Transient Periodic Rotating Waves and Fast Propagation of Synchronization in Linear Arrays of Chaotic Systems

M. A. Matías*
Física Teórica, Facultad de Ciencias, Universidad de Salamanca, E-37008 Salamanca, Spain

J. Güémez
Departamento de Física Aplicada, Universidad de Cantabria, E-39005 Santander, Spain
(Received 20 October 1997)

We study the behavior of arrays of unidirectionally coupled Lorenz systems in the chaotic regime. In the case of rings an instability in the uniform synchronized state leads to the appearance of a periodic rotating wave through a symmetric Hopf bifurcation. From theoretical grounds it is argued that this behavior must also manifest in the behavior of linear arrays. Numerically it is shown that this happens in a transient way, leading to unexpected consequences in the velocity of synchronization, that becomes larger. [S0031-9007(98)07541-3]

PACS numbers: 05.45.+b, 89.70.+c

Recently it has been shown that chaotic synchronization [1,2] occurs in linear arrays of unidirectionally coupled chaotic systems in the form of a synchronization wave that spreads through the system [3,4]. In Ref. [3] we showed that this synchronization wave can be characterized to have a constant velocity that depends linearly on the slowest time scale in the system, quantitatively characterized by the largest conditioned Lyapunov exponent [2] (corresponding to a single drive-response couple).

The aim of the present Letter is to show that, although the scenario presented above is expected to be the *normal* one in this kind of systems, it is possible that interesting dynamical phenomena may occur in some circumstances. In the situation discussed here coherent transient waves appear in the system influencing the way in which the system approaches the synchronized state. This behavior is closely related [5] to the behavior of arrays made up with the same basic cells, but with periodic boundary conditions, i.e., in a ring geometry, although in unidirectionally coupled linear arrays one no longer has a mechanism by which a given oscillator can be affected by those that are down the chain.

The ground for this unexpected relationship between linear and circular arrangements of discretely coupled cells comes from recent studies showing the presence of *hidden* symmetries, i.e., symmetries that appear in systems in which the underlying differential equations have more symmetry than the boundary conditions. In particular, Armbruster and Dangelmayr [6] observed that reaction-diffusion systems satisfying Neumann (free-end) boundary conditions on an interval can be extended to intervals with twice the length that satisfy periodic boundary conditions, i.e., a circle. This extension introduces additional rotational (hidden) symmetries in the system. These ideas allowed Crawford *et al.* [7] to analyze properly the experimental results on parametrically forced waves [8]. Other studies of hidden symmetries on a variety of systems are those of Refs. [9–11].

On the other hand, in recent work [12] it has been shown that rings of chaotic oscillators may exhibit an instability in the uniform synchronized state through a symmetric Hopf bifurcation. Interestingly, this bifurcation leads in some cases, such as in the case of rings of Lorenz systems, to the appearance of a periodic rotating wave with a frequency that is larger than the frequency corresponding to the average distance between peaks in the chaotic state that corresponds to an uncoupled Lorenz system. More recently this behavior has been found experimentally in a ring of Lorenz analog circuits [13]. The goal of the present Letter is precisely to discuss the consequences that have for linear arrays the occurrence of this bifurcation in the corresponding circular arrays. Other authors had considered previously bifurcations of equilibria and periodic solutions in circular arrays of periodic [14–16] and chaotic [17] cells.

The connections in the arrays of Lorenz oscillators [18] to be discussed in the present work are based on a generalization [19(a)] of a previously introduced connection method [19(b)]. Within this scheme one may write the following evolution equations for the array:

$$\left\{\begin{array}{l} \dot{x}_j = \sigma(y_j - x_j) \\ \dot{y}_j = R\,\underline{x_j} - y_j - x_j z_j \\ \dot{z}_j = x_j\,\underline{y_j} - b\, z_j \end{array}\right\} \qquad j = 1, \ldots, N, \qquad (1)$$

where $\underline{x_j} = \alpha\,\overline{x_j} + (1-\alpha)\,x_j$, with $\overline{x_j} = x_{j-1}$ for $j \neq 1$, introduces the coupling and $0 \leq \alpha \leq 1$. The boundary conditions enter through $\overline{x_1}$ that takes the value $\overline{x_1} = x_N$ for circular arrays, while for linear arrays it is $\overline{x_1} = x_1$. The main usefulness of the introduction of the parameter α in this context is that it allows one to control the stability of the connection. The case $\alpha = 1$ corresponds to the original method [19(b)], i.e., $\underline{x_j} = \overline{x_j}$.

