermodynamic limit ($\Delta x \to 0$ and $L \to \infty$), the dynamical equation for the structure function is

$$\frac{\partial S(\boldsymbol{k},t)}{\partial t} = -2\,\omega(k)\,S(\boldsymbol{k},t) + 2\varepsilon + 2\,\frac{\sigma^2}{(2\pi)^d}\int d\boldsymbol{k}'\,\widehat{c}(|\boldsymbol{k}-\boldsymbol{k}'|)\,S(\boldsymbol{k}',t) \quad (21)$$

with

$$\omega(k) = a - \sigma^2\,c(0) + D\,k^2 \quad (22)$$

Looking at this dispersion relation, it is readily seen that perturbations grow when $\omega(k) < 0$ for some interval of k values. Hence, the homogeneous state

$\phi(\boldsymbol{r},t)=0$ is stable if $\omega(k)>0\ \ \forall\, k$. This is satisfied for $a-\sigma^2\, c(0)>0$ and thus we can define an effective control parameter

$$a_{\text{eff}} = a - \sigma^2\, c(0) \tag{23}$$

such that for $a_{\text{eff}} > 0$, the homogeneous state $\phi(\boldsymbol{r},t) = 0$ is stable. The transition point is then

$$a_t = \sigma^2\, c(0)\ , \tag{24}$$

which in discrete space is written as $a_t = \sigma^2 c_0$. This transition point increases with noise intensity and decreases with noise correlation length because $c(0) \sim \lambda^{-d}$. In the deterministic case ($\sigma^2 = 0$ and $\varepsilon = 0$) $a_t = 0$ and thus, for $a > 0$ the disordered homogeneous state $\phi(\boldsymbol{r},t) = 0$ is stable, whereas for $a < 0$ this disordered state is unstable. In the presence of multiplicative noise $a_t = \sigma^2\, c(0) > 0$, and hence the homogeneous disordered state is stable for a *smaller* region of the control parameter than in the deterministic case. Hence, fluctuations in the control parameter *induce order* in the system.

There are other techniques that allow us to find the onset of instability. One of them is the study of the stationary state of the structure function equation given by the linear stability analysis [see (21)]. In contrast with the above discussion, this method takes into account the mode-coupling term which depends also on the multiplicative noise. For the particular case of multiplicative noise white in space [$\widehat{c}(|\boldsymbol{k}-\boldsymbol{k}'|)=1$], the steady state for the structure function can be obtained from (21)

$$S_{\text{st}}(k) = \frac{1}{\omega(k)}\left[\varepsilon + \sigma^2\, G_{\text{st}}(0)\right] \tag{25}$$

where $G_{\text{st}}(\boldsymbol{r}) = (2\pi)^{-d}\int d\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} S_{\text{st}}(\boldsymbol{k})$ is the correlation function. By integrating again the above equation, we find that the value $G_{\text{st}}(0)$ is

$$G_{\text{st}}(0) = \frac{\varepsilon\gamma}{1-\sigma^2\gamma}\ , \qquad \gamma = \frac{1}{(2\pi)^d}\int\frac{d\boldsymbol{k}}{\omega(k)}\ . \tag{26}$$

Hence, the resulting stationary structure function is

$$S_{\text{st}}(k) = \frac{\varepsilon'}{\omega(k)}\ , \qquad \varepsilon' = \frac{\varepsilon}{1-\sigma^2\gamma}\ , \tag{27}$$

where ε' is a renormalized additive noise intensity. In the subcritical region, where nonlinear terms are supposed to be negligible, it is expected that this linear result agrees satisfactorily with the behavior of the full nonlinear model. Nevertheless this stationary solution will diverge for $\gamma\sigma^2 = 1$, which indicates that at that point the result is not valid anymore. Hence, the value of the control parameter at which the stationary structure function diverges is the transition point $a = a_t$. The condition $\gamma\sigma^2 = 1$ explicitly reads,

$$1 = \frac{\sigma^2}{(2\pi)^d}\int_{\Re^d}\frac{d\boldsymbol{k}}{a_t - \sigma^2 + Dk^2} \tag{28}$$

which is the same as the one found by Becker and Kramer [4]. It is possible to see that the contribution of the mode-coupling term [last term in (21)] to the transition point is quite small (of order $\frac{\sigma^4}{D}$ in $d = 1$). Other methods such as mean–field approximations also give corrections to the transition point of order D^{-n}.

