Large-scale transport in oceans

Emilio Hernández-García
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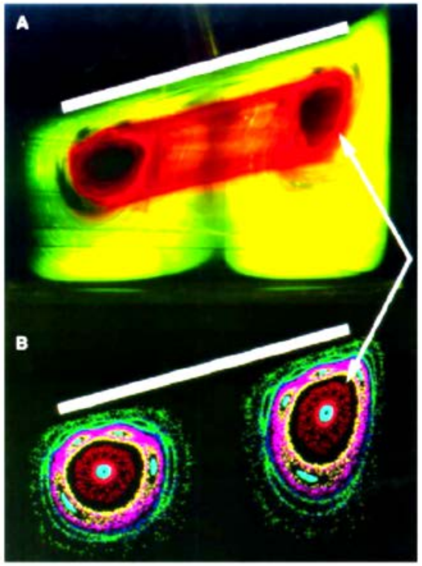
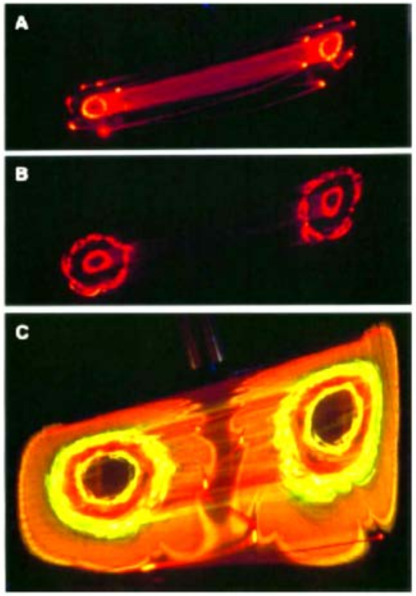
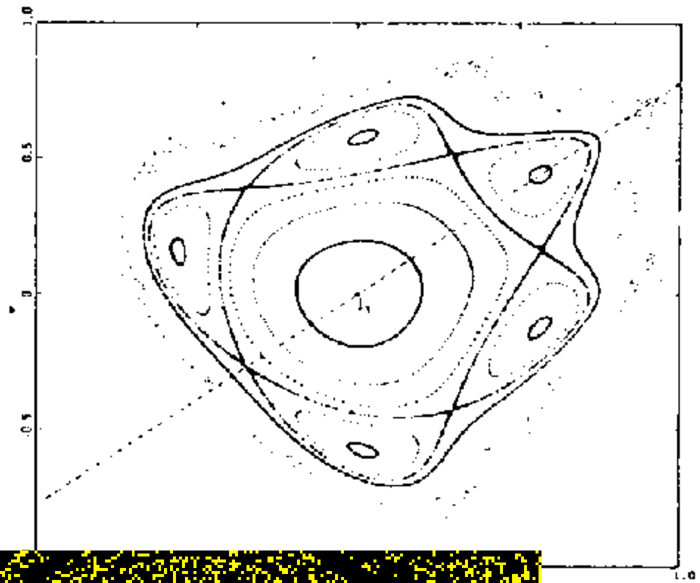
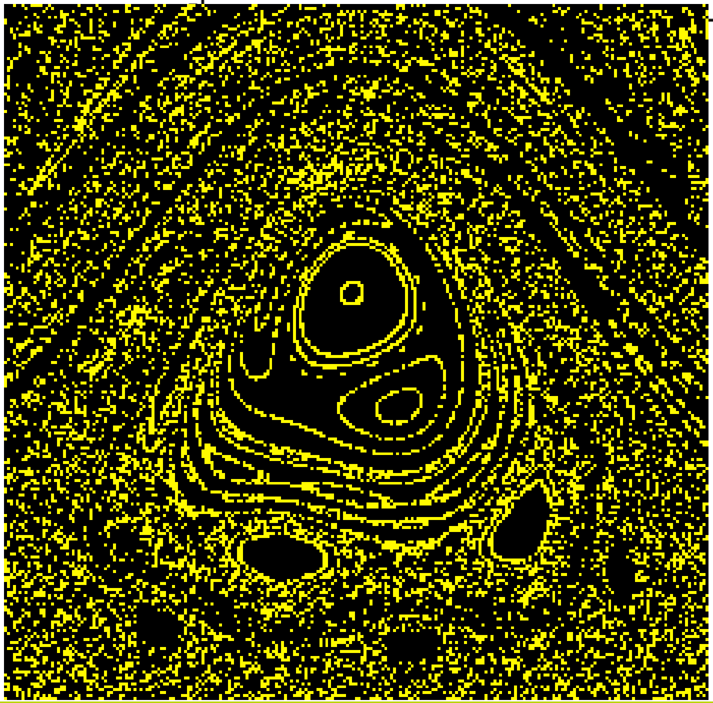
Statistical Physics and Dynamical Systems approaches in Lagrangian Fluid Dynamics

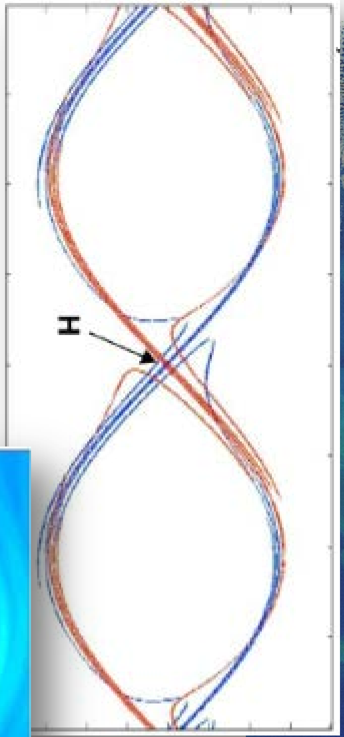


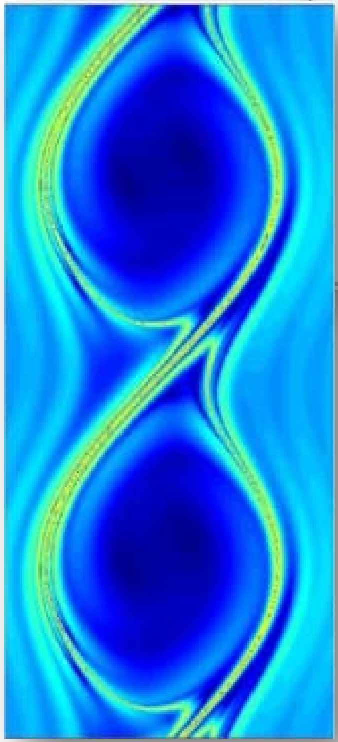


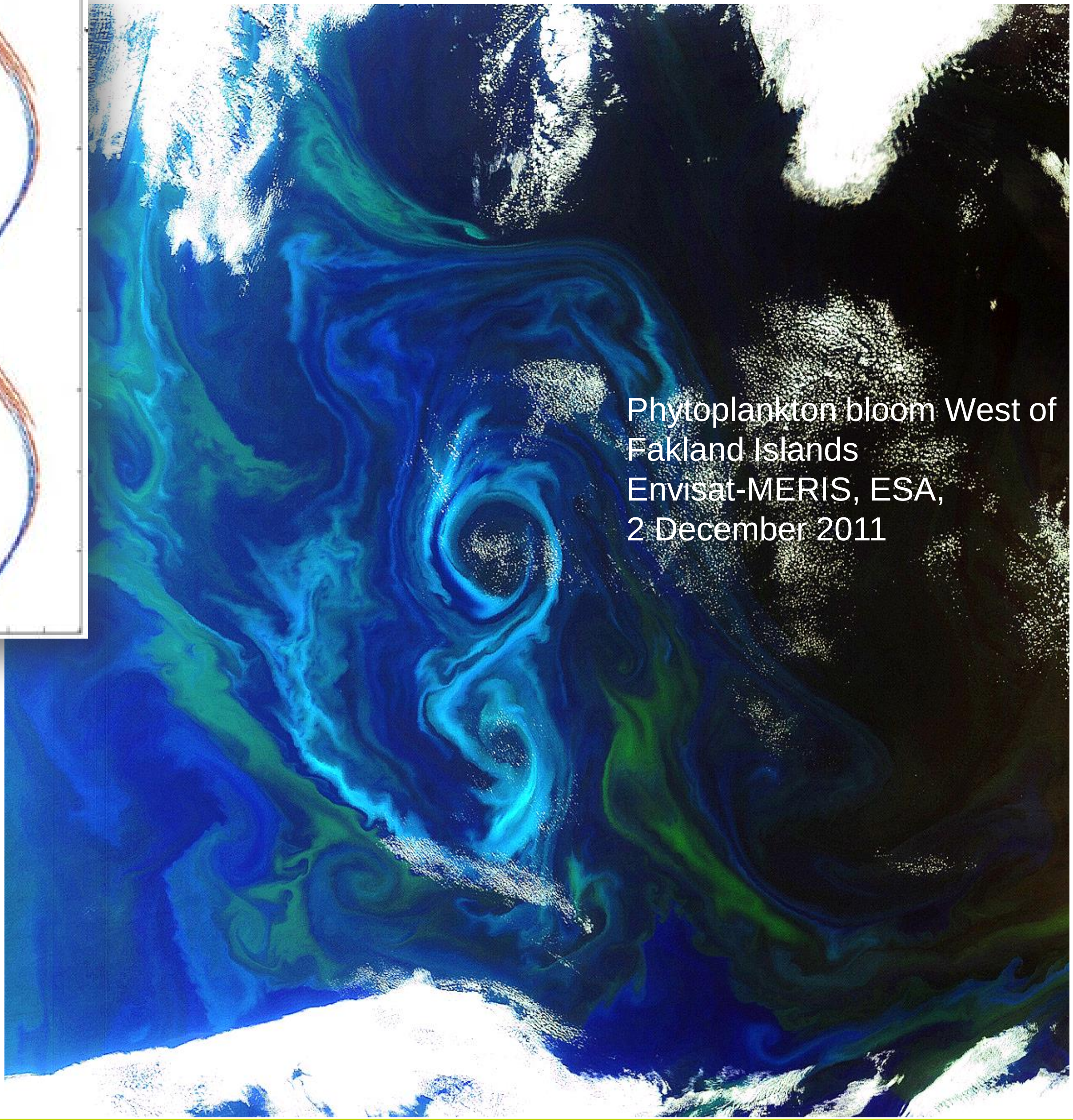
Fountain et al, Science 281, 683 (1998)











Phytoplankton bloom West of Fakland Islands
Envisat-MERIS, ESA,
2 December 2011

STATISTICAL PHYSICS AND DYNAMICAL SYSTEMS APPROACHES IN LAGRANGIAN FLUID DYNAMICS

OUTLINE

1. Lagrangian fluid dynamics and introduction to chaotic advection. Hamiltonian dynamics, KAM tori, Lyapunov exponents, open flows
2. Dispersion, diffusion and coherent structures in flows. Turbulent, pair and chaotic dispersion, gradient production, FTLE, FSLE, Lagrangian Coherent Structures
3. Chemical and biological processes in flows. Fisher and excitable plankton waves, filamental transitions, lamellar approaches, burning manifolds
4. Complex networks of fluid transport. Directed and weighted flow networks. Community detection

STATISTICAL PHYSICS AND DYNAMICAL SYSTEMS APPROACHES IN LAGRANGIAN FLUID DYNAMICS

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- S. Wiggins, The dynamical systems approach to Lagrangian transport in oceanic flows, *Annual Review of Fluid Dynamics* **37**, 295-328 (2005)
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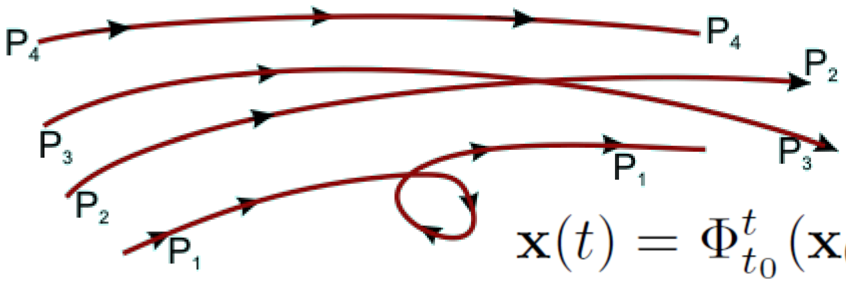
EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

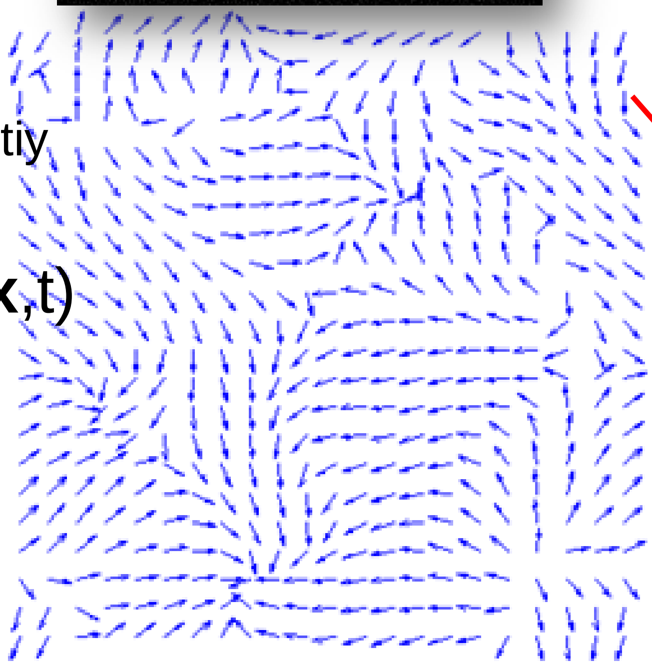


$$\mathbf{x}(t) = \Phi_{t_0}^{t}(\mathbf{x}_0)$$

Trayectories or material lines (in general, material transport and deformation)

