PHYSICAL REVIEW
LETTERS

VOLUME 79 29 SEPTEMBER 1997 NUMBER 13

Disordering Effects of Color in Nonequilibrium Phase Transitions Induced by Multiplicative Noise

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(Received 28 October 1996; revised manuscript received 16 June 1997)

The model introduced by Van den Broeck, Parrondo, and Toral [Phys. Rev. Lett. **73**, 3395 (1994)]—leading to a second-order-like *noise-induced nonequilibrium phase transition* which shows *reentrance* as a function of the (multiplicative) noise intensity σ—is investigated beyond the white-noise assumption. Through a Markovian approximation and within a mean-field treatment it is found that, in striking contrast with the usual behavior for equilibrium phase transitions, for noise self-correlation time $\tau > 0$, the *stable* phase for (diffusive) spatial coupling $D \to \infty$ is always the *disordered* one. Another surprising result is that a large noise "memory" also tends to *destroy* order. These results are supported by numerical simulations. [S0031-9007(97)04161-6]

PACS numbers: 05.40.+j, 05.70.Ln, 47.20.Ky, 64.60.–i

Too often do we resort, in studying nonequilibrium systems, to the paradigmatic body we have inherited from equilibrium thermodynamics. Though most times this way of reasoning is of valuable help for us to interpret the results, we should be more aware of the fact that sometimes it can be seriously misleading. An archetypical example is the intuitive image we have developed of a close relationship between *noise* and *disorder*, and between *spatial coupling* (and also between *time correlation*) and *order*. Regarding the first, whereas it is true that studies on, e.g., Ginzburg-Landau models subject to *additive* noise seem to reinforce this "rule" [1–3], in the last decade we have also witnessed examples of exactly the opposite trend—namely, dynamical systems in which a *multiplicative* noise couples to the system's nonlinearities in such a way that it generates a transition towards an *ordered* state. In fact, it is by now a well-known fact that the noise can induce a unimodal-bimodal transition in some zero-dimensional models [4]; nevertheless, this result can still be argued to be somewhat restricted since in this case there cannot be breakdown of ergodicity, which is required for a true (nonequilibrium, noise-induced) *phase* transition.

Recently, a model was introduced whereby an *extended* system subject to a Gaussian multiplicative noise—white both in space and time—can undergo a noise-induced symmetry-breaking transition towards an ordered state: this became the first example of a *purely* noise-induced, nonequilibrium, ordering *phase* transition [5]. This result was obtained within a mean-field approximation and confirmed afterwards through extensive simulations in $d = 2$ [6]. In this case, and at variance with the case of order-disorder transitions at equilibrium (induced as we know by the spatial coupling constant D and the bistability of the local potential) it is the short-time instability induced by the multiplicative noise intensity σ—reinforced by the spatial coupling D—which induces the transition. Neither the $d = 0$ system ($D = 0$) nor the deterministic one ($\sigma = 0$) show any transition; moreover—and strikingly enough—those systems exhibiting noise-induced transitions in $d = 0$ are automatically ruled out as candidates for this phenomenon. Such a noise-induced phase transition—besides being of a second-order type as a function of the noise intensity—has the noteworthy feature of being *reentrant:* the ordered state can be found only inside

a window determined by two values of σ. A similar reentrant effect has been observed in the Ginzburg-Landau model with multiplicative and additive noises [7].

One can question whether it is realistic enough to consider a genuine multiplicative noise as white. It appears more likely that the kind of fluctuations leading to multiplicative noise—coming in general from the coupling with an external source—will exhibit some degree of spatiotemporal correlation [4,8–10]. Moreover, one should expect new nontrivial effects as a consequence of the color of the multiplicative noise: after all—whereas it was shown in Refs. [2,3] that an *ordering* nonequilibrium phase transition can be induced in a Ginzburg-Landau model by varying the correlation time of the *additive* noise—a *reentrant* behavior has been found recently in a ($d = 0$) colored-noise-induced transition [11].

