Angular minimum uncertainty states with large uncertainties

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Abstract
The uncertainty relation for angle and angular momentum has a lower bound
which depends on the form of the state. Surprisingly, this lower bound can
be very large. We derive the states which have the lowest possible uncertainty
product for a given uncertainty in the angle or in the angular momentum. We
show that, if the given angle uncertainty is close to its maximum value, the
lowest possible uncertainty product tends to infinity.

1. Introduction

If the lower bound in an uncertainty relation is state dependent, states satisfying the equality
in the uncertainty relation need not give a minimum in the uncertainty product. This contrasts
to the Heisenberg uncertainty principle for position and momentum [1], where the lower
bound is a constant. Here, states which satisfy the equality in the uncertainty relation, that
is intelligent states [2], also minimize the uncertainty product. The uncertainty relation for
angular momentum and angle [3], however, has a state-dependent lower bound requiring a
distinction between intelligent states and minimum uncertainty product states [4]. These
two kinds of states are defined as solutions of two different eigenvalue equations. For linear
momentum and position, the solutions to the two differential equations are the same Gaussians.
In the angular case, the two eigenvalue equations have two distinct solutions. Additionally,
the angle is defined on a finite interval, allowing solutions to the differential equation which are
disregarded in the linear case on the grounds that they do not represent physical, normalizable
states on the infinite range on which position and linear momentum are defined. These
solutions are peaked at the edges of the 2π radian interval for the angle and consequently
the angle uncertainty of these states tends to be larger than for cases where the wavefunction
is peaked in the middle of the interval and decays towards the boundaries. The intelligent and
minimum uncertainty product states thus appear in two varieties with small and large angle

uncertainties. The distinction is most apparent for the uncertainty product, which is bounded from above by $\hbar/2$ for the states with small angle uncertainty, but has no upper bound in the large-uncertainty case.

We should stress that in this work we will consider only the uncertainties in the angular momentum and in the associated angular coordinate [3]. This form of the uncertainty relation has been demonstrated to hold in a recent experiment [5] and underlies the security of a lately developed free-space communication system [6]. A range of other uncertainty relations have been derived in which measures of angular spread other than the angle uncertainty are used. These include uncertainties based on trigonometric functions of the angle [7], discrete versions of the uncertainty relation [8] and entropic uncertainty relations [9].

The family of states related to the angular uncertainty relation has been investigated in a series of previous articles. The form of the angular uncertainty relation has been experimentally verified using intelligent states with small angle uncertainties [5]. The distinction between intelligent states and minimum uncertainty states with small uncertainties has been presented in a second article [4], where the possibility has been discussed to distinguish between these in an experiment. Intelligent states with large uncertainties have been introduced in a third article [10], in which we have compared the analytically exact expression for the wavefunction in terms of a special function with approximate expressions for two limiting cases. The present paper completes the study of this family of states by expounding minimum uncertainty product states with large uncertainties. As the angle is defined on a finite interval, its uncertainty is bounded from above and the uncertainty product reaches the global minimum for eigenstates of the angular momentum operator $\hat{L}_z$ with uncertainty $\Delta L_z = 0$. But one can also consider states which minimize the uncertainty product under the constraint of a given uncertainty in either the angle or the angular momentum. It has been shown that states minimizing the uncertainty product are the same whether the additional constraint is a given uncertainty in the angle $\Delta\phi$ or a given uncertainty in the angular momentum ΔL_z [4]. These states are called constrained minimum uncertainty product (CMUP) states.

In this paper, we will extend the analysis of CMUP states to the large-uncertainty regime. An exact analytical solution for the wavefunction of these states will be given in terms of complex confluent hypergeometric functions, but we also present an approximate solution in terms of Airy functions for the limiting case where the given angle uncertainty $\Delta\phi$ tends to its upper bound π. In particular, we will compare this kind of CMUP states with the large-uncertainty intelligent states.