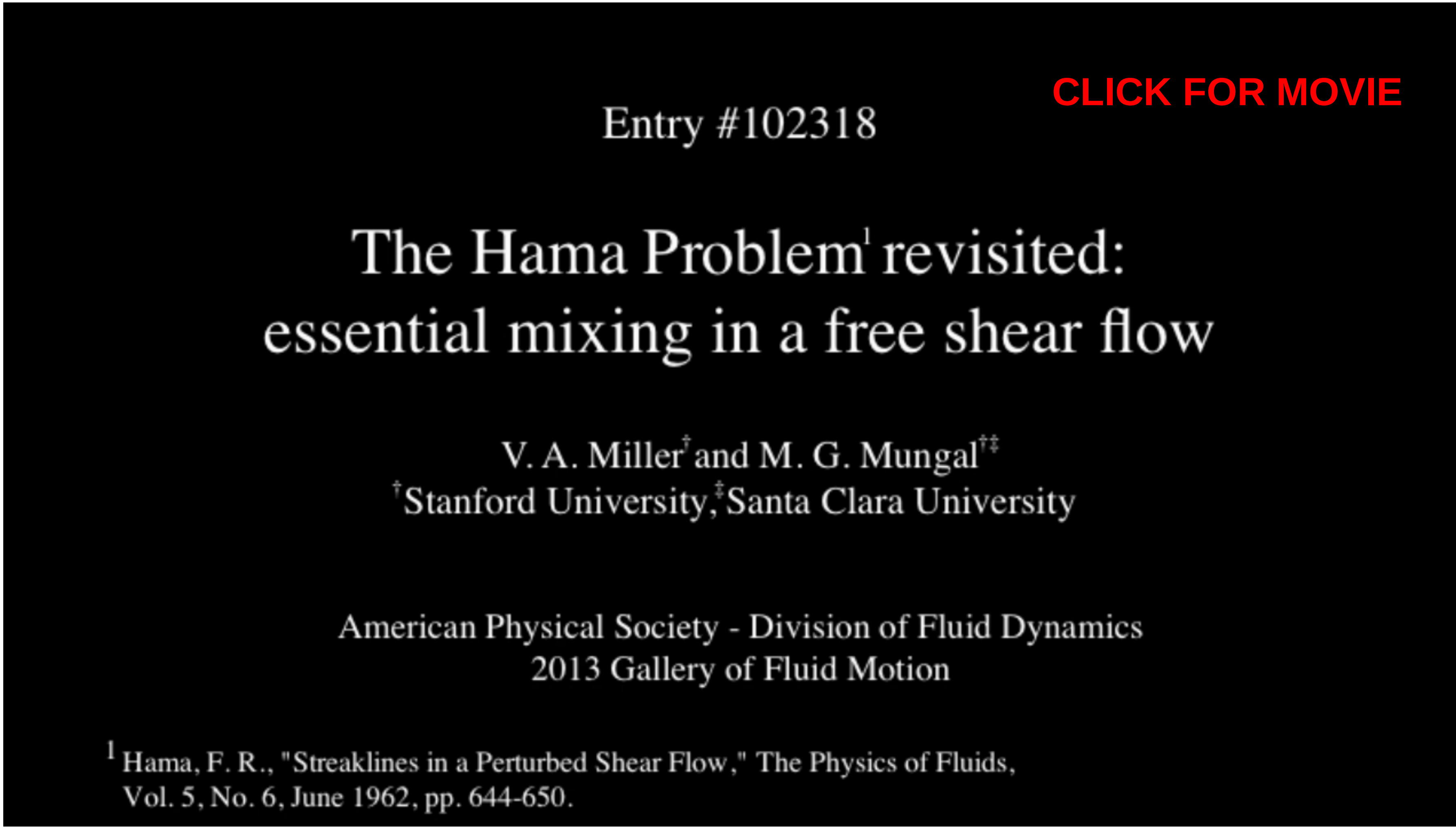
Velocitiy field

$\mathbf{v}(\mathbf{x},t)$



$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t)$$

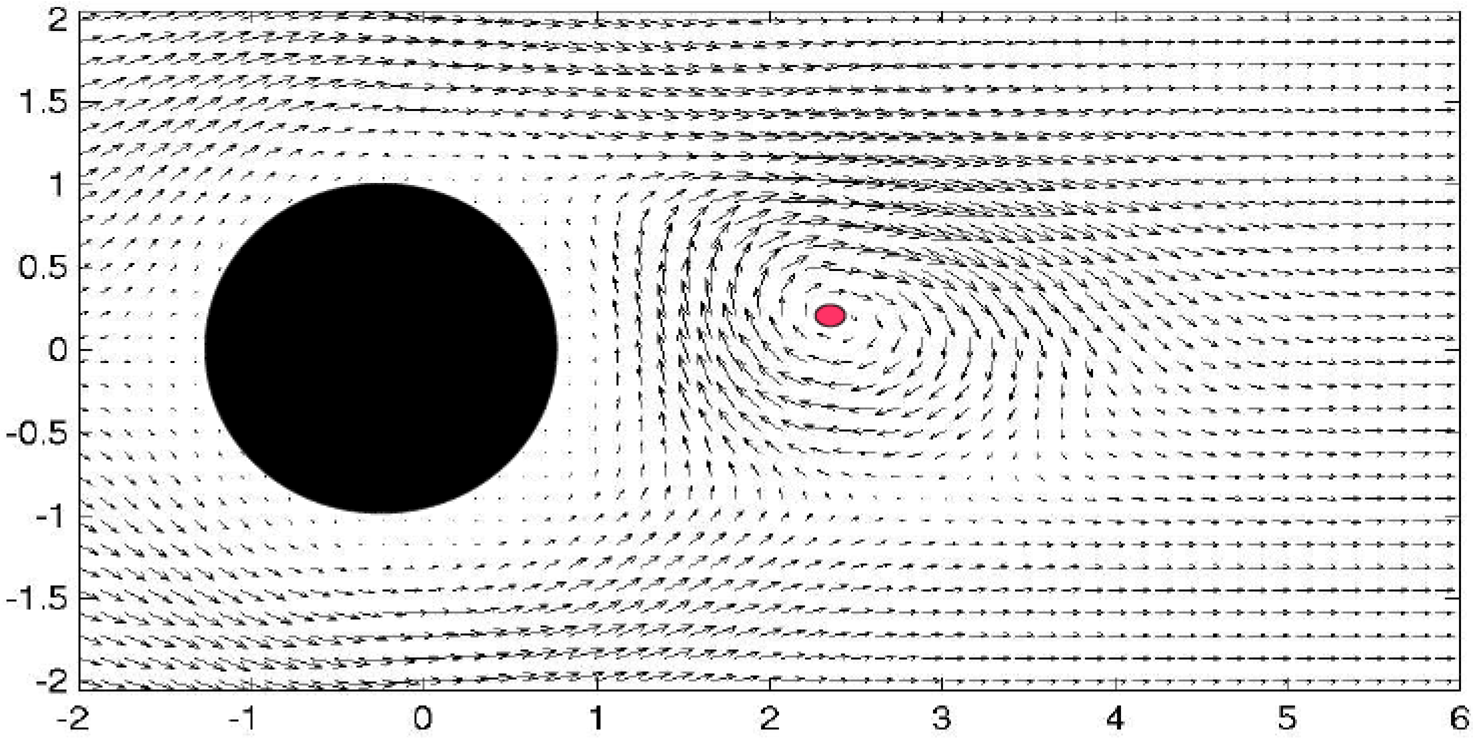
A useful formula: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$



From arXiv:1310.1644

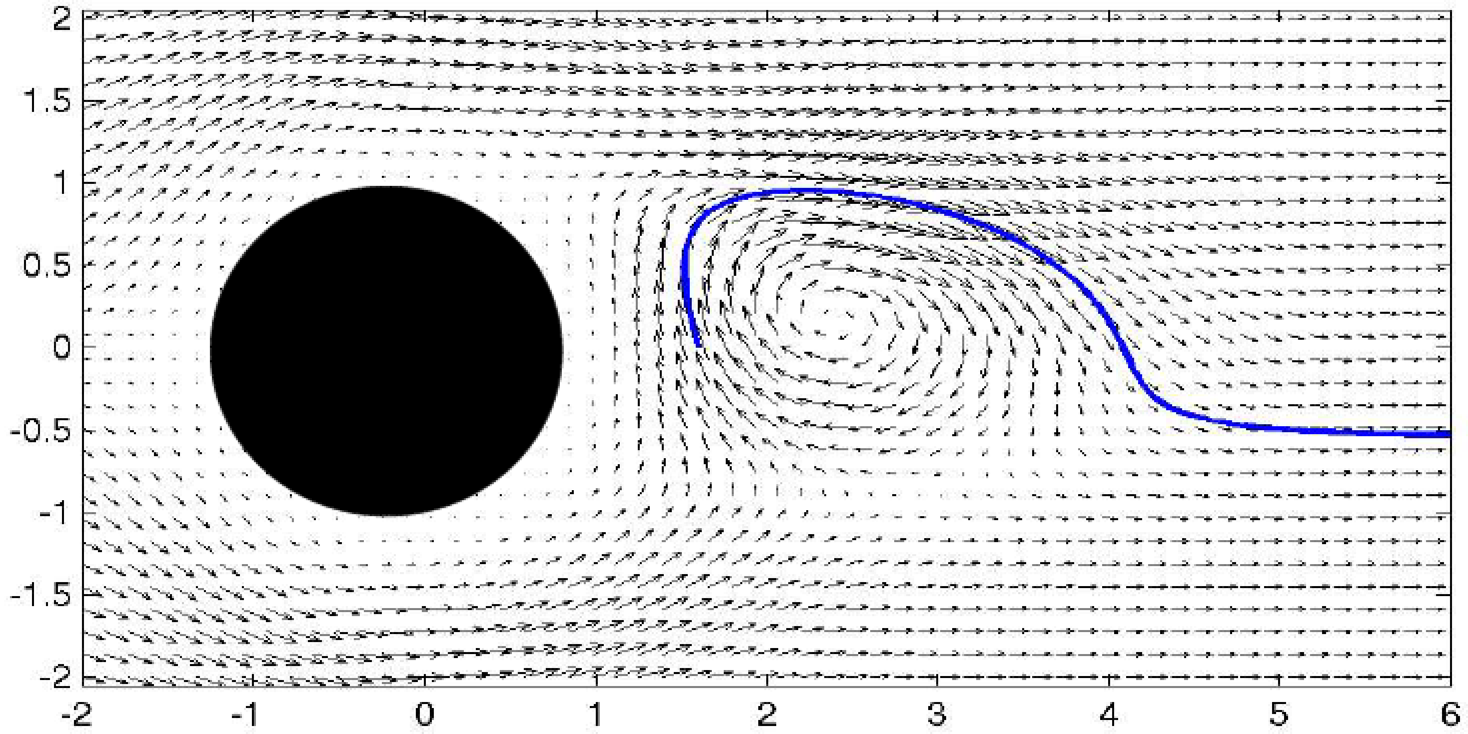
EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

Example: von Kárman cylinder wake flow



EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

Example: von Kárman cylinder wake flow



EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

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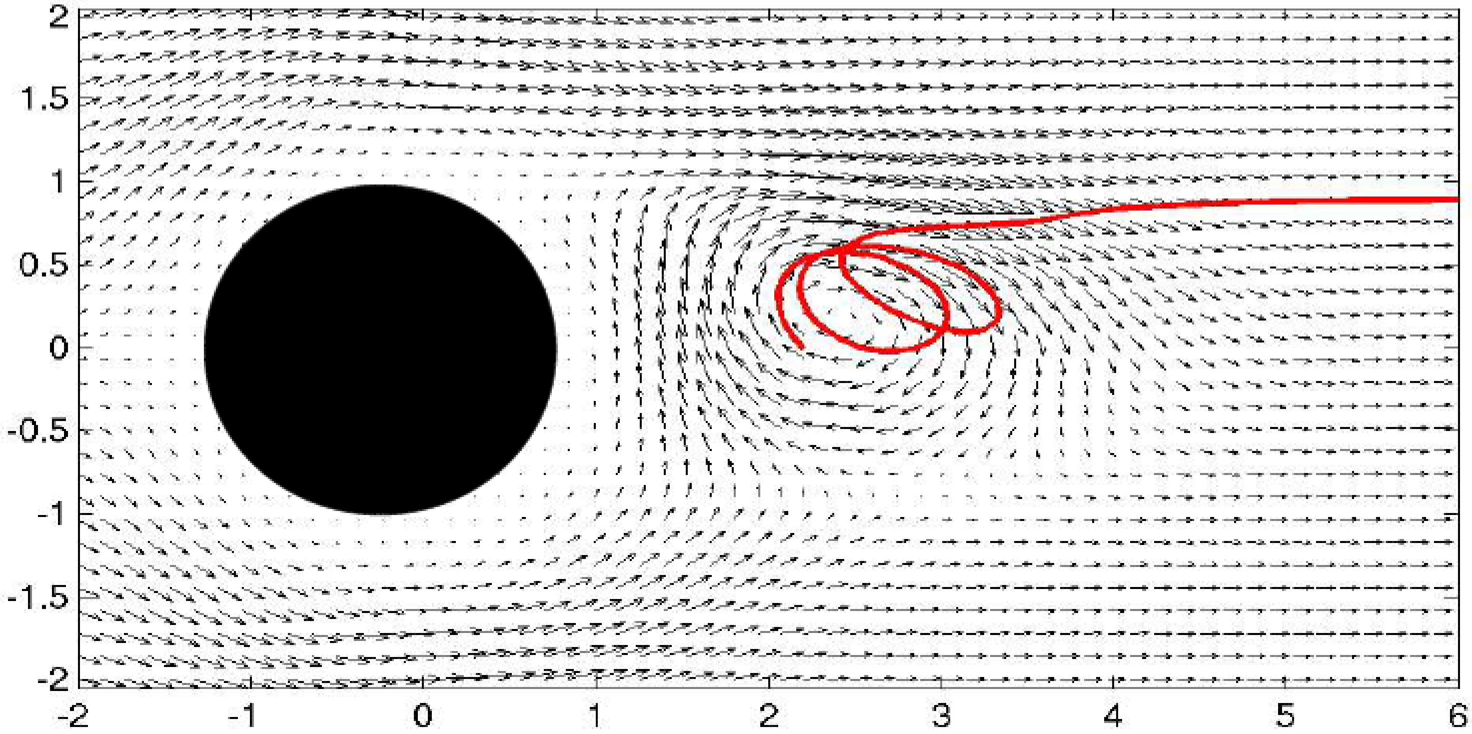
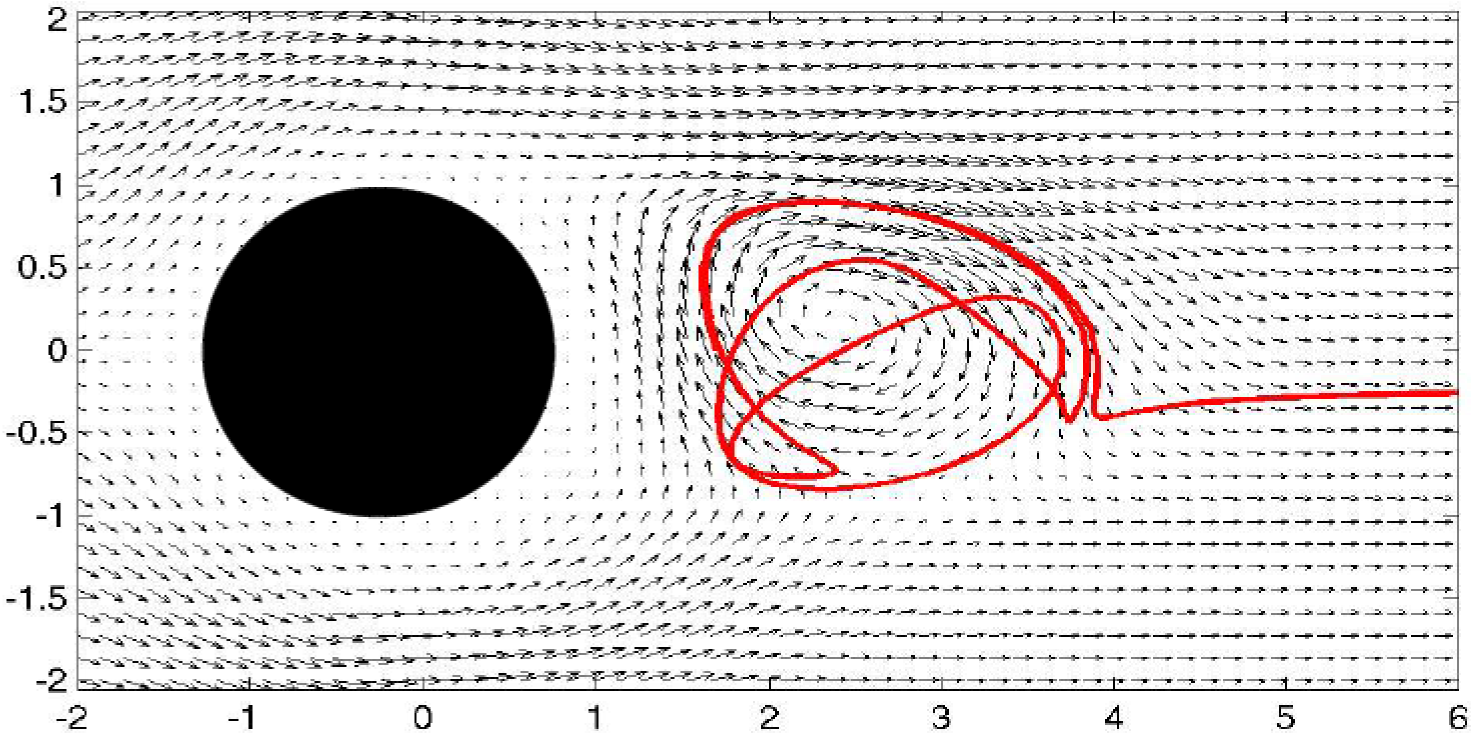


EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

Example: von Kárman cylinder wake flow



EULERIAN VS LAGRANGIAN DESCRIPTION OF FLUID DYNAMICS

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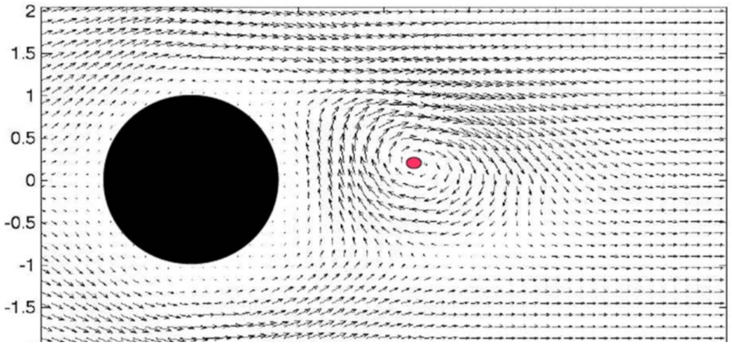
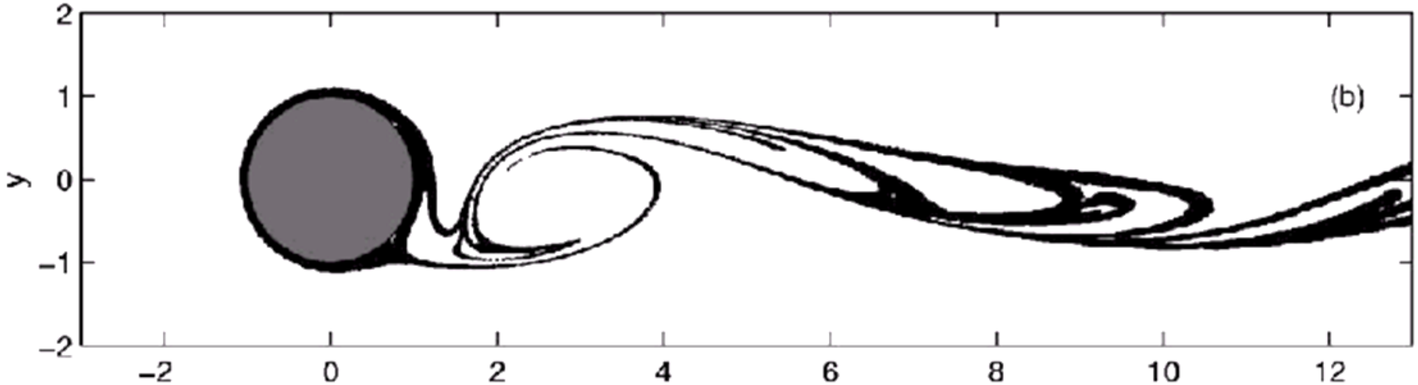
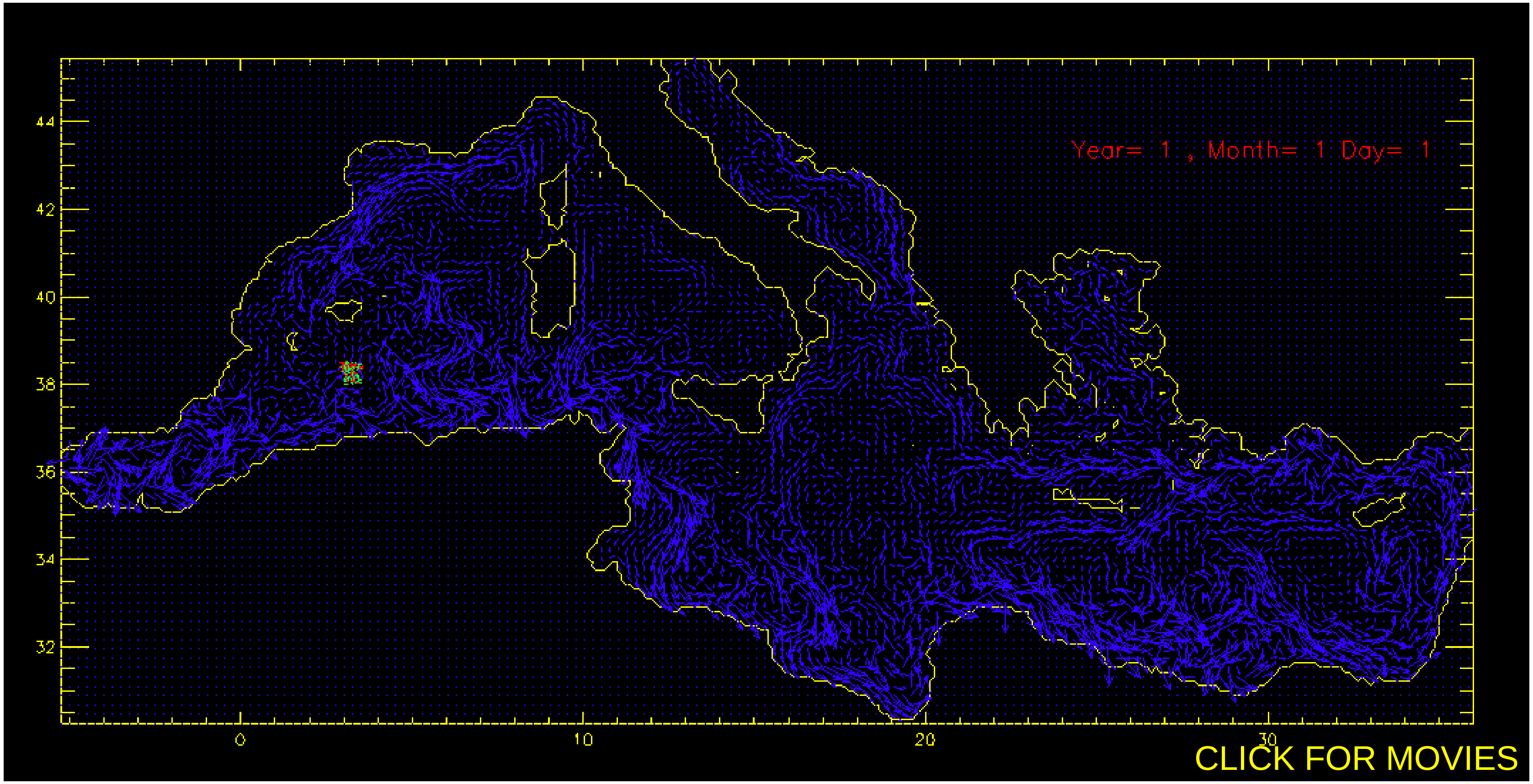






DieCAST model for the full Mediterranean
Primitive equations,
48 vertical levels, 1/8 º horizontal resolution,
climatological forcings … → 5 years of dayly velocity fields

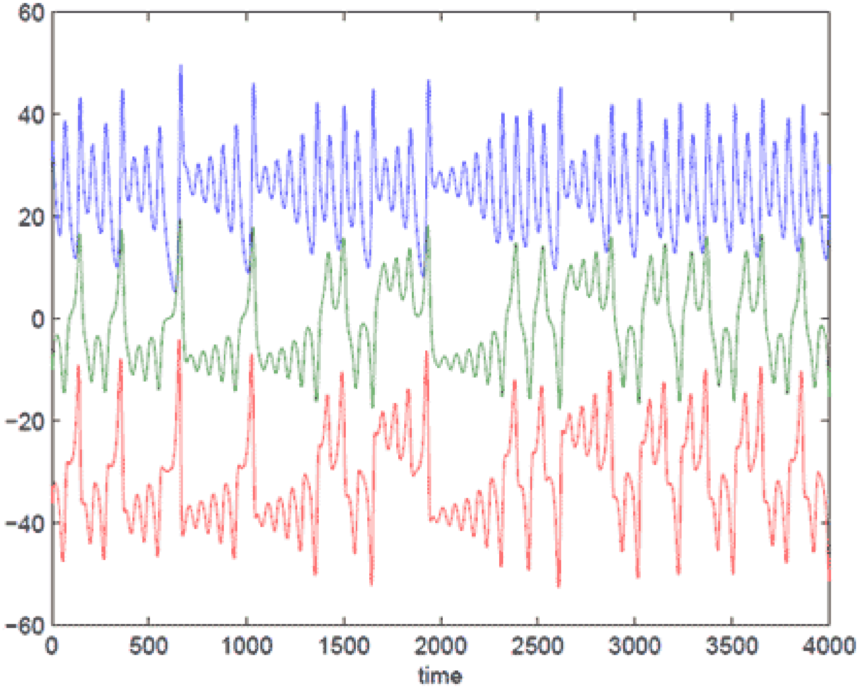
VELOCITIES

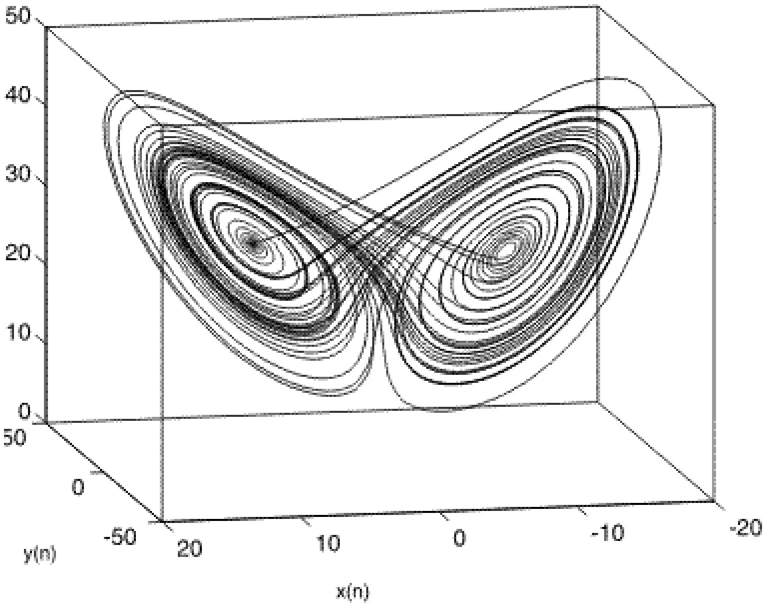
$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t)$$

Motion in simple velocity fields produce very complex trajectories

$$\frac{dx}{dt} = \sigma(y - x) \qquad \frac{dy}{dt} = x(\rho - z) - y \qquad \frac{dz}{dt} = xy - \beta z$$

Example: the Lorentz dynamical system





$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t)$$

Dynamical systems

Fluid transport

ODEs
Autonomous system
KAM tori, stable/unstable manifolds
Chaotic sets
…

→

Fluid flow
Steady velocity field
Transport barriers and avenues
Mixing regions
…

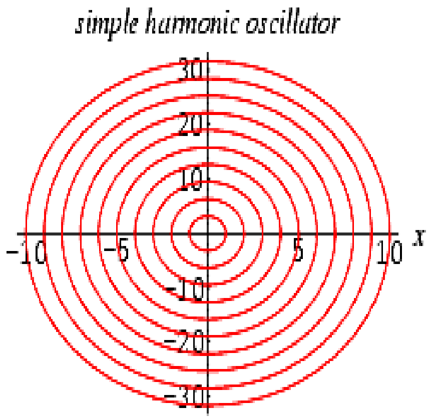
Basics of dynamical systems (I)

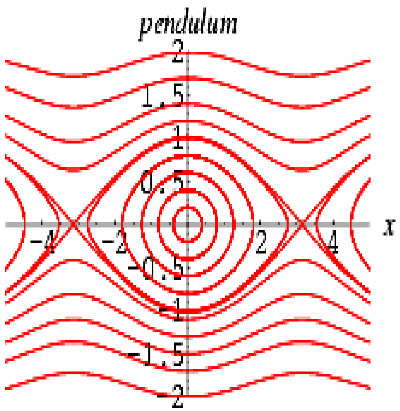
$$\mathbf{v} = \mathbf{v}(\mathbf{x}(\mathrm{t}), \mathrm{t}) \qquad \frac{\mathrm{d}\mathbf{x}(\mathrm{t})}{\mathrm{dt}} = \mathbf{v}(\mathbf{x}(\mathrm{t}), \mathrm{t})$$

$$\mathbf{x}(t) = \Phi_{t_0}^{t}(\mathbf{x}_0)$$

- Properties of a non-autonomous d-dimensional dynamical system are similar those of autonomous (d+1)-dimensional systems
- In autonomous one dimensional systems, motion can just remain, go to, or escape from fixed points
- In autonomous two dimensional systems, motion can just remain, go to or escape from fixed points, or remain, go to or escape from limit cycles (oscillations) [Poincaré-Bendixon theorem]
- Autonomous three-dimensional systems (and thus also non-autonomous 2d), in addition to all the above, can present chaotic trajectories

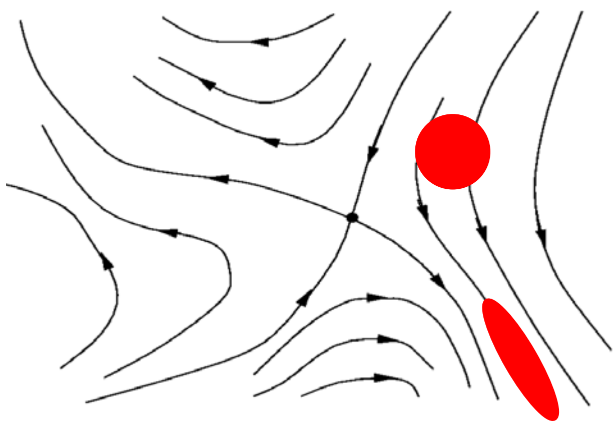
In AUTONOMOUS 2D SYSTEMS (=STEADY 2D FLOWS)





Trajectories are organized by the fixed points, or periodic orbits of the dynamical system