The reason why the bifurcation that leads to stable periodic rotating waves for circular arrays cannot yield this behavior in linear arrays is the unidirectional character of the coupling. In fact, it has been shown [20] that for this type of coupling the first element imposes its behavior to the whole array, in agreement with our previous findings [3,4]. However, the presence of a hidden symmetry leads to an interesting global transient behavior in the linear array. Once the first element starts to drive the array, one would expect [3] that chaotic synchronization would happen as a synchronization wave with constant velocity spreads through the system, i.e., the second oscillator would become synchronized with the first while the other elements in the array evolve freely, then the third with respect to the second one, etc.

However, this is not the behavior of a linear array of Lorenz oscillators connected accordingly to Eq. (1), as can be seen from Fig. 1(a), where the case of a linear array with $N = 11$ Lorenz oscillators is considered. Before attaining the asymptotic synchronous chaotic behavior the oscillators engage in a surprising transient collective behavior in which *all* the driven oscillators participate. Or, in other words, the whole array, while still unsynchronized with the drive (master) oscillator, is under its influence, and the different oscillators perform a kind of coordinated behavior in which neighboring oscillators appear to differ by a $2\pi/N$ phase one to each other. Notice that in the time window shown in the plot a small number of oscillators are already synchronized. Notice also that the time scale of this chaotic synchronized behavior is about 1 order of magnitude longer than that of the periodic behavior, i.e., it is a fast bifurcation, in agreement with the conclusions of Refs. [12,13]. For comparison, Fig. 1(b) displays the behavior exhibited by an array for which the collective behavior does not appear, namely, that in which the connection takes place in the $\dot{x}_j = \sigma(\underline{y_j} - x_j)$ term [details of the connection are along the discussion after Eq. (1)].

Now we shall try to argue that this behavior comes from a hidden symmetry in the linear array. The N coupled oscillators in the linear array exhibit a global symmetric Hopf bifurcation that is related [5] to that exhibited by a circular array with $2N$ oscillators [12], with the difference that this behavior must be transient in the linear array. Although Fig. 1(a) is quite informative, its interpretation is made more difficult by the fact that as one has a relatively large number of oscillators ($N = 11$), more than one mode becomes unstable and the behavior is characteristic of a number of these modes in the form of a beat wave. Figure 2(a) shows results for a ring with $2N = 8$ Lorenz oscillators coupled accordingly to (1), while Fig. 2(b) shows results corresponding to the *analogous* [5] linear array with $N = 4$ Lorenz oscillators. The value of α in the (identical) connections has been chosen such as to excite just one mode in the ring (for $\alpha = 1$ the second mode is excited from $N \geq 6$, what would imply that one should with a linear array of just two oscillators in order to obtain transient waves with approximately constant amplitude). The similarity between both behaviors is striking, although, of course, fine details such as the period and detailed shape of the waves are not identical, due to the presence of the first element in the array that influences the oscillators with its chaotic behavior [in Fig. 1(b) this effect is more apparent in the valleys of the waves].

The previously mentioned global bifurcation occurring in linear arrays of Lorenz oscillators has another effect

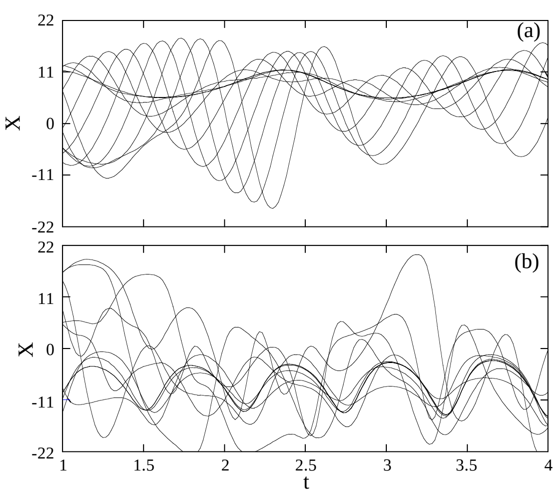

FIG. 1. (a) Representation of variable x vs time in a linear array of 11 Lorenz oscillators (a) when they are connected according to (1), showing the coherent (periodic) in their approach to chaotic synchronization; (b) when they are connected according to Eq. (3) in Ref. [19(b)], showing a *normal* approach to synchronization. The parameters are $\sigma = 18$, $R = 28$, $b = 8/3$, and $\alpha = 1$.