It is our aim in this work to investigate the effects of the self-correlation time τ of the multiplicative noise on the model of Refs. [5,6]. To that end we use an Ornstein-Uhlenbeck (OU) noise and apply, in the framework of a mean-field treatment, a "unified colored-noise approximation" (UCNA) [12], together with an interpolation scheme that extends its range of validity in τ [13]. Our main finding is that—at variance with the usual behavior in *equilibrium* statistical mechanics—a large coupling constant D leads invariably for $\tau > 0$ to a *disordered* state. Since (as discussed thoroughly in Ref. [6]) the clue for the phase transition seems to reside in an instability occurring in the short-time behavior, and the model introduced in Ref. [5] was precisely chosen as a representative (perhaps the simplest one) of a host of systems exhibiting such an instability—and hopefully, a noise-induced phase transition with similar characteristics—this unexpected result warns experimentalists seeking for concrete realizations of this phenomenon not to tune naively the spatial coupling intensity D up to a very large value (as one would do guided by "experience") but to look instead for an optimal value of D for which the order parameter would take its maximum value.

Another striking result is that—consistently with the result in Ref. [11]—increasing the "memory," i.e., the self-correlation time τ of the noise, does not favor (as one would naively expect) the transition towards an ordered phase but all the way around. This is indicated by the following facts: (a) the threshold (critical) value of σ is a strongly increasing function of τ, and (b) the window in D available to the ordered phase strongly shrinks as τ increases. There is nonetheless a hint that (as in Ref. [11]) a small amount of color could induce order slightly beyond the upper critical value of σ corresponding to $\tau = 0$, but since the region in which this phenomenon occurs is somewhat narrow, and the comparison with simulations in $d = 2$—together with finite-size scaling—made in Refs. [5,6] sheds much doubt on the precise location of this value, we prefer not to take this result too seriously for the moment (although it certainly deserves further analysis).

As in Ref. [5] we shall resort to a lattice version of the extended system, whose state at time t will then be given by the set of stochastic variables $\{x_i(t)\}$ ($i = 1, \ldots, L^d$) defined at the sites $\mathbf{r}_i$ of a hypercubic d-dimensional lattice of side L. The variables $\{x_i\}$ obey the following system of ordinary stochastic differential equations (SDE):

$$\dot{x}_i = f(x_i) + g(x_i)\eta_i + \frac{D}{2d} \sum_{j \in n(i)} (x_j - x_i), \quad (1)$$

where D is the lattice version of the diffusion coefficient, $n(i)$ stands for the set of $2d$ sites which form the immediate neighborhood of site $\mathbf{r}_i$, and η_i is the *colored multiplicative* noise acting on site $\mathbf{r}_i$. This coupled set of Langevin-like equations is the discrete version of the *partial* SDE which in the continuum would determine the state of the extended system, the last term being replaced—in the continuum limit—by the Laplacian operator $\nabla^2 x$. The specific case analyzed in Ref. [5] (which the authors conjecture that could be the simplest example exhibiting such a transition) is

$$f(x) = -x(1 + x^2)^2 \quad \text{and} \quad g(x) = 1 + x^2. \quad (2)$$

As in Refs. [2,11], the noises $\{\eta_i\}$ are taken to be OU ones, i.e., Gaussian-distributed stochastic variables with zero mean and the following correlations:

$$\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij} \frac{\sigma^2}{2\tau} \exp\left(-\frac{|t - t'|}{\tau} \right). \quad (3)$$

In the limit $\tau \to 0$ the OU noise $\eta_i(t)$ tends to the white noise $\xi_i^W(t)$ with correlations $\langle \xi_i^W(t)\xi_j^W(t') \rangle = \sigma^2 \delta_{ij} \delta(t - t')$, which is the case studied in Ref. [5].

The non-Markovian character of the process $\{x_i\}$ (due to the colored noise $\{\eta_i\}$) makes it difficult to study. However, there are some approximate Markovian techniques that—whereas capturing some of the essential features of the complete non-Markovian process—strongly simplify the treatment of the equations, allowing us to exploit well-known Markovian techniques [10]. Amongst those approximations, the UCNA and related interpolation schemes are very useful since they can reproduce the limits of small and large correlation time τ [12,13]. As discussed in Ref. [12] for a single SDE, the conditions assumed in the UCNA indicate that its validity should decrease with increasing noise intensity. On the other hand, regarding the τ dependence, the UCNA becomes *exact* for $\tau \to 0$ and for $\tau \to \infty$. Although the interpolation procedure in Ref. [13] extends the validity range of this effective Markovian approximation, it is still not clear how far it does so.