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t))$$



If hyperbolic:
Stable and
unstable manifolds → separatrices

Tracers tend to approach unstable manifolds

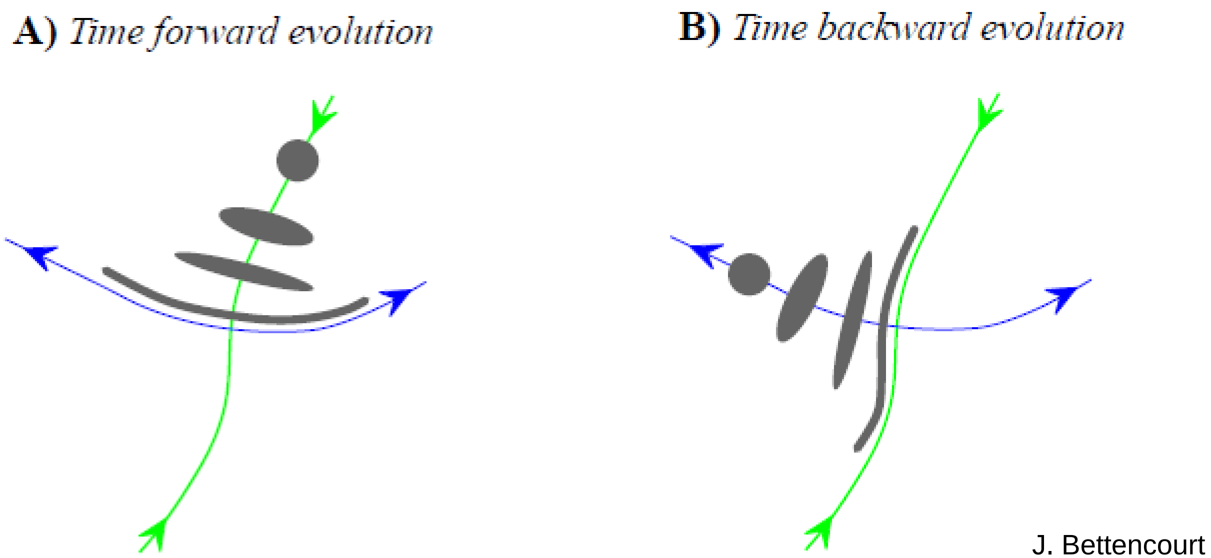


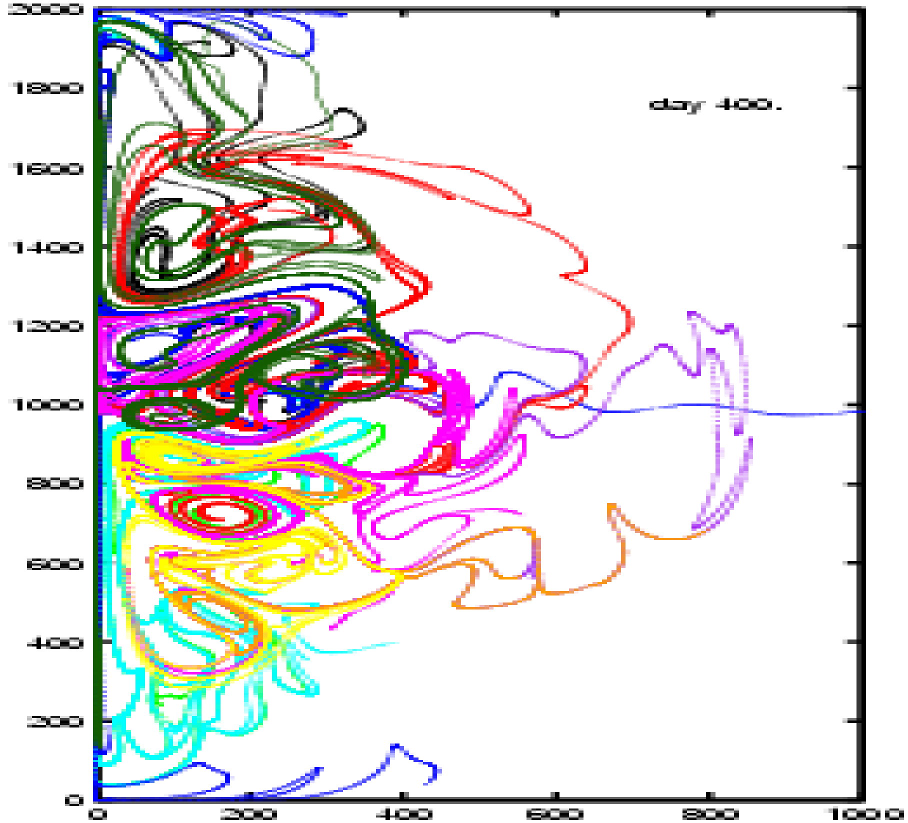
Figure 2.7: Time evolution of a fluid patch (gray) in the vicinity of attracting (blue) and repelling (green) coherent structures. A) Forward in time. B) Backward in time. Arrows indicate forward in time direction of trajectories *on* the coherent structure.

In the forward-time direction (i.e. t > 0), unstable manifolds ATTRACT material
stable ones REPEL material.

The contrary happens in the backwards (t<0) time direction.

But unsteady flows …

is there anything similar?



From Mancho, Small and Wiggins, 2005

Is there any particular subset of hyperbolic points and manifolds organizing the dynamics (the equivalent to the fixed points in autonomous systems) ? How to select them among this mess ?

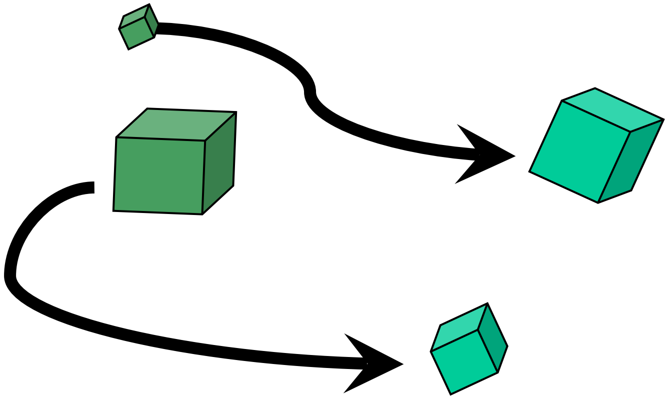
Basics of dynamical systems (II)

$$\mathbf{v} = \mathbf{v}(\mathbf{x}(t), t) \qquad \frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t), t)$$

$$\mathbf{x}(t) = \Phi_{t_0}^{t}(\mathbf{x}_0)$$

- The velocity divergence $\nabla \cdot \mathbf{v}(\mathbf{x}, t)$ gives the relative growth rate of the local volume element:

$$\frac{1}{V(t)} \frac{dV(t)}{dt} = \nabla \cdot \mathbf{v}$$



Then, in incompressible velocity fields $\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0$ volume of phase space (or of fluid) is conserved

Many interesting fluid flows (for example when speeds are much smaller than the speed of sound) are INCOMPRESSIBLE

Any incompressible two-dimensional velocity field (steady or not) can be written in terms of a streamfunction:

$$\frac{dx}{dt} = v_x = \frac{\partial \Psi(x,y,t)}{\partial y}$$
$$\frac{dy}{dt} = v_y = -\frac{\partial \Psi(x,y,t)}{\partial x}$$

Note that 2d incompressible Lagrangian fluid dynamics is fully isomorphic to Hamiltonian mechanics of 1 degree of freedom:

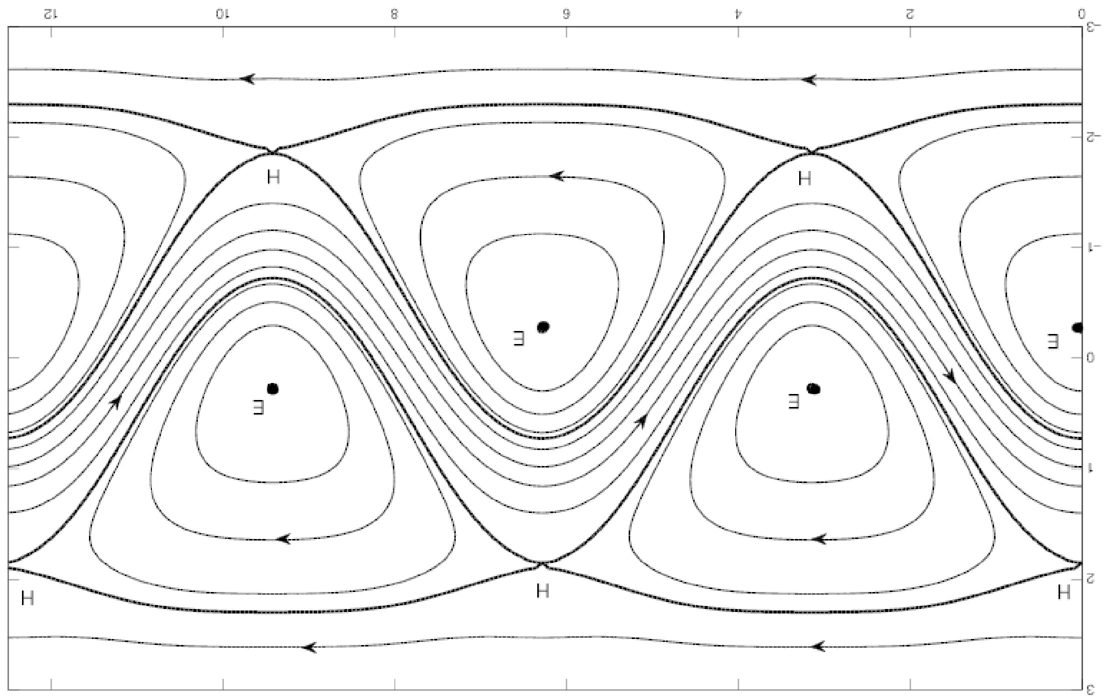
$$\frac{dq}{dt} = \frac{\partial H(p,q,t)}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H(p,q,t)}{\partial q}$$

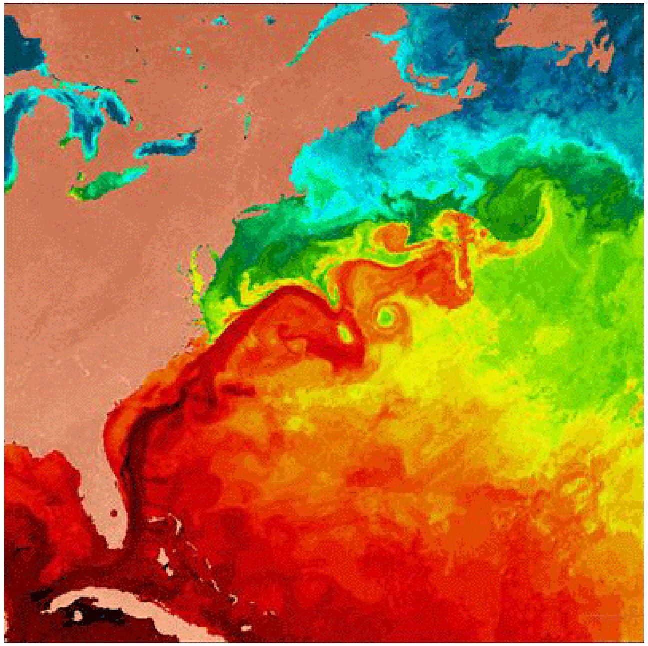
Rate of change of the streamfunction along the path of a fluid particle:

$$\frac{D\Psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\Psi = \frac{\partial\psi}{\partial t}$$

Thus, in a steady flow ($\frac{\partial \Psi}{\partial t}$=0), Ψ is conserved in the fluid elements, so that trajectories are in the isolines of Ψ

Example: a (steady) meandering jet kinematic model





Fixed points and their manifolds organize the flow

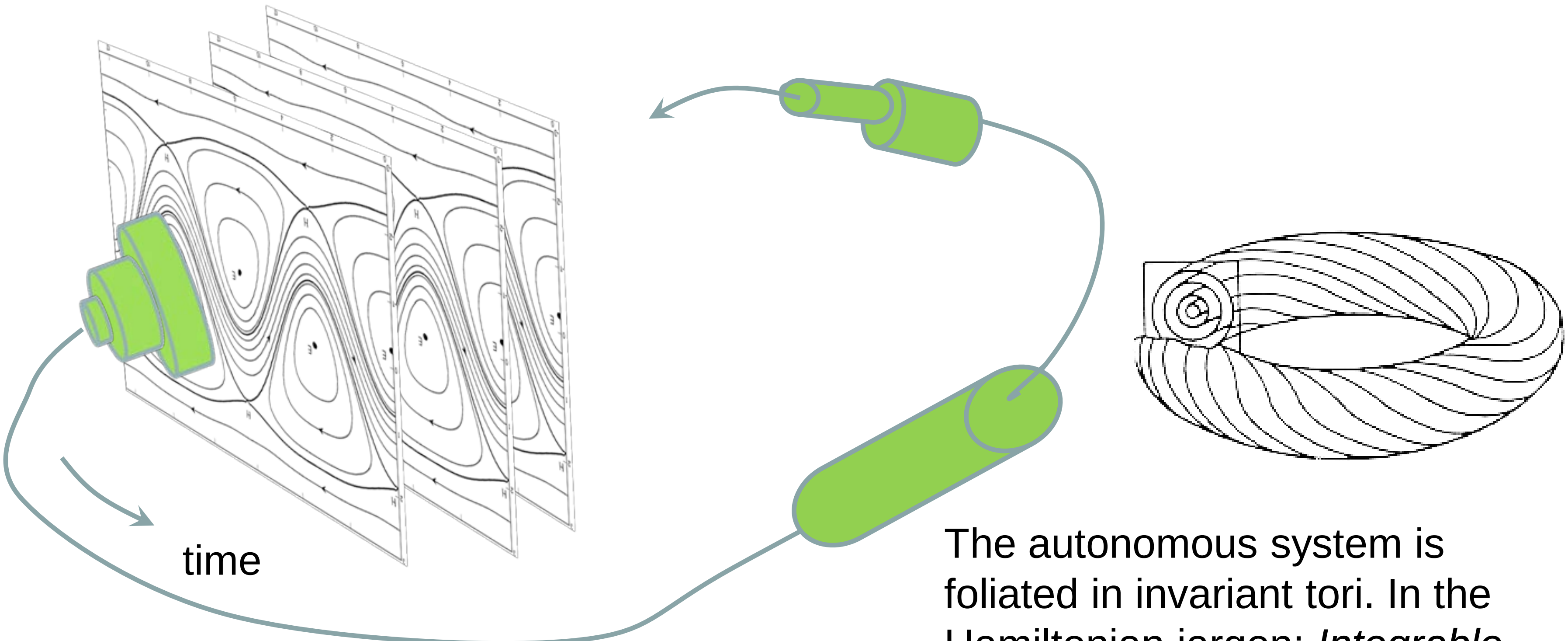
$$\psi(x,y) = Cy - \tanh[(y - A\cos x)/(L\sqrt{1 + A^2 \sin x^2})]$$

What happens if flow becomes unsteady?

$$\psi(x,y) = Cy - \tanh[(y - A\cos x)/(L\sqrt{1 + A^2 \sin x^2})]$$

$$A = A_0 + \epsilon \sin \omega t \ , \ \text{period } T = 2\pi/\omega$$

Useful technique: The *Poincaré map* trick: $\mathbf{x}(t+T) = \Phi_t^{t+T}(\mathbf{x}(t))$



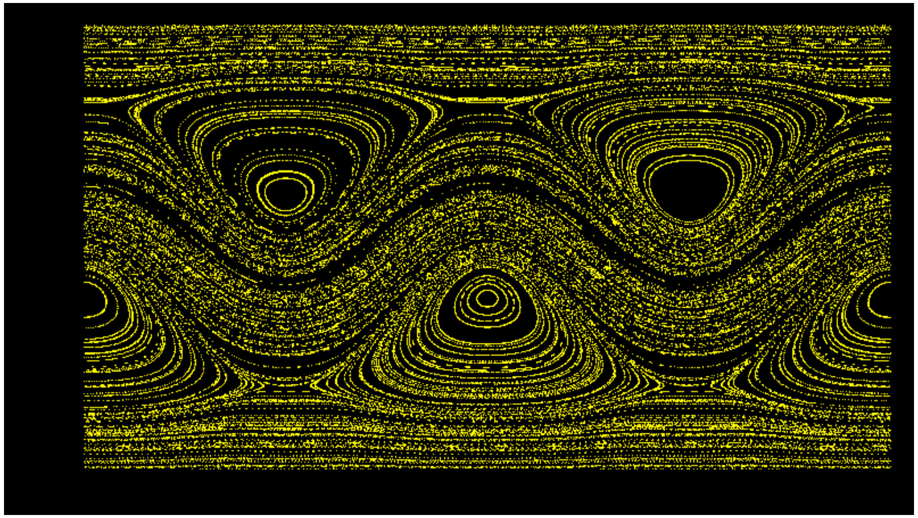
The autonomous system is foliated in invariant tori. In the Hamiltonian jargon: *Integrable*

$$\psi(x,y) = Cy - \tanh[(y - A\cos x)/(L\sqrt{1 + A^2 \sin x^2})]$$

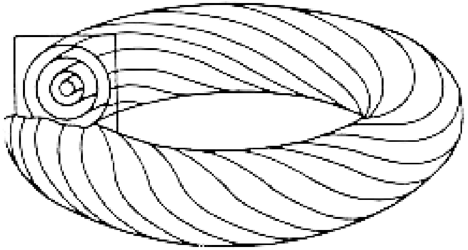
$$A = A_0 + \epsilon \sin \omega t \ , \ \text{period } T = 2\pi/\omega$$