2. Angular uncertainty relation

It is physically impossible to distinguish between two rotation angles differing by 2π radians. Within the quantum mechanical description of rotation angles, this restricts angle eigenvalues to lie within a 2π radian interval $[\theta_0, \theta_0 + 2\pi)$ [3]. Choosing θ_0 determines a particular 2π radian interval and with it a particular angle operator $\hat{\phi}_\theta$ and hence a basis set of angle eigenstates. In the following, we adhere to the choice of θ_0 used in previous work [4, 5, 10] by setting $\theta_0 = -\pi$ and dropping the label on the angle operator $\hat{\phi}$. The lower bound in the general form of the uncertainty relation for two Hermitian operators is given by the expectation value of the commutator of these operators [11]. The commutator for angle and angular momentum operator is rigorously derived in a finite state space Ψ of $2L + 1$ dimensions, spanned by the eigenstates $|m\rangle$ of the angular momentum operator $\hat{L}_z$ with m ranging from $-L, -L + 1, \ldots, L$ [3]. Only after physical results have been calculated in the finite-dimensional space Ψ, L is allowed to tend to infinity. It is in this limit of $L \to \infty$ that

the expectation value of the commutator $[\hat{L}_z, \hat{\phi}]$ can be approximated to an excellent degree for all physical preparable states, which results in the angular uncertainty relation

$$\Delta L_z \Delta \phi \geqslant \tfrac{1}{2} |1 - 2\pi P(\pi)|. \tag{1}$$

Here, we are using units in which $\hbar = 1$ and $P(\pi)$ is the angle probability density at the edge of the chosen 2π radian interval. This corresponds to our choice of $\theta_0 = -\pi$, as the probability density is periodic in the angle and $P(-\pi) = P(\pi)$. In general, different states used to calculate the uncertainties ΔL_z and $\Delta \phi$ will have a different angle probability density $P(\pi)$ rendering the lower bound in the uncertainty relation (1) state dependent. From the uncertainty relation (1), it is evident why angular momentum eigenstates give a global minimum for the uncertainty product. For an eigenstate of the angular momentum operator, the angle probability density takes on the constant value of $P(\phi) = 1/(2\pi)$ for ϕ in $[-\pi, \pi)$. The lower bound in the uncertainty relation thus is equal to zero and so is the uncertainty product as the angular momentum uncertainty vanishes and the angle uncertainty has the finite value of $\pi/\sqrt{3}$. This global minimum in the uncertainty product is also the dividing point between small-uncertainty and large-uncertainty states. Away from this point, the constrained minimum uncertainty product (CMUP) states give a minimum in the uncertainty product for a given $\Delta \phi$ or a given ΔL_z.

3. CMUP states

Seeking states which minimize the uncertainty product for a given uncertainty in the angle or for a given uncertainty in the angular momentum is equivalent to minimizing the uncertainty that is not given. The corresponding equation for CMUP states has been derived in [4] using a variational method [12]. In this approach it is required that a CMUP state $|f\rangle$ minimizes the uncertainty product, but with the constraint of keeping the given variance constant and obeying the normalization condition $\langle f|f \rangle = 1$. The additional constraints are taken into account by introducing undetermined Lagrange multipliers [13]. In [4], it has been shown that for a CMUP state $|f\rangle$ the mean values of angular momentum and angle can be set to zero, that is $\langle \hat{L}_z \rangle = \langle \hat{\phi} \rangle = 0$. Therefore, the variances $(\Delta L_z)^2$ and $(\Delta \phi)^2$ simplify to $\langle \hat{L}_z^2 \rangle$ and $\langle \hat{\phi}^2 \rangle$, respectively. The variation of the uncertainty product with the given constraints thus results in a linear combination of the variations $\delta \langle \hat{L}_z^2 \rangle$, $\delta \langle \hat{\phi}^2 \rangle$ and $\delta \langle f|f \rangle$, in which the Lagrange multipliers are the coefficients:

$$\delta \langle \hat{L}_z^2 \rangle + \lambda \delta \langle \hat{\phi}^2 \rangle = \mu \delta \langle f|f \rangle. \tag{2}$$

The linear combination is the same whether $\langle \hat{L}_z^2 \rangle$ is given and $\langle \hat{\phi}^2 \rangle$ is minimized or $\langle \hat{\phi}^2 \rangle$ is given and $\langle \hat{L}_z^2 \rangle$ is minimized. Furthermore, it has also been demonstrated [4] that it is admissible to consider only real coefficients b_m in the angular momentum decomposition of $|f\rangle$ in the state space Ψ

$$|f\rangle = \sum_{m=-L}^{L} b_m |m\rangle. \tag{3}$$

This allows us to write the variation $\delta \langle f|\hat{A}|f \rangle$ for any Hermitian operator $\hat{A}$ as $2(\delta \langle f|)\hat{A}|f \rangle$ [13]. In particular, this applies to the operators $\hat{L}_z^2$, $\hat{\phi}^2$ and to the identity operator $\hat{I}$ corresponding to the variations $\delta \langle \hat{L}_z^2 \rangle$, $\delta \langle \hat{\phi}^2 \rangle$ and $\delta \langle f|f \rangle$. The linear combination of variations in (2) thus turns into a linear combination of operators applied to $|f\rangle$. This leads to an eigenvalue equation of the form