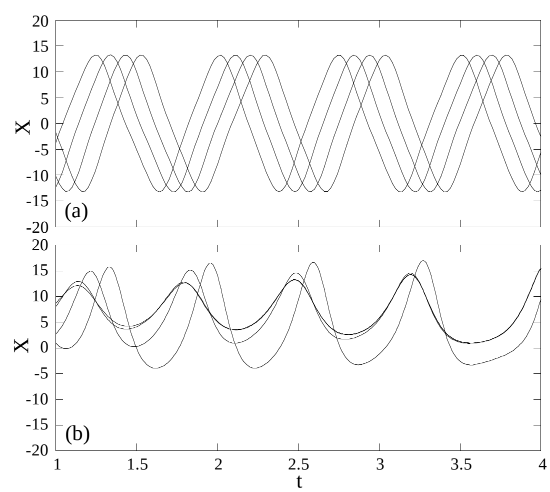

FIG. 2. Representation of variable x vs time for in arrays of Lorenz oscillators coupled following (1): (a) A ring of eight oscillators exhibiting a (stable) periodic rotating wave; (b) a linear array of four oscillators that according to the theoretical analysis is homologous to the latter. The parameters are $\sigma = 10$, $R = 28$, $b = 8/3$, and $\alpha = 0.35$. Notice that α has been chosen such that a ring with this size sustains a single mode, as with $\alpha = 1$ one would have two modes, and accordingly a beat wave.

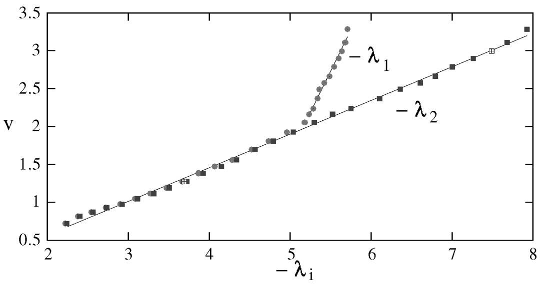

FIG. 3. Dependence of the velocities of synchronization in a linear array of Lorenz oscillators (1) on the two largest conditioned Lyapunov exponents of the system. The parameters are $b = 8/3$, $R = 28$, and $\alpha = 1$, while σ varies in the range $6 \leq \sigma \leq 19$ uniformly spaced by 0.5 ($|\lambda_i|$ grows with σ). The point in which the two largest conditioned Lyapunov exponents are no longer approximately degenerate is located at $\sigma \approx 13$. The velocities corresponding to the *typical* values $\sigma = 10$ and $\sigma = 18$ are represented by using a crossed square.

in the way in which synchronization propagates in these systems: chaotic synchronization now propagates faster than in systems that do not exhibit this transition to periodic discrete rotating waves when they are arranged in ring geometries. In previous work [3,4] we showed that the *normal* situation is that chaotic synchronization propagates at a constant velocity that depends linearly on the largest conditioned Lyapunov exponent of the connection [2]. Figure 3 contains the representation of the dependence of the synchronization velocity in linear arrays of the Lorenz system with different values for the parameter σ in the range $[6, 19]$ versus the largest two conditioned Lyapunov of the connection, λ_1 and λ_2, that vary with σ. The velocity of synchronization is obtained by studying the time that is needed for synchronization as a function of the number of elements in the array, and performing, then, a linear fit. As this velocity depends on the initial conditions it is obtained by carrying out an average over a large number of realizations (different initial conditions).

In the representation versus the largest conditioned exponent, λ_1, it can be clearly seen that there is a point at which the slope changes, located at $\sigma \approx 13$, such that beyond that point synchronization propagates faster. Notice also that the velocity of synchronization scales linearly with the second largest conditioned Lyapunov exponent that defines a shorter time scale. This crossover is associated with the fact that in purity, and due to the type of collective behavior occurring in the array, the stability analysis of a single driven system is no longer representative of the behavior of the array. Instead, one should consider the type of analysis that we carried out in Ref. [12] to characterize the stability of the synchronized state of rings of coupled oscillators. Thus, if one considers a ring comprising N oscillators of dimension m, then the coupled $(N\,m) \times (N\,m)$ dimensional problem can be decoupled through the use of the discrete Fourier transform due to the circulant structure of the problem [21,22] taking the form