Poincaré map : $\mathbf{x}(t+T) = \Phi_t^{t+T}(\mathbf{x}(t))$

$\epsilon = 0$, steady flow



300 particles initially



After 200 periods

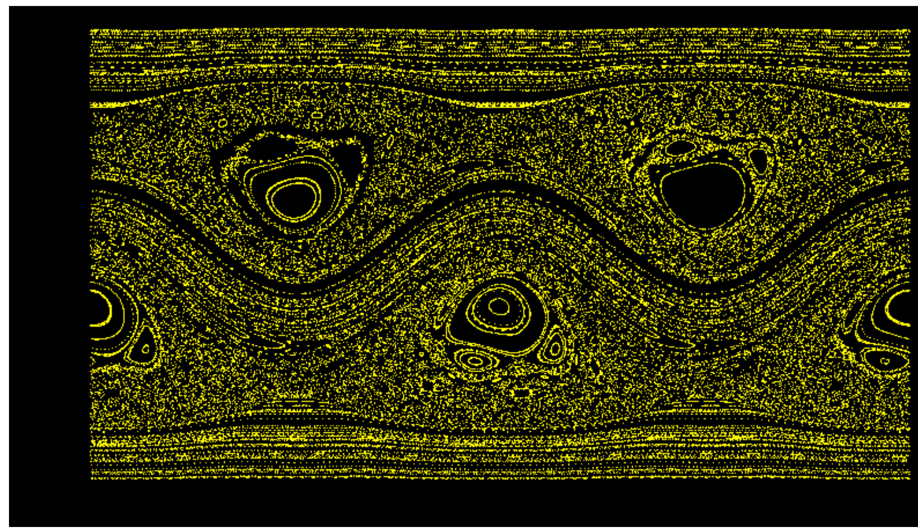
Invariant tori

$$\psi(x,y) = Cy - \tanh[(y - A\cos x)/(L\sqrt{1 + A^2 \sin x^2})]$$

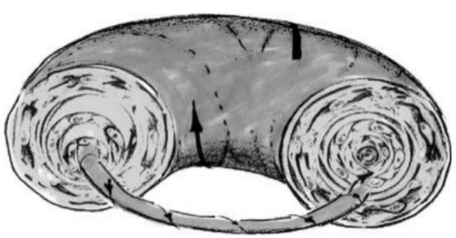
$$A = A_0 + \epsilon \sin \omega t \ , \ \text{period } T = 2\pi/\omega$$

Poincaré map : $\mathbf{x}(t+T) = \Phi_t^{t+T}(\mathbf{x}(t))$

$\epsilon = 0.02$, unsteady flow



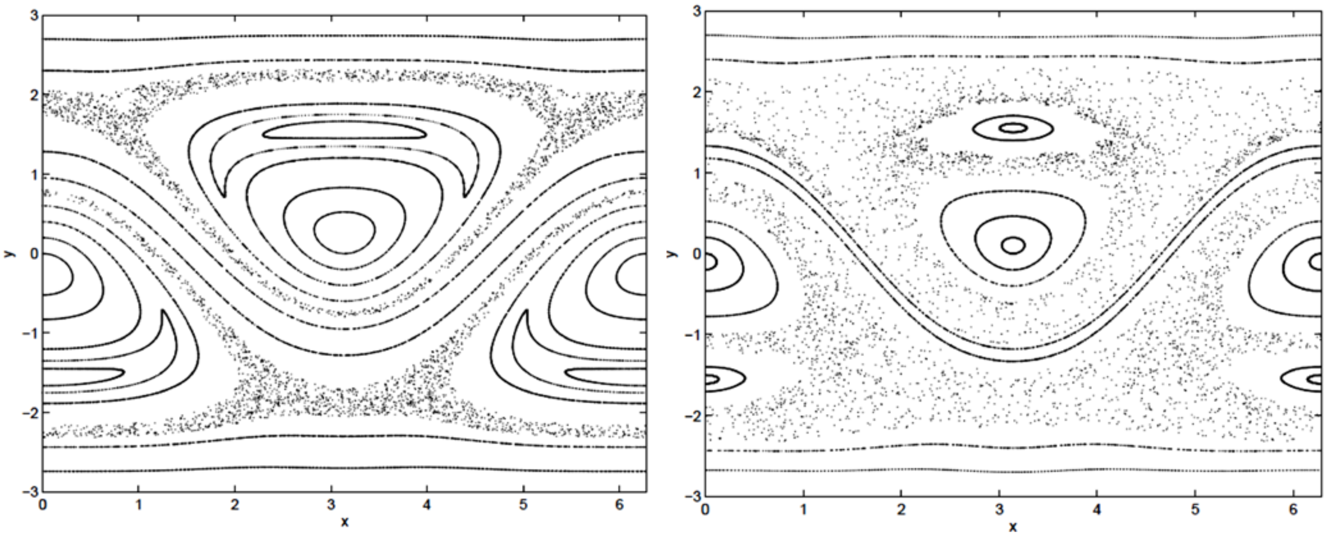
300 particles initially



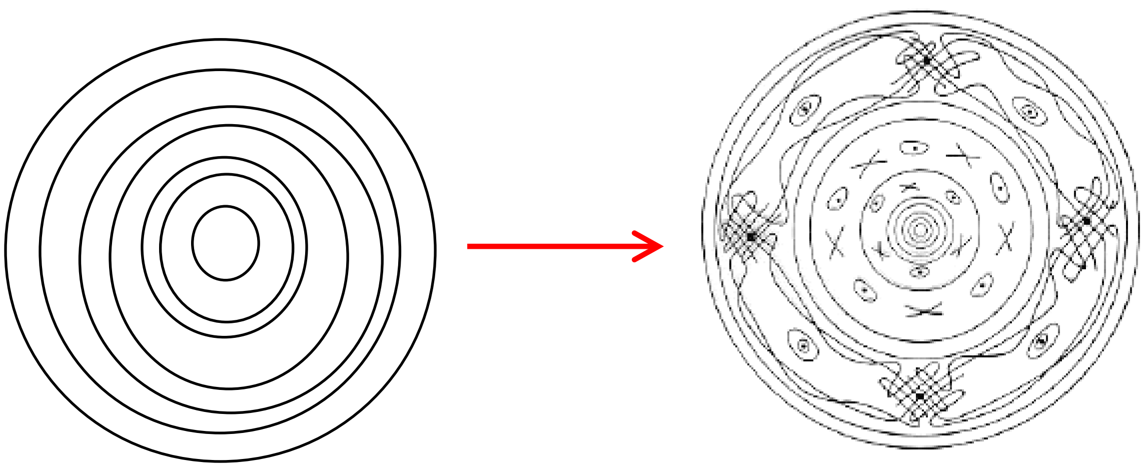
After 200 periods

Some invariant tori are broken

Increasing the time-dependent perturbation to the integrable case



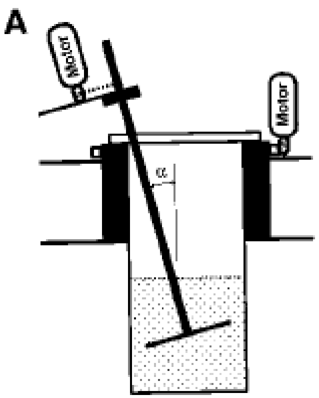
Remaining invariant tori + chaotic sea

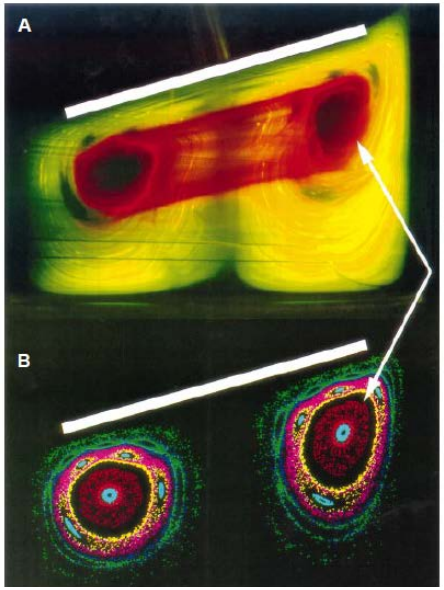


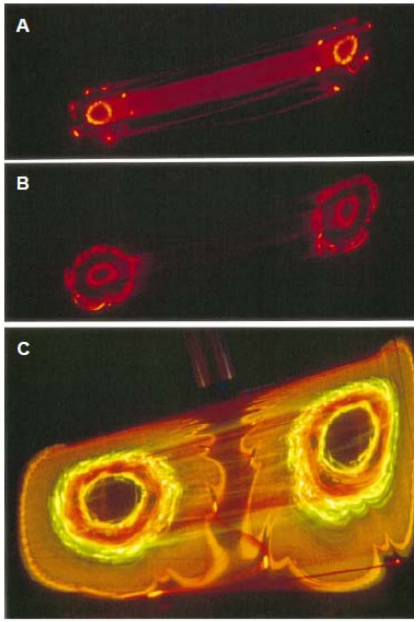
VERY RELEVANT FOR FLUID TRANSPORT !

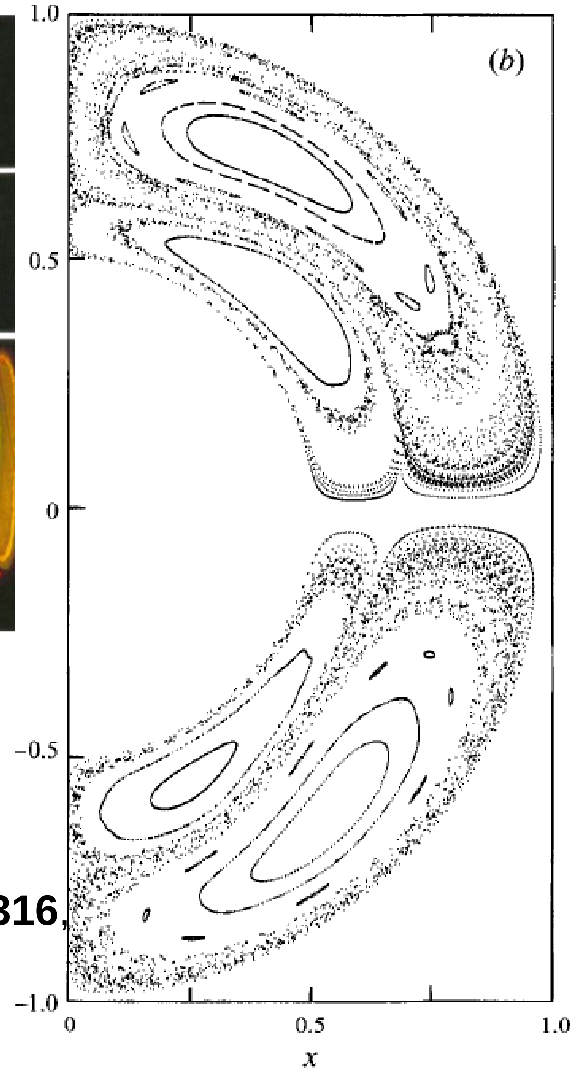
Arnold, 1963

Similar situation in 3D steady and periodically perturbed flows

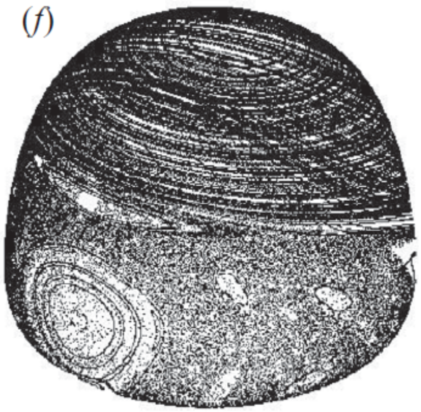








Fountain et al, Science 281, 683 (1998)

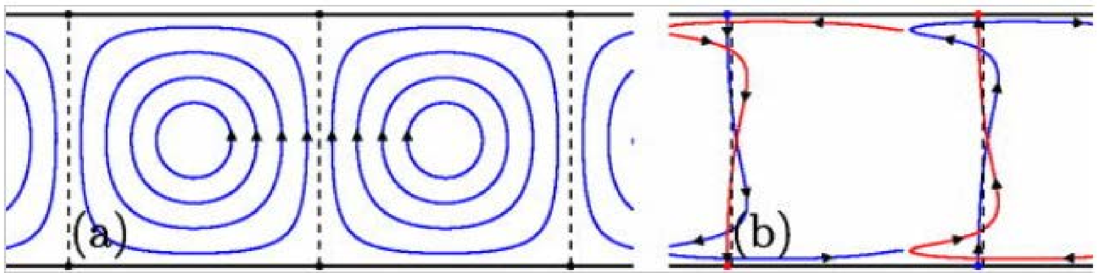


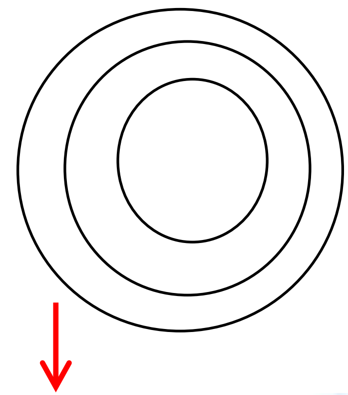
Pouransari et al.
J. Fluid. Mech. **654**, 5-34 (2010)

Cartwright et al.
J. Fluid. Mech. **316**, 259-284 (1996)

Theory (Ott, Chos in Dynamical systems)

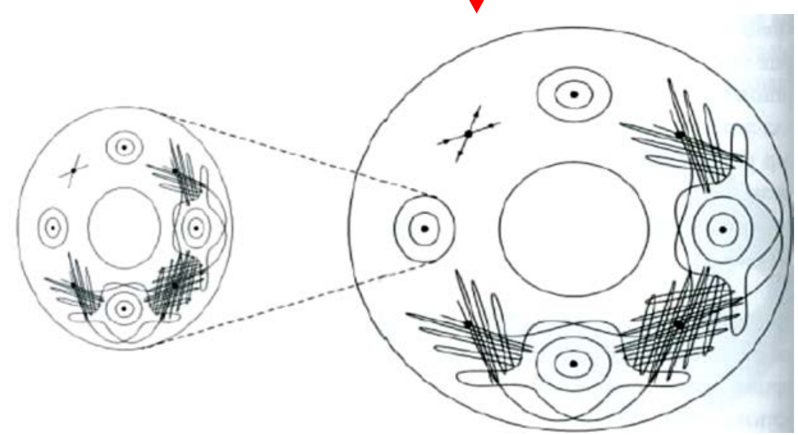
- After all, connection of the unstable manifold of a fixed point with the stable of another is really non-generic, except under the conditions of incompressibility AND steadiness. A perturbation produces separatrix crossing (Melnikov formula)





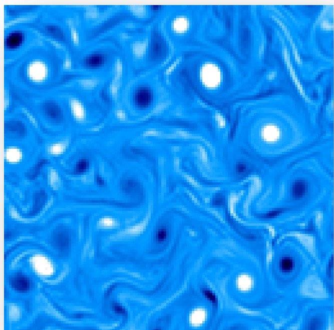
- **Poincaré-Birkoff theorem**: ALL resonant tori (those with $W=T_{\_tori}/T_{\_perturbation}=p/q$ rational) are destroyed and replaced by a chain of elliptic + hyperbolic islands + chaotic sea

- **Kolmogorov-Arnold-Moser (KAM) theorem**: Tori with W sufficiently irrational $\left(\left|W-\frac{p}{q}\right|>\frac{K(\epsilon)}{q^{5/2}}\right)$ are deformed but not destroyed

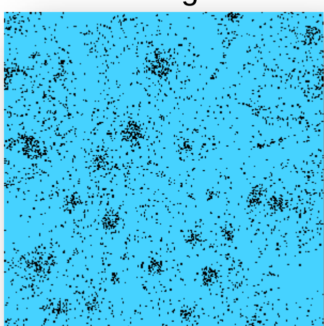


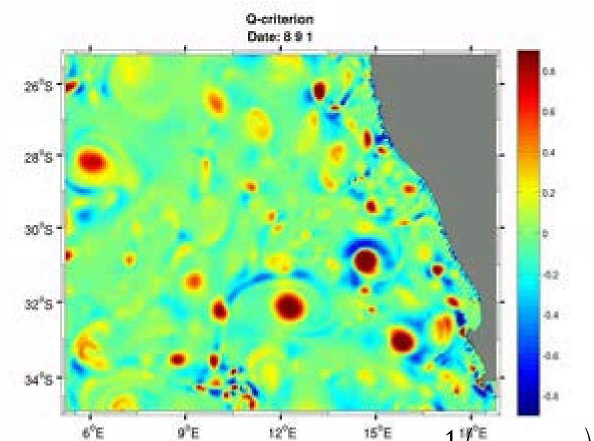
What happens in non-periodically perturbed flows?