$$\left(\hat{L}_z^2 + \lambda \hat{\phi}^2 \right) |f\rangle = \mu |f\rangle, \tag{4}$$

where λ and μ are the Lagrange multipliers. The identification of the angular momentum operator $\hat{L}_z$ as derivative with respect to ϕ sets additional requirements on the wavefunction representing CMUP states [4, 10]. The wavefunction in the angle representation $\psi(\phi) = \langle\phi|f\rangle$ has to be an element of C^1, which is the set of continuously differentiable functions. The question of differentiability is of particular importance at the boundaries of the 2π radian interval, on which the angle wavefunction is defined. Whereas intelligent states are continuous, they do not have a continuous first derivative at $\phi = \pm\pi$. CMUP states, however, do have a continuous first derivative at the boundaries, and therefore representing $\hat{L}_z^2$ by the differential operator $-(\partial^2/\partial\phi^2)$ is well defined. The eigenvalue equation (4) may thus be turned into a differential equation for the CMUP wavefunction $\psi(\phi)$:

$$\frac{\partial^2}{\partial\phi^2}\psi(\phi) = (\lambda\phi^2 - \mu)\psi(\phi). \tag{5}$$

For the small-uncertainty case, a solution of this equation has been given in terms of a confluent hypergeometric function in [4]. The angle wavefunction in this regime is peaked at $\phi = 0$ and $\lambda, \mu > 0$ such that the curvature of the wavefunction around $\phi = 0$ is negative. For the large-uncertainty case, the curvature around the central region is positive and the multipliers $\lambda, \mu < 0$. Formally, we can give the solution to (5) in terms of confluent hypergeometric functions with complex arguments:

$$\psi(\phi) \propto \exp\left(-\frac{i|\lambda|^{\frac{1}{2}}}{2}\phi^2\right) M\left(-\frac{i}{4}\frac{|\mu|}{|\lambda|^{\frac{1}{2}}} + \frac{1}{4}, \frac{1}{2}, i|\lambda|^{\frac{1}{2}}\phi^2\right), \tag{6}$$

where M is Kumer's function [14]. This solution is obtained from (5) by setting $\lambda = -|\lambda|$ and $\mu = -|\mu|$ for $\lambda, \mu < 0$ and using the same scaling as in [4], independent of the sign of λ and μ: $x = \sqrt{2}|\lambda|^{\frac{1}{4}}\phi$ and $a = |\mu|/(2|\lambda|^{\frac{1}{2}})$. With these substitutions (5) takes on the form

$$\frac{\partial^2\psi}{\partial x^2} + \left(\frac{x^2}{4} - a\right)\psi = 0, \tag{7}$$

of which (6) is a solution with the appropriate change of variables. To evaluate the wavefunction and to calculate the angle and angular momentum uncertainty, we have solved the scaled differential equation (7) numerically using a series expansion. In the scaled form, the wavefunction is characterized by the parameter a which takes on positive values for large-uncertainty CMUP states. The appropriate scaling is determined by the condition that the position x_0 of the first maximum of $\psi(x)$ corresponds to $\phi = \pi$, such that

$$\phi = \frac{x}{x_0}\pi \qquad \text{and} \qquad |\lambda| = \frac{x_0^4}{4\pi^4}. \tag{8}$$