$$\dot{\boldsymbol{\eta}}^{(k)} = \mathbf{C}^{(k)}\,\boldsymbol{\eta}^{(k)}, \tag{2}$$

where the $\boldsymbol{\eta}^{(k)}$ are the Fourier transforms of the differences between the variables of contiguous oscillators, and the structure of each block is

$$\mathbf{C}^{(k)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (R^\star - z) & -1 & -x \\ y & x & -b \end{pmatrix}, \tag{3}$$

with $R^\star = R[1 + \alpha(e_k - 1)]$, $e_k = \exp(i\,2\pi\,k/N)$, and $k = 0, \ldots, (N - 1)$.

Because of the presence of time-dependent, chaotically varying coefficients in (3), the characterization of the stability problem associated with rings of chaotic systems requires the determination of the corresponding Lyapunov spectrum that will depend on the discrete wave number k. The uniform (chaotic synchronized) state will be stable whenever the transverse spectrum, corresponding to the modes $k \neq 0$, remains negative. In Ref. [23] we introduced a convenient way of discussing the stability of these rings, through the analysis of the generalized dispersion relation $\lambda(q)$ versus q, see Fig. 4, where $\lambda(q)$ is the largest Lyapunov exponent obtained from (3) and $q = k/N \in [0, 1]$. Notice that this does not imply that we are defining any property related to the behavior of a physical $N = 1$ oscillator, it is just a convenient way of characterizing once for all the behavior of arbitrary sized rings of identical oscillators (for fixed values of the parameters of the oscillators). From the analysis of Fig. 4, an instability in the synchronized state of the ring should occur whenever there is a value of q_c fulfilling $\lambda(q_c) = 0$, implying this will happen for $N \geq N_c$, with $N_c = k/q_c$, and where k is the mode index corresponding to the crossing. In the case that the arrays are designed by using connection (1) the behavior past the instability corresponds to periodic rotating waves [12].

On the other hand, the second transverse Lyapunov exponent is always negative for any value of q. The aspect that this stability analysis illustrates is that the

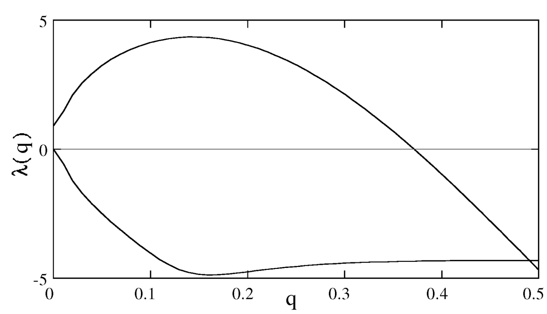

FIG. 4. Representation of the generalized dispersion relation $\lambda(q)$ versus q for the case of a circular array of Lorenz systems connected in the form (1). See text for explanation. The parameters are $\sigma = 10$, $b = 8/3$, $R = 28$, and $\alpha = 1$.

second largest exponent in the array, $\lambda_2(q)$, is quite flat in a range of values of the reduced wave number q, and this appears to be true in the range of values of σ considered in the study of Fig. 3. Instead, as the ring exhibits a periodic wave behavior, it is positive (in the unstable region) near the onset of instability. Afterwards it will be negative, and apparently more negative than the formerly second largest exponent. Thus, when trying to scale the velocity of synchronization with properties of a single driven system, the second largest conditioned exponent appears to be a more suitable parameter to be used as reference. Finally, and as noticed above, the velocity of synchronization scales linearly in a range of values of the parameters, $6 \leq \sigma \leq 13$, with the largest conditioned exponent because for that range the two largest transverse exponents corresponding to a single drive-response couple are approximately degenerate in this range (this can be understood very easily by performing the averaging approximation introduced in Ref. [24]). For instance, for the parameters from which Fig. 4 has been calculated these exponents were reported in Ref. [19(b)] to be $[-3.9513, -4.0420, -5.6734]$; this corresponds to $\sigma = 10$.