- In some cases KAM-like theorems are also valid (quasiperiodic perturbations)
- The idea is that there should be something like ‘travelling KAM tori’. But this is not rigurous in general, and in fact there is leaking from these structures. → **LCSs**

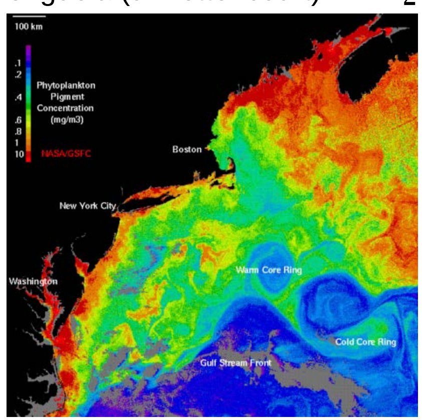


Vorticity in decaying turbule from Cartwright et al.

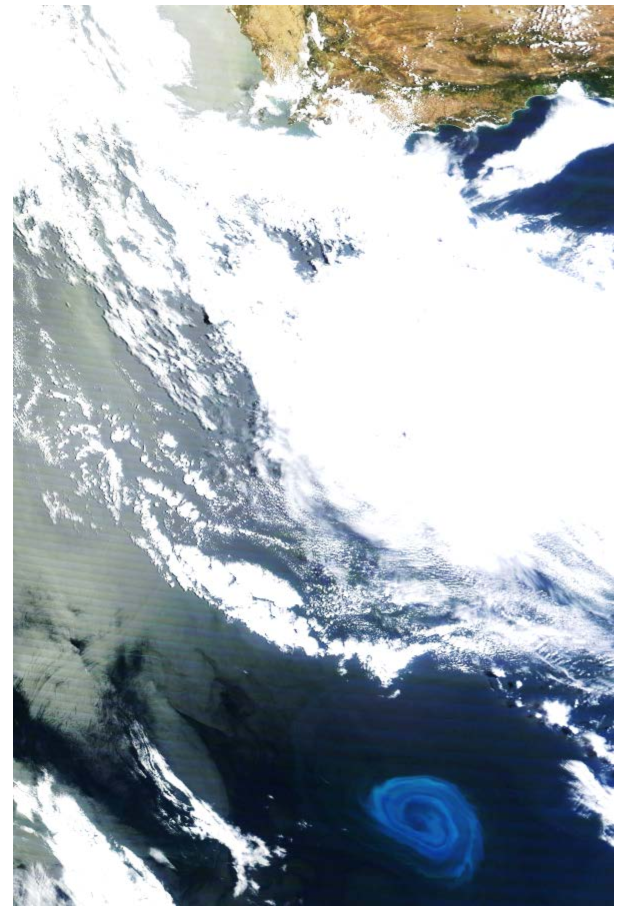




Q-criterion (Hunt et al, 1988): $Q=\frac{1}{2}\left(\|\Omega\|^2-\|S\|^2\right)$
In Benguela (J. Bettencourt)



Temperature

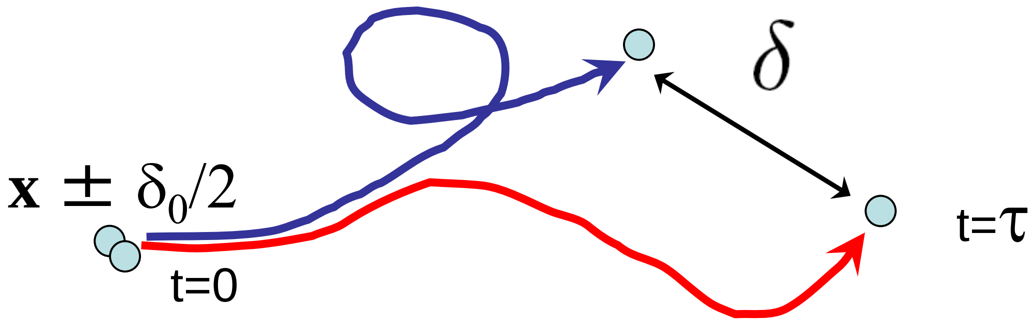


$$\lambda(t) = \lim_{\|\delta(0)\|\to 0} \frac{1}{t} \ln \frac{\|\delta(t)\|}{\|\delta(0)\|}$$

Finite-**time** Lyapunov exponent FTLE

$$\lambda = \lim_{t\to\infty} \lambda(t)$$

Lyapunov exponent



$$\lambda(\delta_0, \delta_f) \equiv \frac{1}{\tau} \log \frac{\delta_f}{\delta_0}$$

Finite-size Lyapunov exponent FSLE

All the quantities are also functions of the initial position and time:

$$\lambda(\mathbf{x}, t, \delta_0, \delta_f)$$

Lyapunov exponent

$\boldsymbol{\delta}(t) = \mathbf{r}'(t) - \mathbf{r}(t)$ satisfies the equation

$$\dot{\boldsymbol{\delta}} = \mathbf{v}[\mathbf{r}'(t), t] - \mathbf{v}[\mathbf{r}(t), t] \approx \nabla\mathbf{v}[\mathbf{r}(t), t]\boldsymbol{\delta}$$

$$\boldsymbol{\delta}(t) = \mathrm{M}_t(\mathbf{x}_0)\boldsymbol{\delta}(0) \qquad M_{ij} = \partial(\Phi_t)_i/\partial r_j$$

$$|\boldsymbol{\delta}(t)|^2 = \boldsymbol{\delta}_0^T \mathrm{M}_t^T \mathrm{M}_t \boldsymbol{\delta}_0$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_d$$

The Oseledec theorem (Eckmann and Ruelle, 1985) implies that under rather general conditions the eigenvectors of $\mathrm{M}_t^T(\mathbf{x}_0)\mathrm{M}_t(\mathbf{x}_0)$ converge, as $t \to \infty$, to a set of Lyapunov vectors $\{\mathbf{v}_i\}$. Since $\mathrm{M}_t^T\mathrm{M}_t$ is symmetric and positive definite the Lyapunov vectors form an orthonormal set and the eigenvalues Λ_i^t are all positive. If the initial separation is chosen to be oriented along one of the Lyapunov vectors, $\boldsymbol{\delta}_0 = \delta_0\mathbf{v}_i$ with $\delta_0 \ll L$, the distance between the particles at a later time t is

$$|\mathrm{M}_t\delta_0\mathbf{v}_i| = (\delta_0\mathbf{v}_i^T\mathrm{M}_t^T\mathrm{M}_t\delta_0\mathbf{v}_i)^{1/2} = \delta_0(\Lambda_i^t)^{1/2}. \qquad (2.70)$$

$$\lambda_i = \lim_{t\to\infty}\lim_{\delta_0\to 0}\frac{1}{t}\ln\left(\frac{|\mathrm{M}_t\delta_0\mathbf{v}_i|}{\delta_0}\right) = \lim_{t\to\infty}\frac{\ln\Lambda_i^t}{2t}.$$

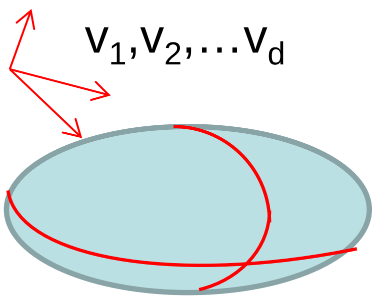
Incompressibility:

$$\det(\mathrm{M}) = 1$$

$$\sum \lambda_i = 0$$

2d incompressible:

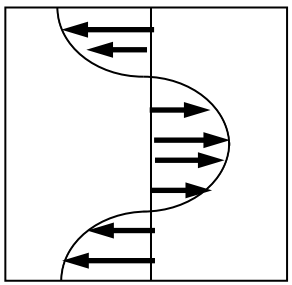
$$\lambda_2 = -\lambda_1$$

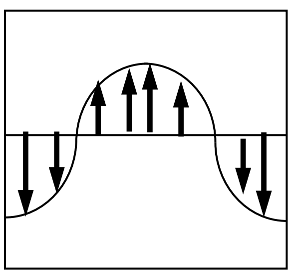


Arbitrary initial displacement:

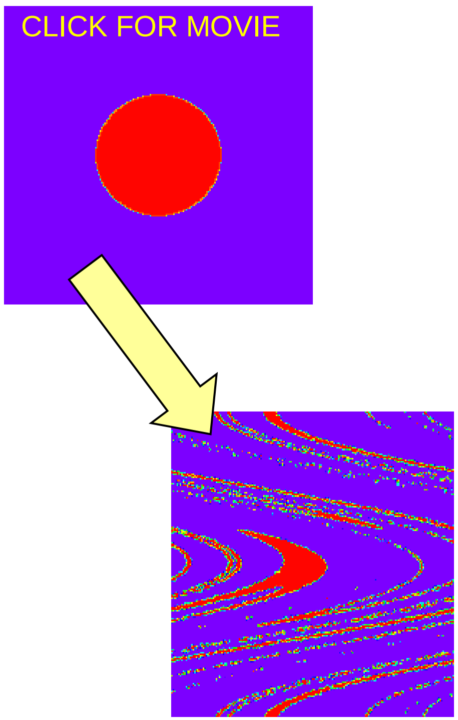
$$\delta_0 = \sum c_i \mathbf{v}_i \qquad \delta(t) = \sum_i c_i \mathbf{M}_t(\mathbf{x}_0)\mathbf{v}_i.$$

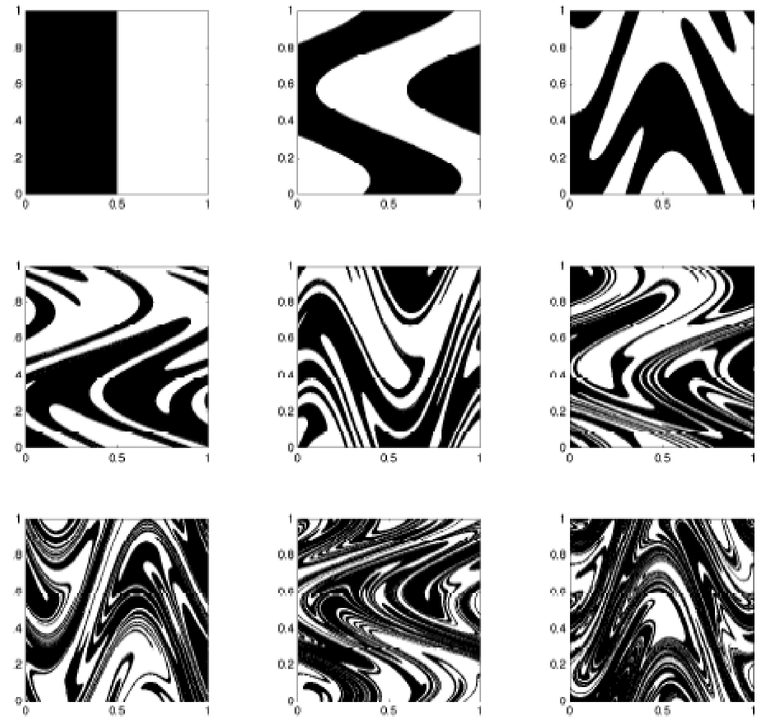
$$\lim_{t\to\infty} \lim_{\delta_0\to 0} \frac{|\delta(t)|}{|\delta_0|} \sim e^{\lambda_1 t},$$





x,y MOD 1





Lyapunov vector fields define unstable (and stable) foliation

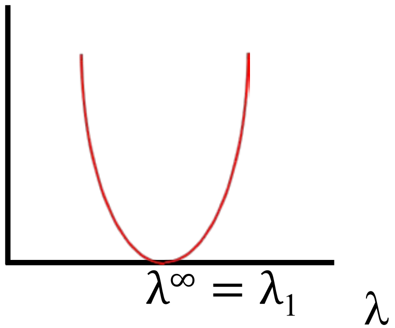
Warning: only in hyperbolic regions

Distribution of Lyapunov exponents at finite time

$$P(\lambda,t) \sim t^{1/2} e^{-G(\lambda)t}$$

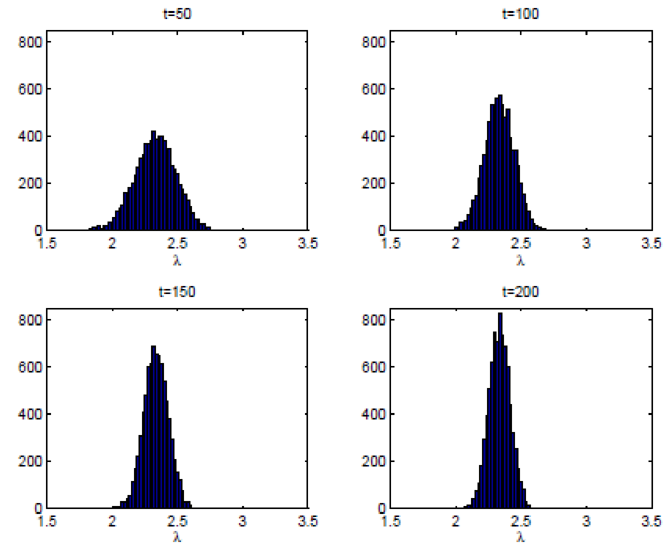
Large deviation theory

G(λ) Cramer function



$$P(\lambda,t)t^{1/2} \sim e^{-\frac{(\lambda-\lambda^\infty)^2}{2\Delta}t}, \quad \text{where } \Delta = \frac{1}{G''(\lambda^\infty)}$$