The value of x_0 is determined numerically, so that the scaled wavefunction can be normalized between $-x_0$ and x_0 allowing a calculation of the expectation value $\langle\hat{x}^2\rangle$ by numerical integration. The transition of the wavefunction for the CMUP states from the small-uncertainty regime to the large-uncertainty regime is shown in figure 1. In connection with figure 1, it is useful to discuss qualitatively the consequences of the positive curvature in the central region of the wavefunction for the large-uncertainty CMUP states on the angle uncertainty. In [4], it has been shown that for CMUP states with small $\Delta\phi$ the single-peaked wavefunction has approximately the form of a Gaussian. On increasing the scaling parameter a towards zero, the wavefunction becomes flatter and deviates from the Gaussian form. For $a = 0$, the wavefunction is uniformly distributed between $-\pi$ and π and the angle uncertainty takes on the value of $\Delta\phi = \pi/\sqrt{3}$. This is the dividing point between the small-uncertainty and large-uncertainty regime. If the parameter a is further increased then the curvature turns positive
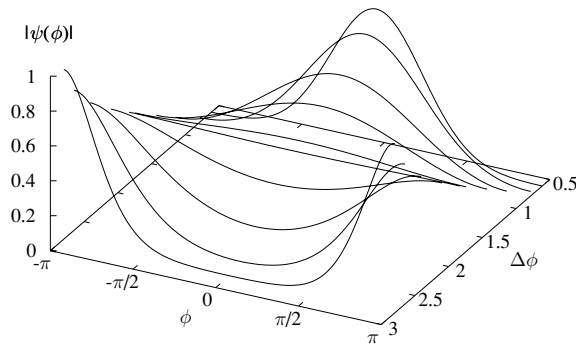
Figure 1. Plot of the wavefunction of CMUP states showing the transition from the small-uncertainty regime to the large-uncertainty regime. This distinction refers to the angle uncertainty $\Delta\phi$, and the dividing point is the flat wavefunction for $\Delta\phi = \pi/\sqrt{3}$.

and the wavefunction is peaked at $\pm\pi$. Calculating the uncertainty for such a CMUP state from the variance $\Delta\phi^2 = \langle\hat{\phi}^2\rangle - \langle\hat{\phi}\rangle^2$ yields values for $\Delta\phi > \pi/\sqrt{3}$. In the limit of $a \to \infty$, the angle uncertainty approaches the maximum value π [10]. In this limit, the 2π-periodic angle probability distribution has a narrow peak centred at $\phi = \pi$. The width of this peak is *not* $\Delta\phi$ but rather the uncertainty in a different angle variable having a different range of allowed angles (for example 0 to 2π). The angle uncertainty for a given state depends on this choice of allowed angles in precisely the same way as does the phase uncertainty for a harmonic oscillator or a single electromagnetic field mode [15].

Using (8) the angle variance $\langle\hat{\phi}^2\rangle$ is given by $\langle\hat{\phi}^2\rangle = \left(\pi^2/x_0^2\right)\langle\hat{x}^2\rangle$ for values of x in $[-x_0, x_0)$. The variance of the angular momentum operator is given by (5) in terms of μ, λ and $\langle\hat{\phi}^2\rangle$, which results in the following expression for the product of the variances $\langle\hat{\phi}^2\rangle$ and $\langle\hat{L}_z^2\rangle$:

$$\langle\hat{\phi}^2\rangle\langle\hat{L}_z^2\rangle = \langle\hat{\phi}^2\rangle(\mu - \lambda\langle\hat{\phi}^2\rangle), \tag{9}$$

$$= \langle\hat{x}^2\rangle\left(-a + \tfrac{1}{4}\langle\hat{x}^2\rangle\right). \tag{10}$$