2.2 Model B

The conserved model corresponding to the one studied in the previous Section is defined by

$$\frac{\partial \phi(\boldsymbol{r},t)}{\partial t} = \nabla^2 \left[a\phi + \phi^3 - D\,\nabla^2\phi + \phi\,\xi(\boldsymbol{r},t)\right] + \eta(\boldsymbol{r},t)\,, \tag{29}$$

where $\phi(\boldsymbol{r},t)$ represents, for instance, the local difference of concentrations of each phase in the case of a binary alloy. Multiplicative noise represents fluctuations in the control parameter a. Both additive and multiplicative noises are Gaussian, with zero mean and a correlation for the additive noise given by

$$\langle \eta(\boldsymbol{r},t)\,\eta(\boldsymbol{r}',t')\rangle = -2\,\varepsilon\,\nabla^2\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(t-t') \tag{30}$$

and (2b) for multiplicative noise. The discrete version of the model in a d–dimensional lattice of mesh size Δx is:

$$\frac{d\phi_i}{dt} = \sum_s \widetilde{D}_{is}\left[a\,\phi_s + \phi_s^3 - D\sum_j \widetilde{D}_{sj}\,\phi_j + \phi_s\,\xi_s(t)\right] + \eta_i(t)\,, \tag{31}$$

where $\phi_i \equiv \phi(\boldsymbol{r}_i)$, as before. The discrete noises $\eta_i(t)$ and $\xi_i(t)$ are still Gaussian, with zero mean and correlations given by

$$\langle \eta_i(t)\,\eta_j(t')\rangle = -2\,\varepsilon\,\frac{\widetilde{D}_{i,j}}{\Delta x^d}\,\delta(t-t') \tag{32}$$

and (5b).

As done in the previous Section, we will study the stability of $\phi_i(t) = 0\ \ \forall i$, taking only into account the linear terms in (31). In this case, the dynamical equation for the two-point correlation function $\langle\phi_i\phi_j\rangle$ is

$$\begin{aligned}\frac{d}{dt}\langle\phi_i\phi_j\rangle &= \sum_s \widetilde{D}_{is}\left[a\langle\phi_j\phi_s\rangle - D\sum_m \widetilde{D}_{sm}\langle\phi_m\phi_j\rangle + \langle\phi_j\phi_s\xi_s\rangle\right] + \langle\eta_i\phi_j\rangle \\ &+ \sum_s \widetilde{D}_{js}\left[-a\langle\phi_i\phi_s\rangle - D\sum_m \widetilde{D}_{sm}\langle\phi_m\phi_i\rangle + \langle\phi_i\phi_s\xi_s\rangle\right] + \langle\eta_j\phi_i\rangle.\end{aligned} \tag{33}$$

Novikov's theorem allows us to calculate all noise terms in the previous equation. The procedure is the same as before [see (7a)–(10b)] and the results are

$$\langle \phi_j \phi_s \xi_s \rangle = \sigma^2 \sum_m \widetilde{D}_{sm} c_{|s-m|} \langle \phi_m \phi_j \rangle + \sigma^2 \sum_m \widetilde{D}_{jm} c_{|s-m|} \langle \phi_m \phi_s \rangle \tag{34a}$$

$$\langle \eta_i \phi_j \rangle = -\frac{\varepsilon \widetilde{D}_{ij}}{\Delta x^d} \tag{34b}$$

Hence, for the second statistical moment we have

$$\begin{aligned} \frac{d}{dt}\langle \phi_i \phi_j \rangle = & \sum_s \widetilde{D}_{is} \Big[a\langle \phi_j \phi_s \rangle - D \sum_m \widetilde{D}_{sm} \langle \phi_m \phi_j \rangle + \sigma^2 \sum_m \widetilde{D}_{sm} c_{|s-m|} \langle \phi_m \phi_j \rangle \\ & + \sigma^2 \sum_m \widetilde{D}_{jm} c_{|s-m|} \langle \phi_m \phi_s \rangle \Big] + \sum_s \widetilde{D}_{js} \Big[-a \langle \phi_i \phi_s \rangle - D \sum_m \widetilde{D}_{sm} \langle \phi_m \phi_i \rangle \\ & + \sigma^2 \sum_m \widetilde{D}_{sm} c_{|s-m|} \langle \phi_m \phi_i \rangle + \sigma^2 \sum_m \widetilde{D}_{im} c_{|s-m|} \langle \phi_m \phi_s \rangle \Big] - 2\varepsilon \frac{\widetilde{D}_{ij}}{\Delta x^d} \, . \end{aligned} \tag{35}$$

We now look for the equation of the structure function. Substituing (35) into (15) and considering relations (14) and (18), we finally find

$$\begin{aligned} \frac{dS_\mu(t)}{dt} = 2\hat{\widetilde{D}}_\mu \left[a - \hat{\widetilde{D}}_\mu \left(D - \sigma^2 c_1 \right) + \sigma^2 \sum_m \widetilde{D}_{0m} c_m \right] S_\mu(t) \\ + 2\sigma^2 \, \hat{\widetilde{D}}_\mu^{\,2} \, (\Delta x L)^{-d} \sum_\nu \widehat{c}_{\nu-\mu} S_\nu(t) - 2\,\varepsilon \hat{\widetilde{D}}_\mu \end{aligned} \tag{36}$$