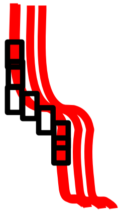
Values of λ are arranged in sets that become Fractal (multifractal) at infinite time



$$N(l) \sim l^{-D_f}$$

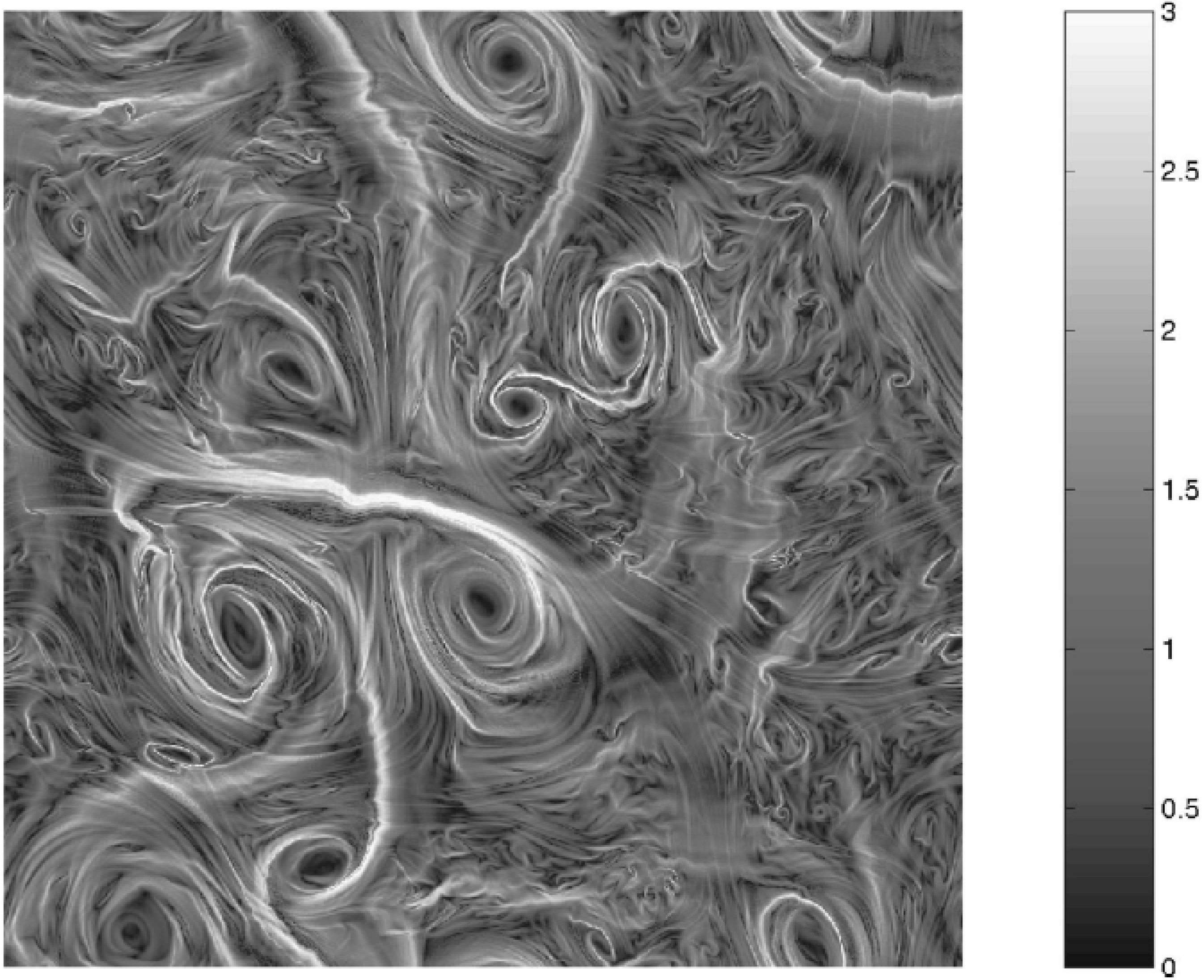
$$A_\lambda(t) \sim \exp(-G(\lambda)t)$$

$$w_\lambda(t) \sim \exp(-\lambda t)$$



$$N_\lambda(l) \simeq \frac{A_\lambda}{(w_\lambda)^2} \sim e^{[2\lambda - G(\lambda)]t} \sim l^{[G(\lambda)/\lambda]-2}$$

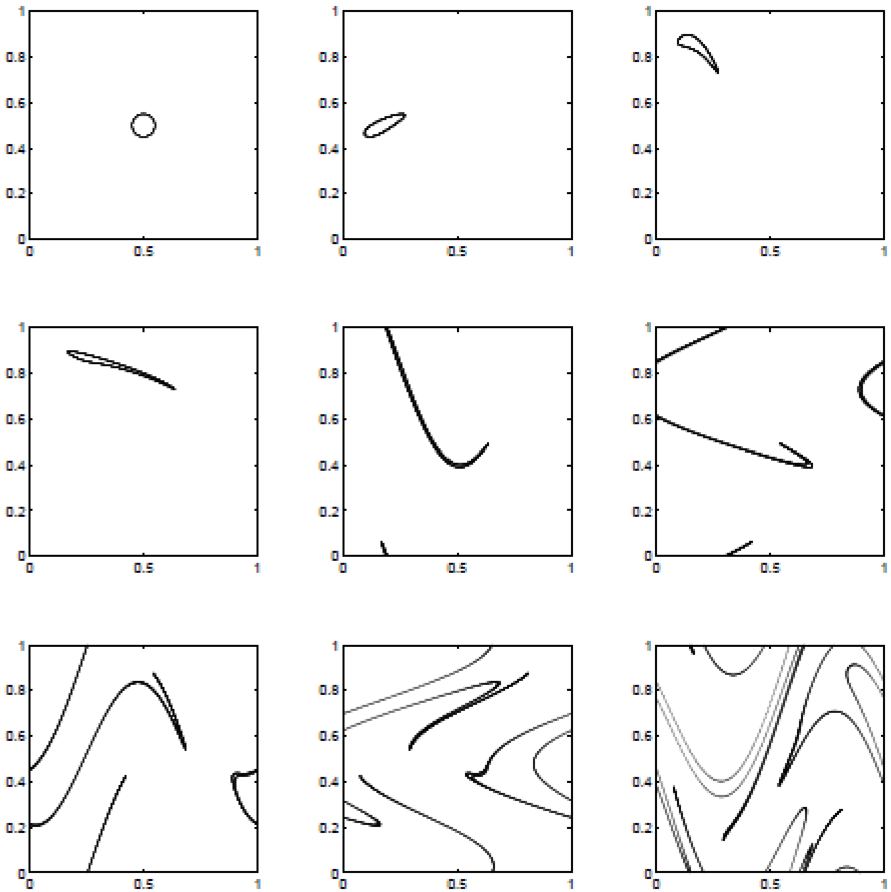
$$D_f(\lambda) = 2 - \frac{G(\lambda)}{\lambda}$$

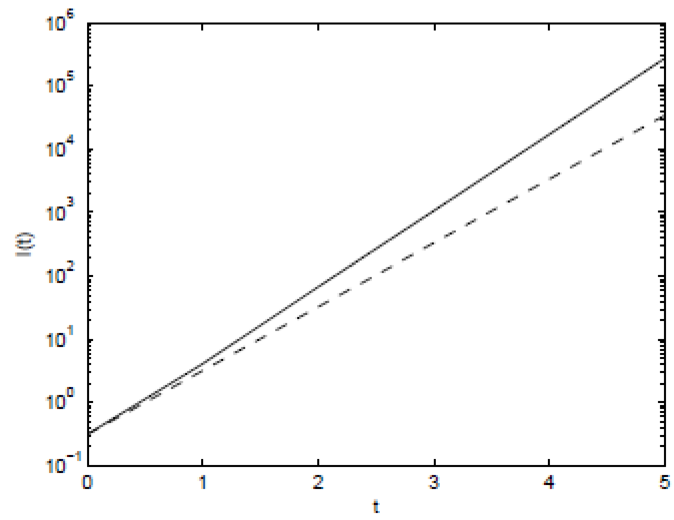


G. Lapeyre, Chaos, Vol. 12, 688 (2002)

Are sets of zero measure relevant for anything?

Length of a material line:





Length of a material line:

$$l(t) \sim l_0 \int e^{\lambda t} P(\lambda, t) d\lambda \sim l_0 \int e^{(\lambda - G(\lambda))t} d\lambda$$

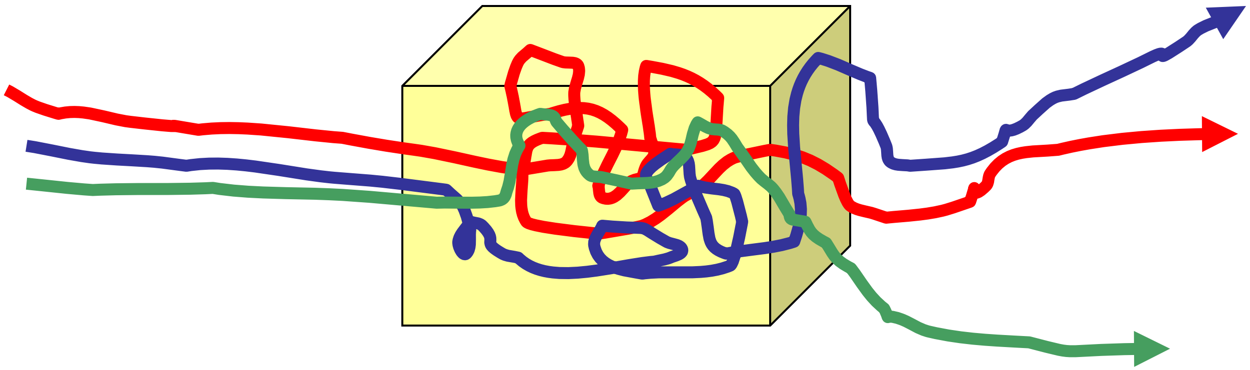
$$\lim_{t\to\infty} l(t) \sim e^{\gamma t} \text{ where } \gamma = \max_{\lambda}[\lambda - G(\lambda)].$$

Topological entropy $\gamma \geq \lambda\infty$

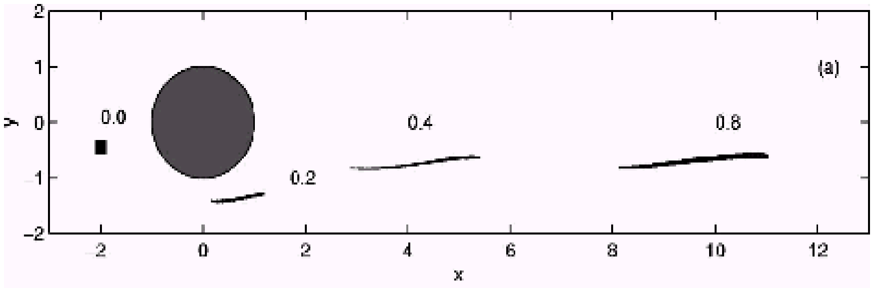
Gaussian approximation: $\lambda^* = \lambda^\infty + \Delta$ $\gamma = \lambda^* - G(\lambda^*) = \lambda^\infty + \frac{\Delta}{2}.$

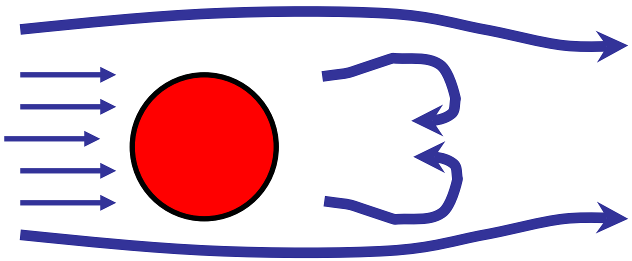
OPEN CHAOTIC FLOWS

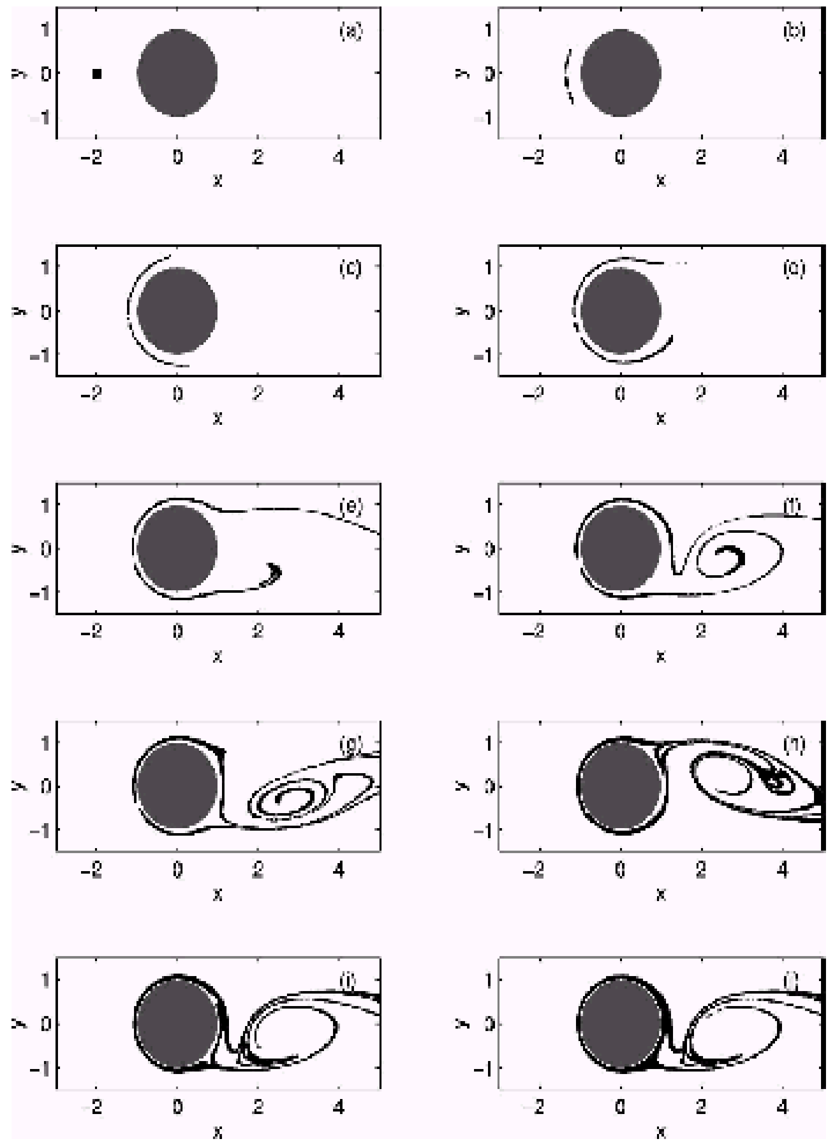
transient chaos (or chaotic scattering)



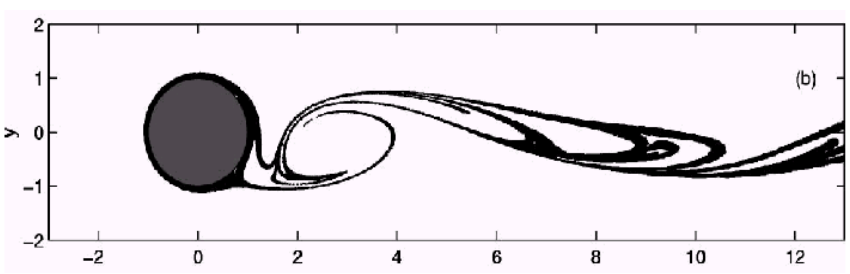
Ott & Tel, *An introduction to chaotic scattering*, CHAOS 3, 417 (1993)
Jung, Tel & Ziemniak, *Applications of scattering chaos to particle transport in a hydrodynamical flow*, CHAOS 3, 555 (1993)