The limiting behaviour of the uncertainty product is directly connected to the behaviour of the ratio μ/λ. For a tending to zero, μ and λ tend to zero individually but their ratio $\mu/\lambda \to \pi^2/3$ (see figure 2). The variance $\langle\hat{\phi}^2\rangle$ takes on the value of $\pi^2/3$ and the overall product of variances vanishes. For $a \to \infty$, both μ and $\lambda \to -\infty$, but the ratio μ/λ approaches π^2. The variance $\langle\hat{\phi}^2\rangle$, however, tends to its maximum π^2 faster than μ/λ such that (9) tends to infinity. The resulting behaviour of the uncertainty product as a function of $\Delta\phi$ is given in figure 3. As in the small-uncertainty case, for $\Delta\phi < \pi/\sqrt{3}$, the uncertainty product is smaller for the CMUP states than for the intelligent states while still obeying the uncertainty relation (1). This is possible because of the smaller probability density $P(\pi)$ at the edge of the chosen 2π radian interval. Also, the difference in the uncertainty product between intelligent states and CMUP states in the large-uncertainty regime is enhanced over the small-uncertainty regime. This goes along with a significant difference in the wavefunction for intelligent states and CMUP states for the same $\Delta\phi$ in this region (see figure 4). In the small-uncertainty regime, the wavefunctions of intelligent and CMUP states both have approximately the same Gaussian form in the region where the uncertainty product is $1/2$ and changes only slowly with $\Delta\phi$. In the large-uncertainty regime, intelligent and CMUP states are of different form and we will discuss an approximate expression for CMUP states in the limit of $\Delta\phi \to \pi$
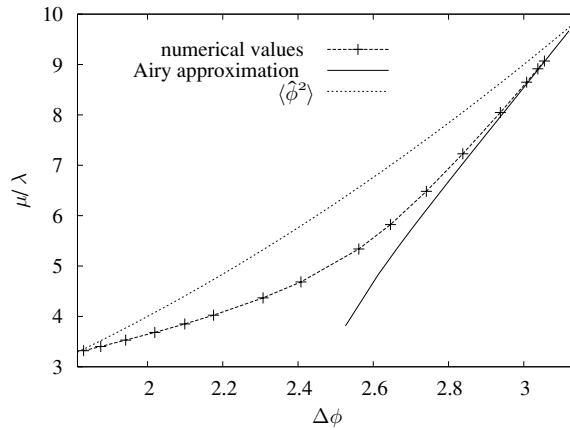
Figure 2. The ratio of the two Lagrange multipliers μ and λ determines the limiting behaviour of the uncertainty product (see equation (9)). For $\Delta\phi \to \pi$ the ratio μ/λ tends to π^2, but more slowly than $\langle\hat{\phi}^2\rangle$. The uncertainty product thus tends to infinity. The plot of μ/λ in the Airy approximation shows the region of validity for this approximation.
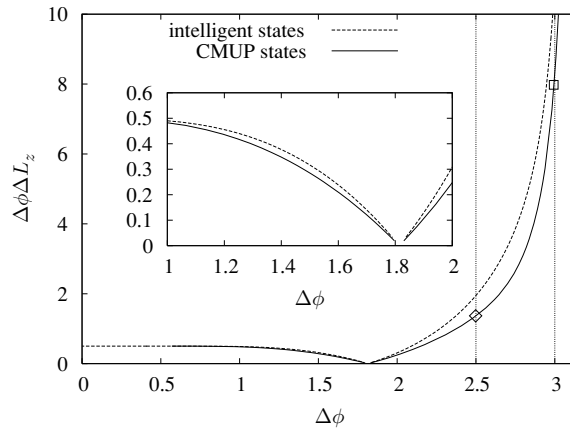

Figure 3. Plot of the uncertainty product as a function of $\Delta\phi$. The graphs of the intelligent states [5, 10] and small-uncertainty CMUP states [4] are shown for comparison (cf plot of intelligent states in [16]). The difference in the uncertainty product between intelligent states and CMUP states is significantly enhanced in the large-uncertainty regime for $\Delta\phi > \pi/\sqrt{3}$. For two values of $\Delta\phi$ (marked by the dotted lines with the symbols ($\diamond$) and ($\square$)), the difference in the wavefunction is shown in figure 4. The inset shows an enlargement around the global minimum in the uncertainty product for $\Delta\phi = \pi/\sqrt{3}$.