To summarize, we have discussed the unexpected consequences of a recently reported transition in circular arrays of unidirectionally coupled chaotic systems to a periodic discrete rotating wave behavior [12,13] on the behavior of linear arrays of the same systems. This connection comes through the existence of the recently studied presence of hidden symmetries that imply the appearance of solutions that have more symmetry than that of the boundary conditions (limited, however, by the symmetry of the equations themselves). The unidirectional character of the coupling implies that ultimately the first oscillator in the array will impose its behavior to the whole system, and, thus, that the influence of the behavior of the circular array will be of transient type (in the circular array the periodic rotating wave is asymptotically stable). An implication of this global bifurcation is that the approach to the synchronized state will be faster than in the *normal* case [3,4], in which the circular array does not exhibit this bifurcation. Indeed, the slowest time scale in the system will approximately depend now on the second largest transverse Lyapunov exponent of a drive-response couple. In addition, interesting transient phenomena occur in the way in which the oscillators approach the synchronized state. It no longer happens that the oscillators approach the synchronized state sequentially, engaging, instead, in a kind of global bifurcation in which the whole array engages in a highly correlated behavior soon before synchronization occurs. This correlated behavior also involves the kind of rotating wave seen for the circular arrays, although now it has a transient nature. An interesting feature of this bifurcation in linear arrays of unidirectionally coupled oscillators is that this induced coherence provides the system with a mechanism by which the oscillators receive an influence from those that are down the chain. Or, in other words, despite the unidirectional character of information transmission, there is a subtle mechanism by which information travels also in the reverse (upstream) direction,

This work was supported in part by DGICYT and by JCL (Spain) under Research Grants No. PB95-0570 and No. SA30/97, respectively.

*Email address: mam@sonia.usal.es

[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983).
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991).
[3] M. N. Lorenzo, I. P. Mariño, V. Pérez-Muñuzuri, M. A. Matías, and V. Pérez-Villar, Phys. Rev. E **54**, R3094 (1996).
[4] E. Sánchez, M. A. Matías, and V. Pérez-Muñuzuri, IEEE Trans. Circuits Syst. I (to be published).
[5] I. R. Epstein and M. Golubitsky, Chaos **3**, 1 (1993).
[6] D. Armbruster and G. Dangelmayr, Math. Proc. Camb. Philos. Soc. **101**, 167 (1987).
[7] J. D. Crawford, J. P. Gollub, and D. Lane, Nonlinearity **6**, 119 (1993).
[8] F. Simonelli and J. P. Gollub, J. Fluid Mech. **23**, 341 (1991).
[9] H. Fujii, M. Mimura, and Y. Nishiura, Physica (Amsterdam) **5D**, 1 (1982).
[10] J. D. Crawford, M. Golubitsky, M. G. M. Gomes, E. Knobloch, and I. Stewart, in *Singularity Theory and its Applications* (Springer, Berlin, 1991), Vol. 2.
[11] J. D. Crawford, Phys. Rev. Lett. **67**, 441 (1991); Physica (Amsterdam) **52D**, 429 (1991).
[12] M. A. Matías, V. Pérez-Muñuzuri, M. N. Lorenzo, I. P. Mariño, and V. Pérez-Villar, Phys. Rev. Lett. **78**, 219 (1997).
[13] E. Sánchez and M. A. Matías, Phys. Rev. E **57**, 6184 (1998).
[14] T. Erneux and M. Herschkowitz-Kaufman, J. Chem. Phys. **66**, 248 (1977).
[15] J. F. G. Auchmuty, Ann. N.Y. Acad. Sci. **316**, 263 (1979).
[16] J. C. Alexander, SIAM J. Appl. Math. **46**, 199 (1986).
[17] J. F. Heagy, L. M. Pecora, and T. L. Carroll, Phys. Rev. Lett. **74**, 4185 (1995).
[18] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[19] J. Güémez and M. A. Matías, (a) Phys. Lett. A **246**, 289 (1998); (b) Phys. Rev. E **52**, R2145 (1995).
[20] I. Aranson, D. Golomb, and H. Sompolinsky, Phys. Rev. Lett. **68**, 3495 (1992)
[21] A. M. Turing, Philos. Trans. R. Soc. London B **237**, 37 (1952).
[22] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[23] M. A. Matías, J. Güémez, V. Pérez-Muñuzuri, M. N. Lorenzo, I. P. Mariño, and V. Pérez-Villar, Europhys. Lett. **37**, 379 (1997).
[24] J. Güémez, C. Martín, and M. A. Matías, Phys. Rev. E **55**, 124 (1997).