We now sketch the main lines of our calculation (a more detailed account is given in Ref. [14]).

(i) For our particular problem, the UCNA proceeds by taking the time derivative of Eq. (1) and—after substitution of the Langevin equations satisfied by the OU noise ($\tau\dot{\eta}_i = -\eta_i + \sigma\xi_i$, where $\{\xi_i\}$ are standard white-noise variables)—setting to zero not only $\ddot{x}_i$ (a usual adiabatic elimination) but also $(\dot{x}_i)^2$, in order to recover a proper Fokker-Planck equation (FPE) description [15].

(ii) In order to reduce the complexity of the resulting system of Markovian SDE, we make the *approximation* of replacing (under the hypothesis that the system is isotropic) *in each equation* of that set the $2d$ variables x_j by a single one y_i.

(iii) Within the Stratonovich prescription we are left with the FPE for a bivariate steady-state probability distribution function (pdf) $P^{st}(x_i, y_i)$. To the drift and diffusion coefficients of this FPE we apply an approximation in the spirit of the Curie-Weiss mean-field type of approach used in Ref. [5], so deriving an effective stationary joint pdf $P^{st}(x, y)$ (we have dropped the subindex i for brevity), from which we derive a one-site pdf $P^{st}(x; \langle x\rangle)$ by assuming $P^{st}(x, y) = P^{st}(x)\,\delta(y - \langle x\rangle)$.

(iv) The value of $\langle x\rangle$ follows then from a self-consistency relation similar to that of Ref. [5]:

$$\langle x\rangle = \int dx\, x P^{st}(x; \langle x\rangle)\,. \tag{4}$$

This equation has always the trivial solution $\langle x\rangle = 0$ corresponding to a disordered phase. When other stable, nontrivial, $\langle x\rangle \neq 0$ solutions appear, the system develops order through a genuine phase transition and $m \equiv |\langle x\rangle|$ can be considered as the order parameter (due to the symmetry of the problem, both $\pm\langle x\rangle$ are solutions of the previous equation). In the white-noise limit $\tau = 0$ this is known to be the case for sufficiently large values of the coupling D and for a window of values of the noise intensity $\sigma \in [\sigma_1, \sigma_2]$.

We now discuss how the presence of ordered states is altered by nonzero values of τ in the mean-field study. Figure 1 shows, in the parameter subspace σ-D, the boundaries separating the ordered and disordered phases for different values of τ. The noteworthy aspects of this graph are the following.

(i) For fixed $\sigma > 1$ and $\tau > 0$ the ordered states can exist *only within a window* of values for D. In other words, the noise-induced nonequilibrium phase transition exhibits reentrance not only with respect to σ (as in the $\tau = 0$ case) but also with respect to D.

(ii) For fixed D and σ inside the $\tau = 0$ phase boundary—as indicated, for example, by point (a) in Fig. 1—there always exists a value of the correlation time τ beyond which the system becomes disordered. Furthermore, there seems to exist a value of $\tau > 0.123$ beyond which order is impossible, whatever the values of σ and D.

(iii) For fixed (and large enough) values of D, and for values of σ that would correspond to the disordered phase for $\tau = 0$, a small increase in τ induces a transition

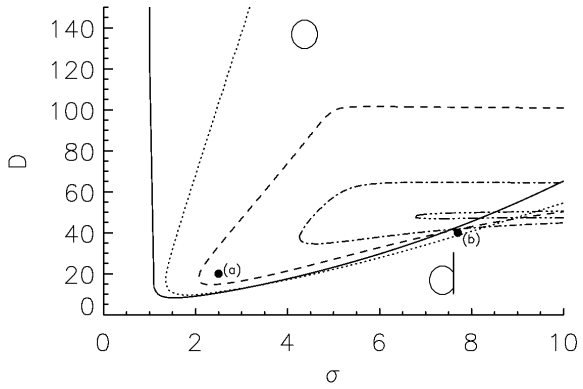

FIG. 1. Projection of the mean-field phase diagram onto the σ-D plane, for $\tau = 0$ (continuous line), $\tau = 0.015$ (dotted line), $\tau = 0.05$ (dashed line), $\tau = 0.1$ (dot-dashed line), and $\tau = 0.123$ (triple dot-dashed line). For each curve, the ordered zone is the area inside the curve (for $\tau = 0$ we have marked the ordered and disordered regions with "o" and "d," respectively). Points (a) and (b) correspond to the transitions referred to in the text.