later in this paper. In connection with figure 3, it is appropriate to clarify the meaning of minimizing the uncertainty product under a constraint. For CMUP states with small and large angle uncertainties, the angular momentum uncertainty can take on all positive real values. ΔL_z is zero for the angular momentum eigenstates at $\Delta\phi = \pi/\sqrt{3}$ and it approaches infinity for $\Delta\phi \to 0$ and $\Delta\phi \to \pi$. Minimizing the uncertainty product for a given ΔL_z yields two constrained minima. The smaller constrained minimum is obtained for the small-uncertainty CMUP states and corresponds to an angle uncertainty $\Delta\phi < \pi/\sqrt{3}$. A secondary minimum, however, is obtained for the large-uncertainty CMUP states corresponding to $\Delta\phi > \pi/\sqrt{3}$ (see figure 5). On the other hand, minimizing the uncertainty product for a given $\Delta\phi$ results
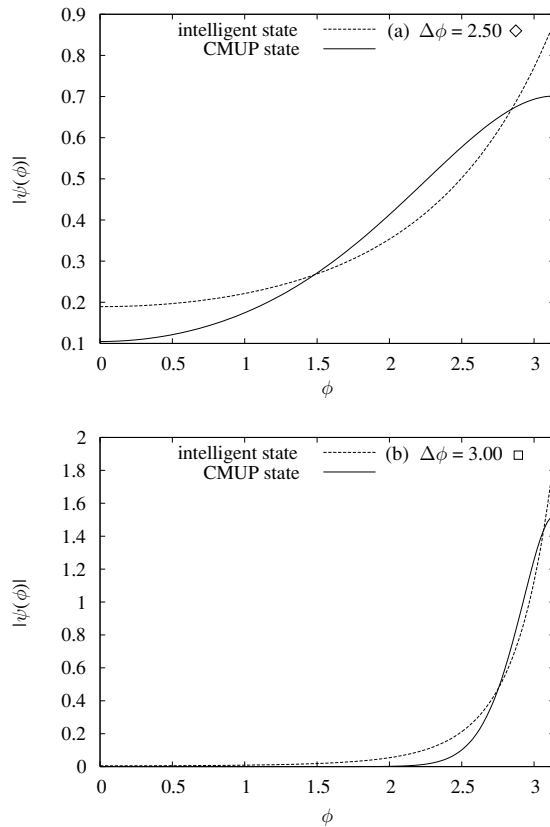
Figure 4. Comparison of the wavefunction for intelligent states and CMUP states for the same $\Delta\phi$ in the large-uncertainty regime. The difference in the uncertainty product for the two values (a) $\Delta\phi = 2.5(\diamond)$ and (b) $\Delta\phi = 3.0(\square)$ can be seen in figure 3. The position of the two values for (a) and (b) is marked in figure 3 by dotted lines with the respective symbols.
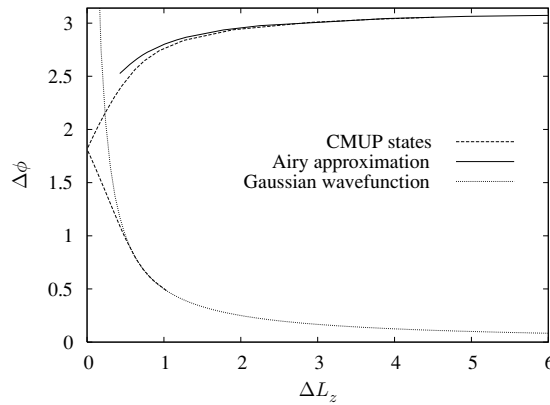

Figure 5. If the uncertainty product is minimized for a given uncertainty in the angular momentum, two minima can be obtained. The first and smaller minimum is obtained for the small-uncertainty CMUP states while a secondary minimum is found for the large-uncertainty CMUP states. For comparison, the limiting cases for these two kinds of states are shown. For $\Delta\phi \to 0$, the small-uncertainty states become Gaussians [4], whereas the large-uncertainty states are approximatively given by Airy functions for $\Delta\phi \to \pi$.

in a unique minimum. Whether this minimum is obtained for small-uncertainty or large-uncertainty CMUP states depends on the given $\Delta\phi$.

Owing to the complexity of the CMUP states we are not able to give an analytical explanation of the limiting behaviour in simple terms. Also, our method to determine the first maximum of the wavefunction numerically fails for very sharply peaked wavefunctions. In the following, we therefore present an approximate expression for the wavefunction in terms of Airy functions, which allows us to calculate the variance $\Delta\phi$ analytically.

4. Airy approximation

The defining differential equation for the CMUP states (5) can be approximated and the resulting equation solved to give an analytical expression for the CMUP wavefunction in the limiting case $\Delta\phi \to \pi$, as we now describe.

The behaviour of the solution for a general differential equation of the form

$$\frac{\partial^2 \psi}{\partial x^2} = P(x)\psi(x) \tag{11}$$

is partly determined by the sign of the function $P(x)$. Should $P(x)$ be purely positive we would expect an exponential behaviour, whereas for a purely negative $P(x)$ the solution would be oscillating. Of particular importance, therefore, are the values of x where $P(x)$ exhibits a change of sign, that is the turning points of equation (11). We can restrict the analysis of the differential equation (5) to the half interval $[0, \pi)$ due to the symmetry of the equation. In this range, equation (5) has one turning point at $\phi = \sqrt{\mu/\lambda}$. The equation is approximated by expanding $P(x) = \lambda\phi^2 - \mu$ around this turning point. Setting $\phi = \sqrt{\mu/\lambda} + x$ and neglecting quadratic terms in x turns (5) into Airy's equation [17]

$$\frac{\partial^2 \psi}{\partial y^2} = y\psi, \qquad y = -(2\sqrt{\mu\lambda})^{\frac{1}{3}}x = -(2\sqrt{\mu\lambda})^{\frac{1}{3}}(\phi - \sqrt{\mu/\lambda}). \tag{12}$$