In the continuous and thermodynamic limit, we have

$$\frac{\partial S(\boldsymbol{k},t)}{\partial t} = -2\,k^2\,\omega(k)\,S(\boldsymbol{k},t) + 2\varepsilon\,k^2 - 2\,k^2 \frac{\sigma^2}{(2\pi)^d} \int d\boldsymbol{k}'\,\widehat{c}(|\boldsymbol{k}-\boldsymbol{k}'|)\,S(\boldsymbol{k}',t) \tag{37}$$

with the dispersion relation

$$\omega(k) = a + \sigma^2 \left[\nabla^2 c(|\boldsymbol{r}|) \right]_{r=0} + \left(D - \sigma^2\,c(0) \right) k^2 \; . \tag{38}$$

In discrete space, this relation reads

$$\omega_\mu = a + 2d\,\sigma^2 (c_1 - c_0) - (D - \sigma^2\,c_1) \hat{\widetilde{D}}_\mu \; . \tag{39}$$

The dispersion relation indicates that for $\omega(k) > 0 \;\forall\, k$, the homogeneous null state is stable. This occurs for $a + \sigma^2 \left[\nabla^2 c(|\boldsymbol{r}|) \right]_{r=0} > 0$ so that, as in the previous Section, we can define an effective control parameter

$$a_{\text{eff}} = a + \sigma^2 \left[\nabla^2\,c(|\boldsymbol{r}|) \right]_{r=0} \; , \tag{40}$$

such that the homogeneous null state is stable for positive values of a_{eff}. Hence, the onset of stability is now given by

$$a_t = -\sigma^2 \left[\nabla^2\,c(|\boldsymbol{r}|) \right]_{r=0} \; , \tag{41}$$

which in a discrete space is written as

$$a_t = 2d\,\sigma^2(c_0 - c_1)\ . \tag{42}$$

As in model A, the transition point in the presence of multiplicative noise is *positive* and therefore, fluctuations in the control parameter *induce order* in the system. However, the effective control parameter is different between the two models. In model A, the dependence of the transition point on the spatial correlation of the noise is merely due to a natural "softening" effect of noise correlation [when the noise is spatially correlated, its effective intensity is $\sigma^2\,c(0) \sim \sigma^2\,\lambda^{-d}$]. In model B, the Laplace operator introduces a more complicated dependence on the spatial correlation of the noise, $a_t \sim \sigma^2\lambda^{-(2+d)}$. Moreover, the expression of $\omega(k)$ for model B (38) has a noise dependence term that can be considered as a modification of the spatial coupling parameter D. We can thus define an effective spatial coupling parameter $D_{\text{eff}} = D - \sigma^2\,c(0)$. These results are consistent with those coming from a mean-field approximation in the limit of infinite coupling [14].

As it has been done in the previous Section, we can look for the stationary state of the structure function and find a condition for the onset of instability which takes into account, in contrast with the above discussion, the coupling term between Fourier modes. For the case of multiplicative noise white both in time and space ($c(|\boldsymbol{k}-\boldsymbol{k}'|) = 1$), the stationary structure function is

$$S_{\text{st}}(k) = \frac{\varepsilon'}{\omega(k)}\,, \tag{43}$$

with ε' given by expressions (27) and (26). This solution differs from that corresponding to model A in the expression of $\omega(k)$ [compare (22) and (38)]. The transition point is given by $\gamma\sigma^2 = 1$, for which the stationary structure function diverges, and is equal at first order to the result given in (41).

For the particular case of a spatial correlation $c(|\boldsymbol{r}|)$ of gaussian type

$$c\,(|\boldsymbol{r}|) = \frac{1}{(\lambda\sqrt{2\pi})^d}\ \exp\left(-\frac{|\boldsymbol{r}|^2}{2\lambda^2}\right)\ , \tag{44}$$

whose width λ characterizes the correlation length of the noise, and which becomes a delta function for $\lambda \to 0$, the transition points for models A and B are, respectively, [see (24) and (41)]

$$a_t^A = \frac{\sigma^2}{(2\pi)^{d/2}\,\lambda^d}\,, \qquad a_t^B = \frac{d\,\sigma^2}{(2\pi)^{d/2}\,\lambda^{d+2}}\,. \tag{45}$$

3 Conclusions

Noise-induced phase transitions for which linear destabilization is the dominant mechanism have been examined in detail by means of a linear stability analysis. The study is performed on both non-conserved and conserved models. In the two cases, noise is seen to have an ordering effect in the system, although the influence is seen to be different in each situation. The role of spatial correlation of the noise is somewhat simple in the non-conserved case, but clearly non-trivial in the conserved model.

Acknowledgments

This work has been financially supported by the Dirección General de Enseñanza Superior, under projects PB94-1167, PB96-0241, PB97-0141-C02-01, and PB98-0935.

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