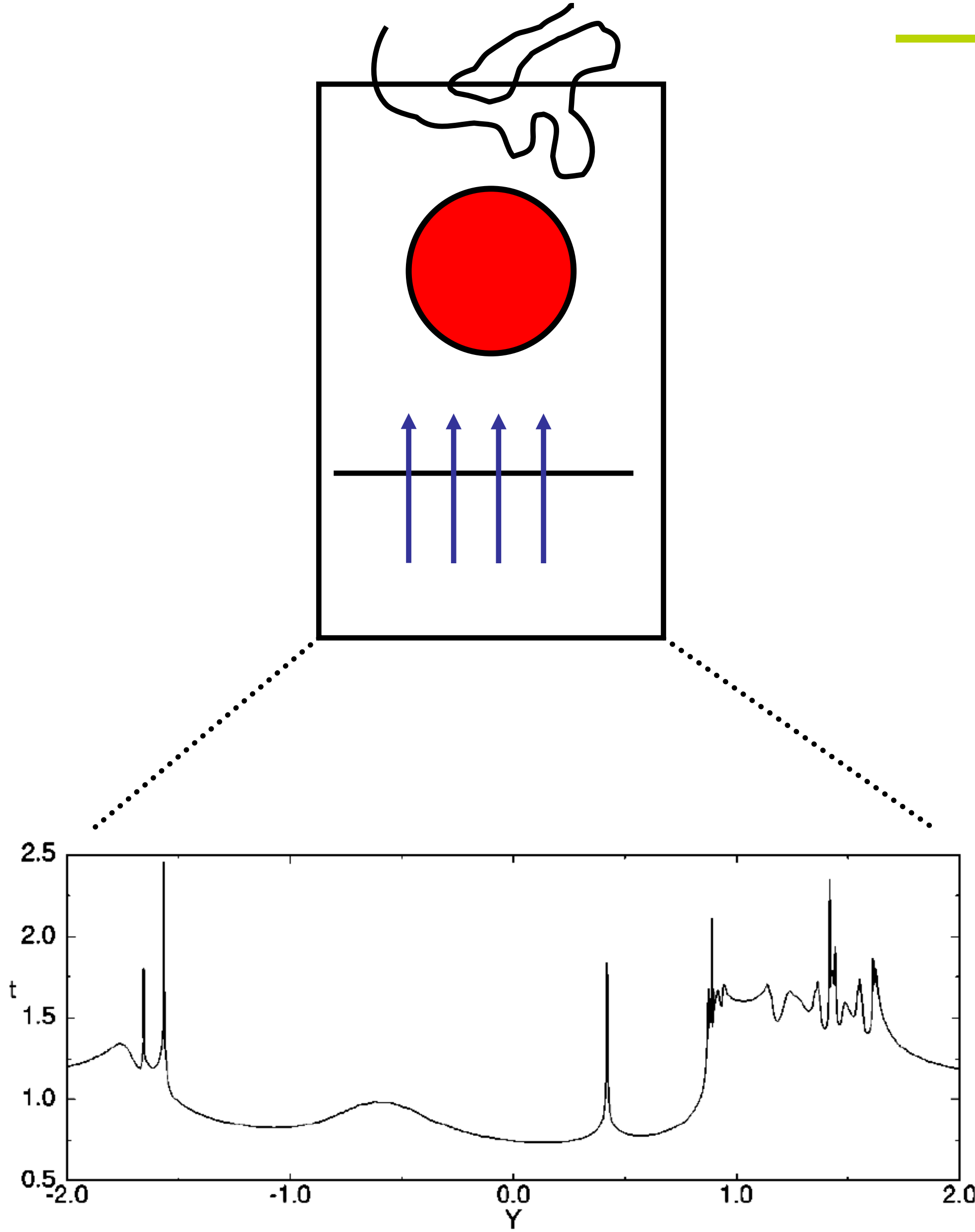


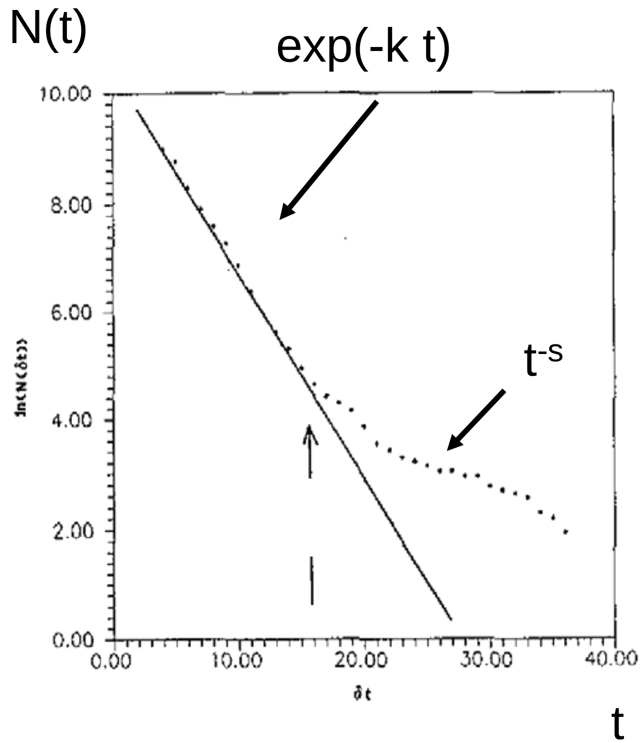




A von Karman vortex street flow







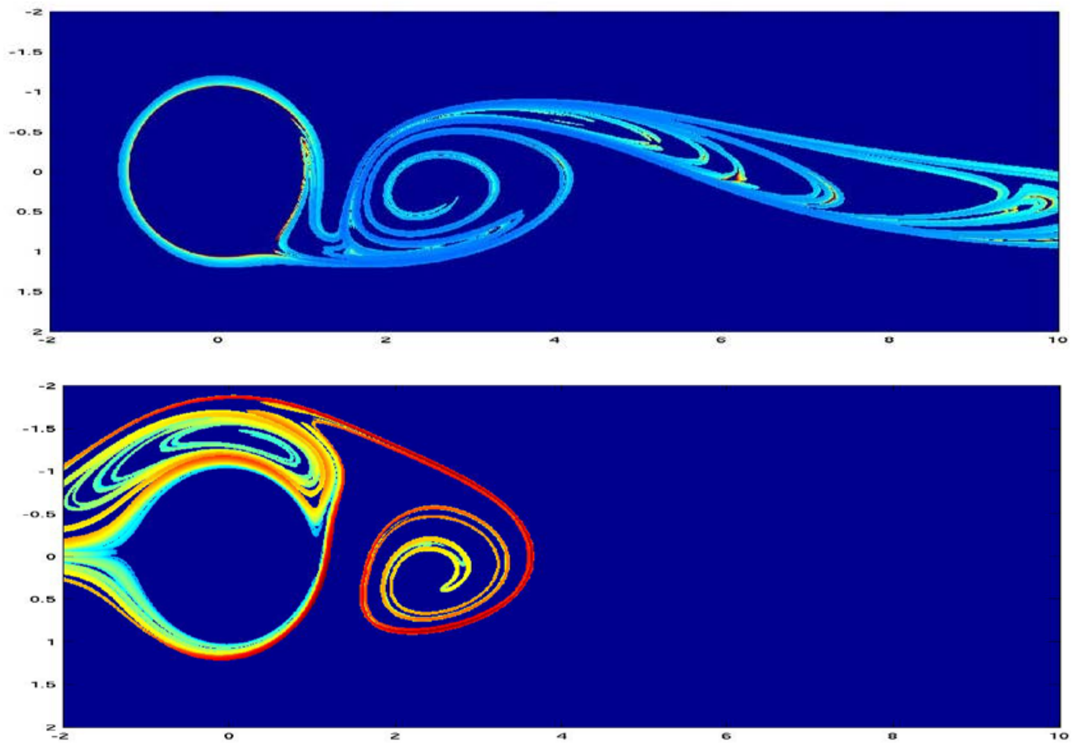
• Exponential escape of almost all trajectories

• Presence of a particular (fractal, zero measure) set of long remaining trajectories

OPEN CHAOTIC FLOWS: wakes, jets, sinks ...

Stable and unstable manifolds of the chaotic saddle

CHAOTIC SADDLE:
set of trajectories never living the chaotic area $\Rightarrow \lambda > 0$
Close trajecctories escape at a rate κ
Fractal set of zero measure.
Dimension D=2(1-κ/λ)
Tracers accumulate at its unstable manifold (dimension 2-κ/λ)



$D_u = 2 - \kappa/\lambda$

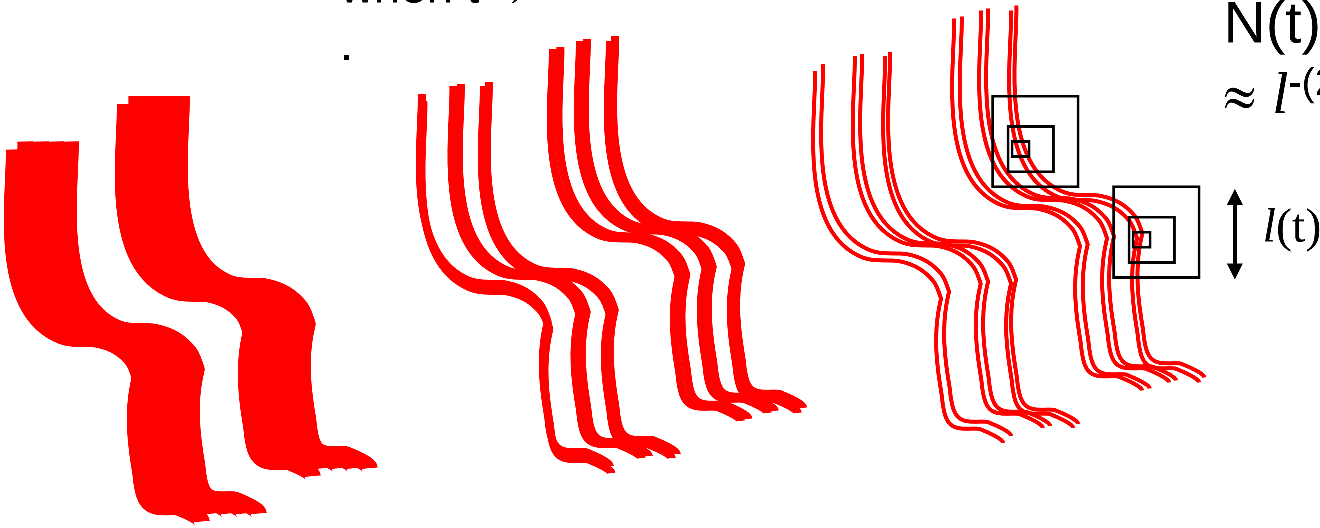
Dimension of the unstable manifold of the saddle (where particles are at long times)

$N \approx l^{-D}$, or $D = \lim_{l \to 0} -(\log N / \log l)$

Perform the limit with boxes of size $l(t) = l_0 \exp(-\lambda t)$, when $t \to \infty$

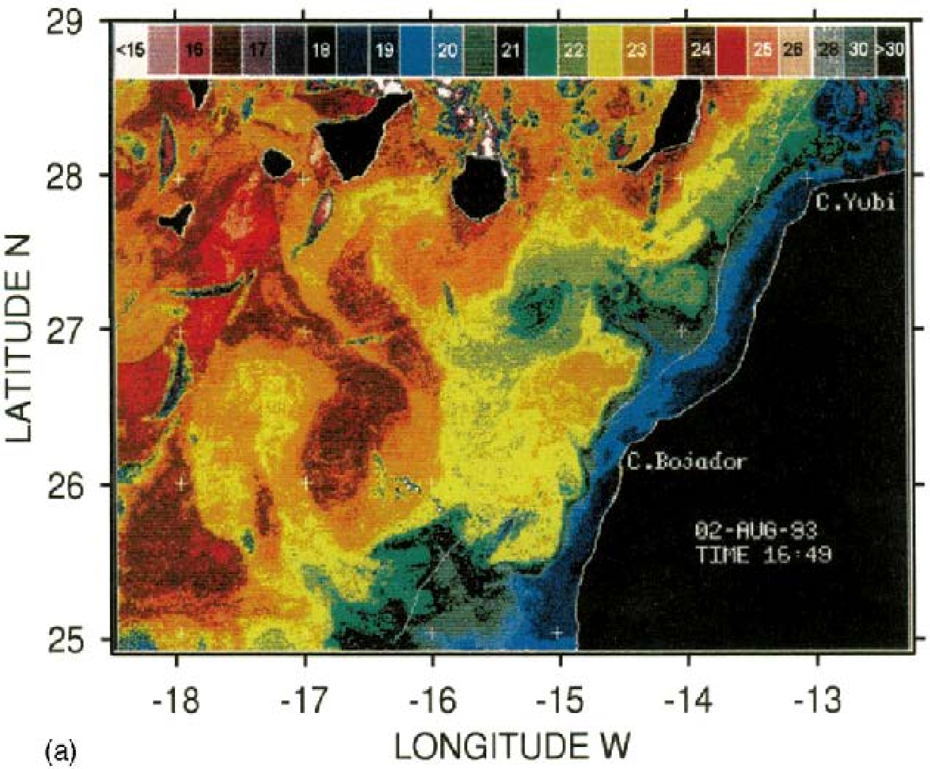
.

$N(t) = A_0 \exp(-\kappa t) / l(t)^2 \approx l^{-(2-\kappa/\lambda)}$.



Dynamics: area decreasing as $A_0 \exp(-\kappa t)$.
Widths decreasing as $l_0 \exp(-\lambda t)$.

'2 E. Barton et al./Progress in Oceanography 41 (1998) 455–504



(a)

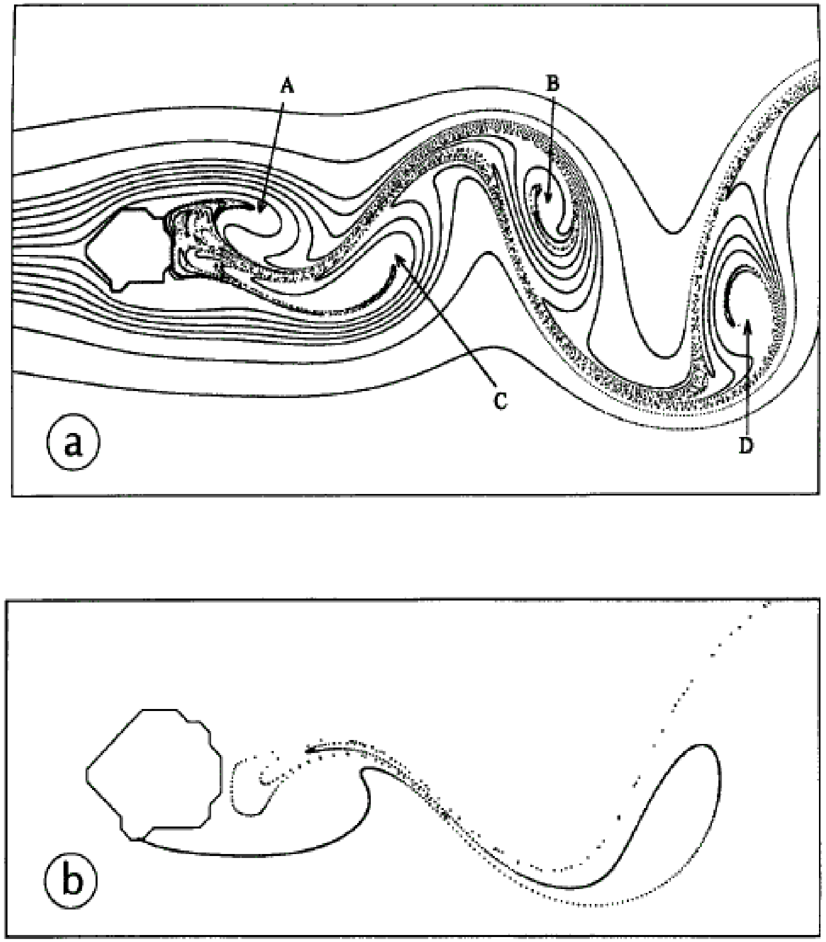


Fig. 10. (a) Simulated streak-line eddies street downstream of Gran Canaria for Reynolds number = 100. (A) Anticyclonic eddy in stage of formation; (B) mature anticyclonic eddy; (C) cyclonic eddy being shed by the obstacle; (D) mature cyclonic eddy. (b) Structure of one streak line that originated on the west side of the island at a different instant than in (a). The incident flow comes from the left side of the picture (northeast). The density of dots is a function of the residence time of the tracer. Regions with dispersion of dots indicate shorter residence times (after Sangrá, 1995).