This equation is solved exactly by the Airy function $\mathrm{Ai}(y)$ which results in

$$\psi(\phi) = C\,\mathrm{Ai}\big(-(2\sqrt{\mu\lambda})^{\frac{1}{3}}(\phi - \sqrt{\mu/\lambda})\big) \tag{13}$$

on substituting the appropriate variables. Here, C is the normalization constant. To fulfil the boundary condition $\psi'(\pi) = 0$, the argument of the Airy function in (13) is required to have the value of the first zero of Ai' for $\phi = \pi$. This leads to the equation

$$-(2\sqrt{\mu\lambda})^{\frac{1}{3}}(\pi - \sqrt{\mu/\lambda}) = -1.0188. \tag{14}$$

Choosing a particular λ gives a quartic equation for $\sqrt{\mu/\lambda}$ and for values of $\sqrt{\mu/\lambda}$ close to π an approximate solution is given by

$$\frac{\mu}{\lambda} \approx \pi - \left(\frac{-1.0188}{2\lambda\pi}\right)^{\frac{1}{3}}. \tag{15}$$

In the Airy approximation, a particular CMUP state can thus be characterized by the Lagrange multiplier λ. The normalization constant can be determined by analytical evaluation of the normalization integral

$$1 = 2\int_0^\pi \psi^2(\phi)\,\mathrm{d}\phi \approx 2C^2(2\sqrt{\mu\lambda})^{-\frac{1}{3}}\int_{-1.0188}^\infty \mathrm{Ai}^2(y)\,\mathrm{d}y. \tag{16}$$

In the last step, we have extended the range of integration from $y(\phi = 0) = (2\sqrt{\mu\lambda})^{\frac{1}{3}}\sqrt{\mu/\lambda}$ to infinity. In the region where the Airy approximation is applicable ($\Delta\phi \to \pi$), the wavefunction decays to zero sufficiently quickly for small angles so that extending the upper bound in the
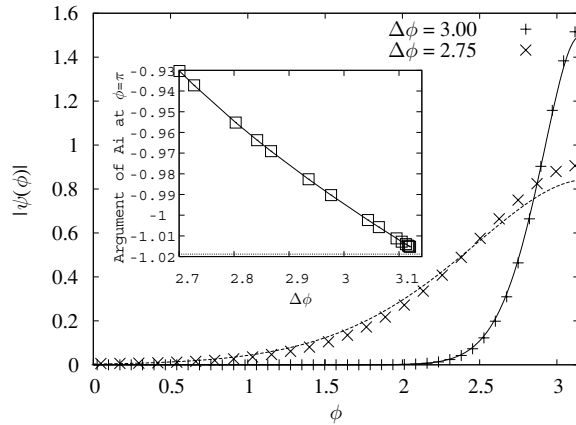
Figure 6. Plot showing a comparison of the Airy approximation (continuous lines) with the numerical calculated wavefunction (individual points). For $\Delta\phi = 3$ (+), the Airy approximation shows a good agreement with the numerical results. The inset shows the deviation of the argument of the Airy function Ai at $\phi = \pi$ from -1.0188 (marked by the horizontal dotted line), the position of the first maximum of the Airy function Ai.

integral does not significantly change the normalization integral. Primitives of products of Airy functions can be calculated using the method of Albright [18]. This results in

$$C = (\mu\lambda)^{\frac{1}{12}}\left((1.0188)^{\frac{1}{2}}(0.5357)2^{\frac{1}{3}}\right)^{-1}, \tag{17}$$

where $\mathrm{Ai}(y = -1.0188) = 0.5357$. In figure 6, a comparison of the numerically calculated wavefunction and the wavefunction in the Airy approximation is shown. The approximation becomes better for values of $\Delta\phi$ closer to π. An inset in figure 6 gives the deviation of the argument of the Airy function in (13) from the exact value of $y = -1.0188$ due to the approximation (15) of the quartic equation (14) determining μ/λ.