towards an ordered phase—as indicated by the point marked (b) in Fig. 1. However, a further increase in τ can again lead to disorder. In other words, the transition can also be reentrant with respect to τ. Regarding the reentrant nature of the transition with respect to D, in Fig. 1 it can be seen that—as τ increases from zero—the maximum value of D compatible with the ordered phase reaches, for σ large enough, a "plateau" which is a decreasing function of τ. At the same time, the minimum value of D (that at $\tau = 0$ goes like $D \propto \sigma_2^2$) tends also to become constant as a function of σ as τ increases, so shrinking the window available for the ordered phase until it virtually disappears.

Regarding the character of the transition at $\tau = 0$, we have checked that as τ decreases the lower-left elbow climbs up the σ_1 branch corresponding to $\tau = 0$, but the slope of the corresponding $\tau \neq 0$ branch is always positive.

The previous features can also be inferred from the behavior of the order parameter m. Plotting m vs σ for a fixed value of D and different values of τ, we would see a general trend of the ordered zone to shrink and disappear with increasing τ. Whereas the lower critical value σ_1 increases monotonically with τ, the upper value σ_2 first increases a little and then becomes a monotonically decreasing function of τ. This is, of course, consistent with what occurs around point (b) in Fig. 1.

Since the previous results have been obtained in the mean-field and UCNA approximations, and their range of validity is somewhat unclear, we have also performed numerical simulations in order to have an independent check of the predictions. As a representative example (corresponding to phenomenon (i) above—namely, the destruction of the ordered phase by an increasing coupling constant D) we plot in Fig. 2 m vs D as predicted by our

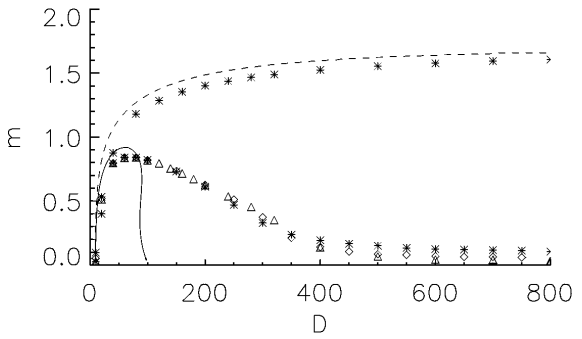

FIG. 2. Mean-field prediction for the order parameter m as a function of the spatial coupling D, for noise intensity $\sigma = 2$ and self-correlation times $\tau = 0$ (dashed line) and $\tau = 0.01$ (continuous line). Notice that whereas for $\tau = 0$ the curve tends to the asymptotic value $(\sigma^2 - 1)^{1/2} = 1.73$, for $\tau = 0.01$ the order parameter seems to fall off rather abruptly to zero for a large value D_{MF} of the coupling. Simulation results for different system sizes: $L = 16$ (asterisks), $L = 32$ (diamonds), and $L = 64$ (triangles) are also included, showing that m indeed decays to zero, although much slower and for a much larger "critical" value D_c.

mean-field theory, and results coming from a numerical integration of the SDE, for σ fixed and two values of τ. Although for $\tau \neq 0$ the numerical results do not follow the mean-field theory, it is obvious that there is an optimal value of the coupling D for which the order parameter takes a maximum value, and that order *disappears* for D large enough. From Fig. 2 one cannot decide whether the maxima will accompany the $\tau = 0$ curve as $\tau \to 0$. It could well be that the phase transition at $\tau = 0$ for σ fixed and D large enough be even a *first-order* one. This certainly calls for further investigation.