Within the Airy approximation, the integral for the angle variance can be calculated analytically using the method of Albright [18]:

$$(\Delta\phi)^2 = \frac{\mu}{\lambda} + \frac{2}{3}(1.0188)(2\sqrt{\mu\lambda})^{-\frac{1}{3}}\sqrt{\mu/\lambda} + \frac{1}{5}(1.0188^{-1} + 1.0188^2)(2\sqrt{\mu\lambda})^{-\frac{2}{3}}. \tag{18}$$

As in the calculation of the normalization constant (16), the upper boundary in the integration has been extended to infinity. On multiplying (18) by λ, one can see in (9) that $\lambda\langle\hat{\phi}^2\rangle$ will always be smaller than μ resulting in an unbounded uncertainty product. Within the Airy approximation, the uncertainty product can be calculated for values of $\Delta\phi$ much closer to π than in the numerical calculation. This is due to the fact that our numerical determination of the first maximum fails for large values of a. In the Airy approximation, a numerical search for the first maximum is not necessary. The uncertainty product calculated in the Airy approximation is compared with the numerical results in figure 7.

5. Conclusion

In this paper, we have completed the study of states related to the angular uncertainty relation. This particular uncertainty relation differs from the Heisenberg uncertainty principle for linear position and momentum in two ways. The lower bound in the uncertainty relation is state dependent which causes intelligent states, that is states satisfying the equality in the uncertainty
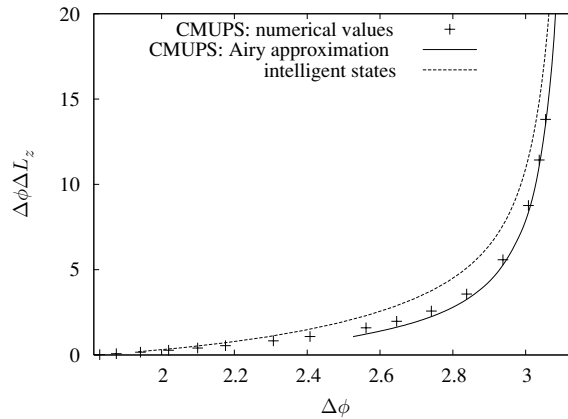
Figure 7. Comparison of the uncertainty product calculated in the Airy approximation and for numerical results. In difference to figure 3, the graph shows only the large-uncertainty region ($\Delta\phi > \pi/\sqrt{3}$) and the ordinate is extended to larger values of $\Delta\phi\Delta L_z$. The Airy approximation explains the behaviour of $\Delta\phi\Delta L_z$ in a region where our numerical calculation fails.

relation, to be distinct from constrained minimum uncertainty product (CMUP) states. These states minimize the uncertainty product for a given variance in angle or for a given variance in angular momentum. Also, in contrast to the linear position, the angle is defined on a bounded interval. Therefore, wavefunctions peaked at the edge of the interval are normalizable and can represent physical states. Intelligent and CMUP states are defined by two different eigenvalue equations. For the angular uncertainty relation, the solutions to the eigenvalue equations are angle wavefunctions which are peaked in the middle or peaked at the edge. This gives rise to two varieties of states with small and large angle uncertainties respectively. Intelligent states with large angle uncertainties may have arbitrarily large-uncertainty products while still satisfying the equality in the uncertainty relation. Similarly, CMUP states with large angle uncertainties minimize the uncertainty product locally or globally for a given constraint. The obtained minimum uncertainty product may be very large. It depends on the given uncertainty whether the minimum is local and the small-uncertainty CMUP states give a smaller uncertainty product or if the large-uncertainty CMUP state gives the smallest possible constrained uncertainty product.

Intelligent states with small and large angle uncertainties have been discussed previously in two papers [5, 10], while the CMUP states with small angle uncertainties have been studied in a third paper [4]. Here, we have examined CMUP states with large angle uncertainties. We have found an analytically exact solution for the CMUP eigenvalue equation in terms of confluent hypergeometric functions with complex arguments. We also have solved the equation numerically and have calculated the angle uncertainty from this solution by numerical integration. To explain the limiting behaviour for sharply peaked wavefunctions, we have developed an analytical approximation using an expansion about a turning point. The approximate solution is given as the decaying tail of the Airy function Ai. Within this approximation, we were able to calculate the uncertainty product analytically.

We have found that the difference in the uncertainty product between intelligent states and CMUP state is enhanced in the large-uncertainty regime. In [4], the possibility of distinguishing between intelligent states and CMUP states in an experiment was discussed. While it might be more difficult to prepare large-uncertainty states in an experiment, the greater difference in the uncertainty product could simplify the experimental evaluation significantly.

The difference between CMUP and intelligent states in the large-uncertainty regime is also a clear indication for the necessary distinction between the two regimes and shows that large-uncertainty states cannot be transformed into small-uncertainty states by shifting the 2π radian range, but should be treated separately.

Acknowledgments

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