We stress again the fact that these effects of a colored multiplicative noise on an extended dynamical system (unable to undergo any phase transition in the absence of noise) are *qualitatively* different to the ones observed in (nonequilibrium) phase transitions driven by a colored additive noise on a prototypic model for equilibrium phase transitions [2,3]. Whereas in the last case the role of the correlation time is to *stabilize* the ordered phase and/or induce order in systems that are disordered for $\tau = 0$, the main effect of color in our case is to *destroy* order. Also, we should not be turned back by the quantitative disagreement between the mean-field theory and the numerical simulations: it is known that in equilibrium phase transitions, mean-field theory overestimates the ordered region and, for example, in the previous study of the same model with white noise [5,6], the mean-field prediction for the upper critical value σ_2 for the reentrant transition was thrice the one found numerically. Although the numerical results are affected by finite-size effects—as one would expect in a second-order phase transition—one can see unambiguously in Fig. 2 the decrease of the order parameter with increasing coupling D, for τ as small as 0.01.

In order to understand this sudden change in behavior as soon as a tiny self-correlation is present, we have studied the time evolution equation for $\langle x \rangle$ (which is small when the parameters are around the phase boundary) within the mean-field approximation, as $\frac{D}{\sigma^2} \to \infty$. In Ref. [6] this simple *linear* (i.e., up to first order in $\langle x \rangle$) criterion of stabilization of the disordered phase was introduced as a way of determining the region of appearance of ordered phases. For $\tau \ll 1$ and $\frac{D}{\sigma^2} \to \infty$ [we indeed assume τD and σ^2 to be $O(1)$], it reads

$$\langle \dot{x} \rangle = -\alpha \langle x \rangle, \quad \text{with } \alpha = \frac{1 + \tau D - \sigma^2}{1 + \tau D}. \qquad (5)$$

When $1 + \tau D > \sigma^2$ it is $\alpha > 0$, and hence the disordered phase ($\langle x \rangle = 0$) is stable. On the other hand, if $1 + \tau D < \sigma^2$ it is $\alpha < 0$, and it is the ordered phase ($\langle x \rangle \neq 0$) which becomes stable. In summary, whereas the noise intensity σ has a stabilizing effect on the ordered phase, as soon as $\tau \neq 0$ the spatial coupling D tends to *destabilize* it. For $\tau = 0$ the last effect is not present, being then the condition for ordering that $\sigma > 1$ (this is the effect that was reported in Refs. [5,6]). Considering that the effect of even a tiny correlation is enhanced by D, we can understand the abrupt change shown in Figs. 1 and 2 as soon as $\tau \neq 0$.

This work has focused on the effects of a self-correlation in the multiplicative noise on the reentrant noise-induced phase transition reported in Ref. [5]. It appears that for $\tau \neq 0$, a strong enough spatial coupling is capable of destroying the order established as a consequence of the multiplicative character of the noise. The foregoing result can be understood by recalling the fact that the ordered phase arises as a consequence of the collaboration between the multiplicative character of the noise and the presence of spatial coupling. When no self-correlation is present, the disordering effect of D cannot be felt. This explains the results in Ref. [6], which have been rightly interpreted in terms of a "freezing" of the short-time behavior by a strong enough spatial coupling. As τ increases, the minimum value of D required to destabilize the ordered phase becomes lower and lower. In this way, the region in parameter space available to the ordered phase shrinks further and further until it vanishes.

The main lesson one can draw from the present results is that the conceptual inheritance from equilibrium thermodynamics (though often useful) is not always applicable. By following the equilibrium-thermodynamic lore, one would tend to think that as $D \to \infty$ an ordered situation is favored. This is certainly true for the Curie-Weiss-type models, since in that case the deterministic potential is itself bistable and an increase of spatial coupling has the effect of rising the potential barrier between the stable states. In the case we are dealing with, the deterministic potential is monostable and it is the combined effects of the multiplicative noise and the spatial coupling that induce the transition.

R. Toral acknowledges financial support from DGICyT, Projects No. PB94-1167 and No. PB94-1172 and H. S. Wio from Fundación Antorchas, Project No. A-13396/1 and CONICET PIP-4593/96.

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